

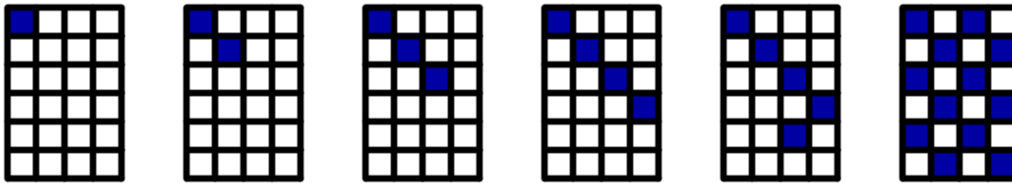
# Solving the Cross Diagonal Cover Problem

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<http://www.msccroggs.co.uk/blog/32>

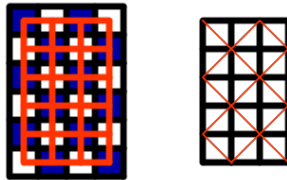
In five blog posts Gaurish Korpall present the Cross diagonal cover problem, some ideas about how to solve it and some conjectures. In this post, I will present my solution to this problem. But first, the problem itself:

Draw with an  $m \times n$  rectangle, split into unit squares. Starting in the top left corner, move at 45 degrees across the rectangle. When you reach the side, bounce off. Continue until you reach another corner of the rectangle:



## Restating the Problem

When I first saw this problem, it reminded me of Rebounds, a puzzle I posted here in 2014. To restate the problem in a similar way, we place a point in the centre of each unit square, then creating a second grid. I will call this the dual grid. The original problem is equivalent to asking, if a line bounces around the dual grid, how many corners will it pass through.

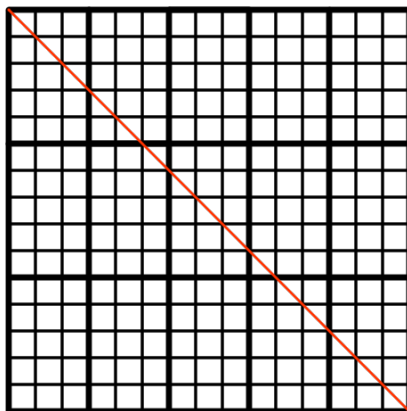


It is worth noting here that the dual grid is  $n - 1 \times m - 1$ : each side is one shorter than the original grid. A corner cannot be travelled over more than twice: otherwise, the line would be retracing its past path; to do this requires it to have already hit a corner of the rectangle. Therefore we can calculate the number of distinct corners travelled to using:

$$\text{Distinct corners visited} = \text{Corners visited} - \text{Corners visited twice}$$

## Introducing Mirrors

When I solved Rebounds, I imagined the line passing through mirror images of the rectangle, rather than bouncing. For our example above, it would look like this:



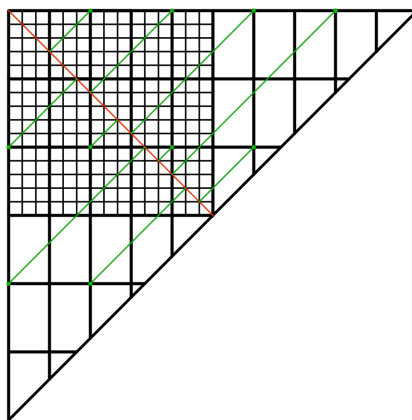
Looking at the puzzle in this way, it can be seen that the line will travel through  $\text{lcm}(n-1, m-1)$  squares, and so hit  $\text{lcm}(n-1, m-1) + 1$  corners (the  $+1$  appears due to fence panels and fence posts).

We have shown that:

$$\text{Corners visited} = \text{lcm}(n-1, m-1) + 1$$

### Collisions in the Mirror

To solve the problem, we need to work out how many corners are visited twice; or the number of times the line crosses itself in the rectangle. To do this, imagine the mirror images of the red line in the mirrors. Ignoring the images parallel to the red line, and terminating the lines when they hit the red line gives the following diagram:



I have added extra rectangles to the diagram so that all the reflections that hit the red line and their starting points can be seen. The diagonal black line has been added because all lines outside that clearly cannot intersect the red line. We now need to justify two claims:

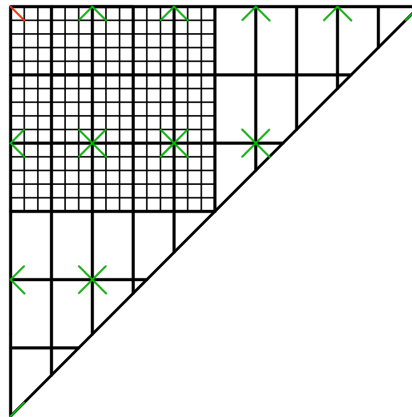
- Each intersection of a red line and a green line corresponds to the line bouncing off the side of the rectangle or crossing itself (in the dual rectangle). Each bounce or crossing matches with exactly one red-green intersection.
- The green lines starting on the outer border are those that correspond to the bounces.

The first claim can be seen by reflecting the green lines and the parts of the red line they hit back into the top left rectangle. The second claim can be shown in two parts: First, each line starting from the border will meet the red line on the edge of a rectangle: this is because the green lines all start a multiple of two rectangles away, and meet at half this distance away (and half a multiple of two is a whole number). Conversely, if a red line meets a green line at the edge of a rectangle, then the reflection of the red line in the edge (ie. the green line) must go back to a starting point on the boundary.

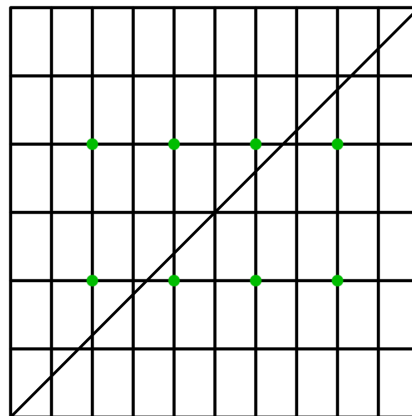
These two claims show that the points where the line crosses itself in the dual rectangle match up one-to-one with the interior points at which the green lines start. So if we can count these points, we can solve the problem.

### Counting the Interior Points

The green lines will start from all points that are a multiple of two rectangles (in both directions) away from the top left. The reason for this can be seen by reflecting the first little bit of the red line in all the mirrors:



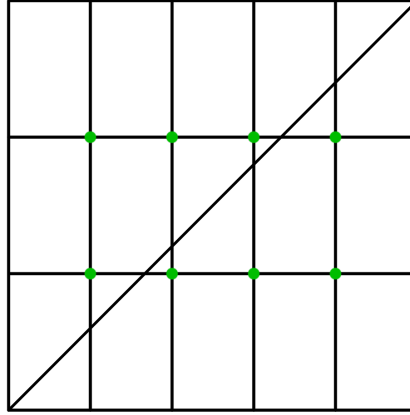
We see that the perpendicular green lines that we are interested in, plus many other irrelevant lines, start from a grid of points on the corner of every other rectangle. To count these points, we will extend them into the full square:



To make for clearer counting, I have not drawn the unit squares.

Alternatively, this square can be thought of as being made up from two copies of the triangle.

We next notice that these points can never lie on the diagonal (the diagonal drawn in the diagram): a green point lying on the diagonal would imply that the red line met a corner before it did. Taking out the lines on which the green dots never lie, we get:



We can now count the green dots. In the square above, there are  $\frac{\text{lcm}(m-1, n-1)}{m-1}$  rectangles vertically and  $\frac{\text{lcm}(m-1, n-1)}{n-1}$  rectangles horizontally. Therefore there are  $\frac{\text{lcm}(m-1, n-1)}{m-1} - 1$  columns of green dots and  $\frac{\text{lcm}(m-1, n-1)}{n-1} - 1$  rows of green dots. (Again, we take one due to fence posts and fence panels.) Of these green dots, half are in the triangle of interest, so:

$$\text{Corners visited twice} = \frac{1}{2} \left( \frac{\text{lcm}(m-1, n-1)}{m-1} - 1 \right) \left( \frac{\text{lcm}(m-1, n-1)}{n-1} - 1 \right)$$

### Putting it together

We can now put the two parts together to get:

$$\text{Distinct corners visited} = \text{lcm}(m-1, n-1) + 1 - \frac{1}{2} \left( \frac{\text{lcm}(m-1, n-1)}{m-1} - 1 \right) \left( \frac{\text{lcm}(m-1, n-1)}{n-1} - 1 \right)$$

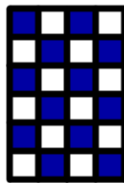
And we have solved the problem.

### Example

For the  $4 \times 6$  rectangle given, our formula gives:

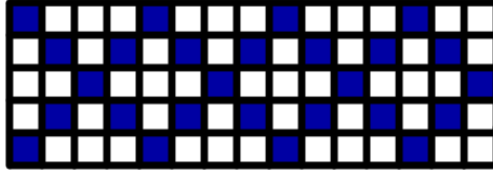
$$\begin{aligned} \text{Distinct corners visited} &= \text{lcm}(3, 5) + 1 - \frac{1}{2} \left( \frac{\text{lcm}(3, 5)}{3} - 1 \right) \left( \frac{\text{lcm}(3, 5)}{5} - 1 \right) \\ &= 15 + 1 - \frac{1}{2}(5-1)(3-1) \\ &= 16 - 4 = 12 \end{aligned}$$

This is correct:



## Proving the Conjecture

In Gaurish's most recent post, he gave the following conjecture: The highest common factor (or greatest common divisor) of  $m$  and  $n$  always divides the number of coloured squares. After trying to prove this for a while, I found that my attempted proof required that  $\text{hcf}(n-1, m-1)^2 - 1$  is a multiple of  $\text{hcf}(n, m)$ . However this is not in general true (eg. 15,5). In fact, 15,5 provides us with a counterexample to the conjecture:



In this diagram, 26 squares are coloured. However  $\text{hcf}(15, 5) = 5$  and 5 is not a factor of 26.