Bachet - 1

Lectures on Mathematical Card Tricks

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Abstract

The earliest discussion of card magic by a mathematician seems to be in Problèmes plaisans et delectables, by Claud Gaspard Bachet, a recreational work published in France in 1612.

1 Monge's Shuffle

In 1773, Gaspard Monge[1] (1746-1818) published a theory for this type of shuffling. In this, a pack of n cards is shuffled by placing the second card on the first, the third below these, the fourth above them, and so on. The following are some of the results:

- If the pack contains $6k+4$ cards, the $(2k+2)$th card will occupy the same position in the shuffled pack. For instance, if a complete pack of 52 cards is shuffled as described above, the 18th card will remain the 18th card.

- If a pack of $10k+2$ cards is shuffled in this way, the $(2k+1)$th and the $(6k+2)$th cards will interchange places. For instance, if a pack of 32 cards is shuffled as described above, the 7th and the 20th cards will change places.

- One shuffle of a pack of $2k$ cards will move the card which was in the x_0th place to the x_1th place, where

$$x_1 = \begin{cases} \frac{2k+x_0+1}{2} & \text{if } x_0 \text{ is odd} \\ \frac{2k-x_0+2}{2} & \text{if } x_0 \text{ is even} \end{cases}$$

  From this, previous results can be deduced.

- By repeated applications of the above formulae, the effect of m such shuffles is to move the card which was initially in the x_0th place to the x_mth place where

$$2^{m+1}x_m = (4k+1)(2^{m-1} \pm 2^{m-2} \pm \ldots \pm 2 \pm 1) \pm 2x_0 + 2^m \pm 1$$

  the sign $\pm$ representing an ambiguity of sign.

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[1] Mémoires de l'Académie des Sciences, Paris, 1773, pp. 390-412

- In any pack of n cards after a certain number of shufflings, not greater than n, the cards will return to their primitive order. This will always be the case as soon as the original top card occupies that position again.
- The number of shuffles required for a pack of $2k$ cards can be determined by putting $x_m = x_0$ and finding the smallest value of m which satisfies the resulting equation for all values of x_0 from 1 to $2k$. It follows that, if m is the least number which makes $4^m - 1$ divisible by $4k+1$, then m shuffles will be required if either $2^m + 1$ or $2^m - 1$ is divisible by $4k+1$, otherwise $2m$ shuffles will be required. The number for a pack of $2k+1$ cards is the same as that for a pack of $2k$ cards. For instance, with a pack of 32 cards, six shuffles are sufficient; with a pack of 2^n cards, $n+1$ shuffles are sufficient; with a full pack of 52 cards, twelve shuffles are sufficient; with a pack of 13 cards ten shuffles are sufficient; while with a pack of 50 cards fifty shuffles are required; and so on.

Theorem 1 (Hudson Law[2]). *Whatever is the law of shuffling, yet if it is repeated again and again on a pack of n cards, the cards will ultimately fall into their initial positions after a number of shufflings not greater than the greatest possible least common multiple of all numbers whose sum is n.*

Proof. Suppose that any particular position is occupied after the 1st, 2nd, ..., kth shuffles by the cards $A_1, A_2, \ldots, A_k$ respectively, and that initially the position is occupied by the card A_0 . Suppose further that after the kth shuffle A_0 returns to its initial position, therefore $A_0 = A_k$. Then at the second shuffling A_2 succeeds A_1 by the same law by which A_1 succeeded A_0 at the first; hence it follows that previous to the second shuffling A_2 must have been in the place occupied by A_1 previous to the first. Thus the cards which after the successive shuffles take the place initially occupied by A_1 are $A_2, A_3, \ldots, A_k, A_1$; that is, after the kth shuffle A_1 has returned to the place initially occupied by it: and so for all the other cards $A_2, A_3, \ldots, A_{k-1}$.

Hence the cards $A_1, A_2, \ldots, A_k$ form a cycle of k cards, one or other of which is always in one or other of k positions in the pack, and which go through all their changes in k shufflings. Let the number n of the pack be divided into $k, l, m, \ldots$ such cycles, whose sum is n; then the l.c.m. of $k, l, m, \ldots$ is the utmost number of shufflings necessary before all the cards will be brought back to their original places. □

In the case of a pack of 52 cards, the greatest l.c.m. of numbers whose sum is 52 will be found by trial to be 180180. Moreover, 8 perfect *riffle shuffle* will return a full deck of 52 cards to its original starting arrangement. Also, note that the top & bottom cards never change position; and 18th & 35th cards alternate positions with each shuffle.

2 Spelling the Spades

This trick was sold in 1920 by Charles T. Jordan, under the title of "The Improved Chevalier Card Trick." The magician asks the spectator to give the deck a single *riffle shuffle*, followed by a cut. The performer takes the deck and runs through it face up, removing all the spades. This packet of spades is handed to the spectator, face down, with the request that he shift one card at a time from top to bottom, spelling A-C-E. The card on the last letter is turned face up. It is the Ace of Spades. The ace is tossed aside and the same spelling procedure repeated for T-W-O. Again, the Two of Spades turns up at the end of the spelling. This continues until all the spades from ace to king have been spelled in this manner.

Method: The magician prepares for the trick in advance by arranging the thirteen spades as follows. He/she holds a king in his left hand. He/she then picks up the queen and places it on the king, saying "Q" to himself. The cards are now shifted one at a time from bottom to top as he/she spells "U-E-E-N." The same procedure is followed for jack, ten, nine, and so on down to ace. In other words, he simply performs in reverse the spelling process which the spectator will perform later. At the finish,

[2] W. H. H. Hudson, Educational Times Reprints, London, 1865, vol. ii, p. 105.

he will have a packet of thirteen cards so arranged that they will spell from ace to king in the manner already described. This packet is placed in the center of the deck and the trick is ready to begin.

> The spectator's riffle shuffle will not disturb the order of the series. It will merely distribute an upper portion of the series into the lower part of the deck and a lower portion of the series into the upper part of the deck.

A cut made near the center of the pack will now bring the series back to its original order, although the cards will of course be distributed throughout the deck. As the magician runs through the pack to remove the spades, he takes the cards one at a time, beginning at the bottom of the deck. As each spade is removed it is placed face down on the table to form a packet of thirteen cards. This packet will then be in proper order for the spelling.

3 Gergonne's Pile Problem

In 1813, Joseph Diez Gergonne[3], extensively studied this trick. Take 27 cards and deal them into three piles, face upwards. By "dealing" is to be understood that the top card is placed as the bottom card of the first pile, the second card in the pack as the bottom card of the second pile, the third card as the bottom card of the third pile, the fourth card on the top of the first one, and so on: moreover it is assumed that throughout the problem the cards are held in the hand face upwards. The result can be modified to cover any other way of dealing.

Request a spectator to note a card, and remember in which pile it is. After finishing the deal, ask in which pile the card is. Take up the three piles, placing that pile between the other two. Deal again as before, and repeat the question as to which pile contains the given card. Take up the three piles again, placing the pile which now contains the selected card between the other two. Deal again as before, but in dealing note the middle card of each pile. Ask again for the third time in which pile the card lies, and you will know that the card was the one which you noted as being the middle card of that pile.

The trick can be finished then in any way that you like. The usual method – but a very clumsy one – is to take up the three piles once more, placing the named pile between the other two as before, when the selected card will be the middle one in the pack, that is, if 27 cards are used it will be the fourteenth.

> The trick is often performed with 15 cards or with 21 cards, in either of which cases the same rule holds.

The trick works by a simple process of elimination. The first deal narrows the card to a group of nine, the second deal narrows it to three, and the last deal to one.

Theorem 2 (Gergonne's equation)**.** *For a pack of twenty-seven cards, three successive deals, each into three piles of nine cards, is sufficient to determine the card. Moreover, if after the deals the pile indicated as containing the given card is taken up ath, bth, and cth respectively, then the card will be the* $(9c - 3b + a)$*th in the pack or will be the* $(9 - 3b + a)$*th card in the pile indicated after the third deal as containing it.*

Proof. Suppose that, after the first deal, the pile containing the selected card is taken up ath: then

(i) at the top of the pack there are $a - 1$ piles each containing nine cards;

(ii) next there are 9 cards, of which one is the selected card;

(iii) lastly there are the remaining cards of the pack.

The cards are dealt out now for the second time: in each pile the bottom $3(a - 1)$ cards will be taken from (i), the next 3 cards from (ii), and the remaining $9 - 3a$ cards from (iii). Suppose that the pile now indicated as containing the selected card is taken up bth: then

[3] Annales de Mathématiques, Nismes, 1813-4, vol. iv, pp. 276-283.

(I) at the top of the pack are $9(b-1)$ cards;

(II) next are $9-3a$ cards;

(III) next are 3 cards, of which one is the selected card;

(IV) lastly are the remaining cards of the pack.

The cards are dealt out now for the third time: in each pile the bottom $3(b-1)$ cards will be taken from (I), the next $3-a$ cards will be taken from (II), the next card will be one of the three cards in (III), and the remaining $8-3b+a$ cards are from (IV). Hence, after this deal, as soon as the pile is indicated, it is known that the card is the $(9-3b+a)$th from the top of that pile. If the process is continued by taking up this pile as cth, then the selected card will come out in the place $9(c-1)+(8-3b+a)+1$ from the top, that is, will come out as the $(9c-3b+a)$th card. □

Since, after the third deal, the position of the card in the pile then indicated is known, it is easy to notice the card, in which case the trick can be finished in some way more effective than dealing again.

If we put the pile indicated always in the middle of the pack we have $a=2, b=2, c=2$, hence $n=9c-3b+a=14$, which is the form in which the trick is usually presented, as was explained above.

We saw that if a, b, c are known, then n is determined. You can permit the spectator to assemble the piles after each deal, picking them up in any order he chooses. The packet of 27 cards is always held face down for dealing and the cards are always dealt into face-up piles. The dealing may also be done by the spectator. In fact, it is not necessary for the performer to touch the cards at any time. He merely watches the procedure and after the third and final assembly, correctly states the numerical position of the chosen card in the packet of 27.

Example 1. Suppose the spectator assembles the piles after the first deal in such a way that the pile containing his card goes on the bottom. The second time he picks up the piles, the pile with his card goes in the middle, and the last time he picks them up the pile goes on top. Determine the position of the chosen card.

Solution. This gives you the key numbers $a=3, b=2$, and $c=1$, hence $n=9-6+3=6$. The card will therefore be sixth from the top of the packet. ◊

We may modify the rule so as to make the selected card come out in any assigned position, say the nth. In this case we have to find values of a, b, c which will satisfy the equation $n=9c-3b+a$, where a, b, c can have only the values 1, 2, or 3.

Hence, if we divide n by 3 and the remainder is 1 or 2, this remainder will be a; but, if the remainder is 0, we must decrease the quotient by unity so that the remainder is 3, and this remainder will be a. In other words a is the smallest positive number (exclusive of zero) which must be subtracted from n to make the difference a multiple of 3. Next let q be this multiple, i.e. q is the next lowest integer to $n/3$: then $3q=9c-3b$, therefore $q=3c-b$. Hence b is the smallest positive number (exclusive of zero) which must be added to q to make the sum a multiple of 3, and c is that multiple.

Example 2. For a pack of twenty-seven cards, modify the rule of three successive deals, each into three piles of nine cards so as to make the selected card come out in the

1. 21st position from the top.

2. 22nd position from the top.

Solution. We just need to find the solutions for Gergonne's equation.

1. We have, $21=9c-3b+a$. Therefore a is the smallest number which subtracted from 21 makes a multiple of 3, therefore a $=3$. Hence $6=3c-b$. Therefore b is the smallest number which added to 6 makes a multiple of 3, therefore $b=3$. Hence $9=3c$, therefore $c=3$. Thus $a=3, b=3, c=3$.

2. We have, $22 = 9c - 3b + a$. The smallest number which must be subtracted from 22 to leave a multiple of 3 is 1, therefore a = 1. Hence $22 = 9c - 3b + 1$, therefore $7 = 3c - b$. The smallest number which must be added to 7 to make a multiple of 3 is 2, therefore $b = 2$. Hence $7 = 3c - 2$, therefore $c = 3$. Thus $a = 1, b = 2, c = 3$. ♢

If any difficulty is experienced in this work, we can proceed thus. Let $a = x+1, b = 3-y, c = z+1$; then x, y, z may have only the values 0, 1, or 2. In this case Gergonne's equation takes the form $9z + 3y + x = n - 1$. Hence, if $n - 1$ is expressed in the ternary scale of notation, x, y, z will be determined, and therefore a, b, c will be known.

The rule in the case of a pack of m^m cards is similar. We want to make the card come out in a given place. Hence, in Gergonne's formula, we are given n and we have to find $a, b, \ldots, k$. We can affect this by dividing n continually by m, with the convention that the remainder are to be alternately positive and negative and that their numerical values are to be not greater than m or less than unity.

Theorem (Gergonne's generalization)**.** *Suppose a pack of m^m cards is arranged in m piles, each containing m^{m-1} cards, and that, after the first deal, the pile indicated as containing the selected card is taken up ath; after the second deal, is taken up bth; and so on, and finally after the mth deal, the pile containing the card is taken up kth. Then when the cards are collected after the mth deal the selected card will be nth from the top where*

$$n = \begin{cases} km^{m-1} - jm^{m-2} + \ldots + bm - a + 1 & \textit{if } m \textit{ is even} \\ km^{m-1} - jm^{m-2} + \ldots - bm + a & \textit{if } m \textit{ is odd} \end{cases}$$

The method of proof for this theorem is same as seen by considering the usual case of a pack of twenty-seven cards, for which $m = 3$, which are dealt into three piles each of nine cards.

If a pack of 256 cards (i.e. $m = 4$) is given, and anyone selects a card out of it, the card can be determined by making four successive deals into four piles of 64 cards each, and after each deal asking in which pile the selected card lay. The reason is that after the first deal you know it is one of sixty-four cards. In the next deal these sixty-four cards are distributed equally over the four piles, and therefore, if you know in which pile it is, you will know that it is one of sixteen cards. After the third deal you know it is one of four cards. After the fourth deal you know which card it is.

Moreover, if the pack of 256 cards is used, it is immaterial in what order the pile containing the selected card is taken up after a deal. For, if after the first deal it is taken up ath, after the second bth, after the third cth, and after the fourth dth, the card will be the $(64d - 16c + 4b - a + 1)$th from the top of the pack, and thus will be known. We need not take up the cards after the fourth deal, for the same argument will show that it is the $(64 - 16c + 4b - a + 1)$th in the pile then indicated as containing it. Thus if $a = 3, b = 4, c = 1, d = 2$, it will be the 62nd card in the pile indicated after the fourth deal as containing it and will be the 126th card in the pack as then collected.

References

[1] W. W. Rouse Ball (1905). *Mathematical Recreations and Essays.* London: Macmillan and Co. Ltd. http://www.gutenberg.org/ebooks/26839

[2] Martin Gardner (1956). *Mathematics, Magic and Mystery.* Dover Publications.