Bachet - 2

Lectures on Mathematical Card Tricks

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Contents

1 Peirce Curiosity 1

2 Pairs Repaired 2

3 The Royal Pairs 3

4 The Cyclic Number 3

Abstract

The earliest discussion of card magic by a mathematician seems to be in Problèmes plaisans et delectables, by Claud Gaspard Bachet, a recreational work published in France in 1612.

1 Peirce Curiosity

In 1860, Charles Sanders Peirce (1839-1914) "cooked up" a number of unusual card effects based upon what he calls "cyclic arithmetic". Following trick illustrates how the principle of an old trick has been transformed in such a way as to increase enormously its entertainment value.

The magician is seated at a table with four spectators. He/she deals five hands (one for himself/herself) of five cards each. Each person is asked to pick up his hand and mentally select one card among the five. The hands are gathered up and once more dealt around the table to form five piles of cards. The magician picks up any designated pile and fans it so that the faces are toward the spectators. He/she asks if anyone sees his/her selected card. If so, the magician (without looking at the cards) immediately pulls the chosen card from the fan. This is repeated with each hand until all the selected cards are discovered. In some hands there may be no chosen cards at all. Other hands may have two or more. In all cases, however, the performer finds the cards instantly.

This trick depends on the obvious fact that if n^2 cards are arranged in the form of a square of n rows, each containing n cards, then any card will be defined if the row and the column in which it lies are mentioned. Let's illustrate a clever usage of this principle.

Example 1. Sixteen cards are placed face up on the table, in the form of a square with four cards on each side. Someone is asked to select any card in his mind and tell the performer which of the four vertical columns his card is in. The cards in each column are then taken up, face upwards, one at a time beginning with the lowest card of each column and taking the columns in their order from right to left – each card taken up being placed on the top of those previously taken up. Once more the cards are dealt to the table to form a square. This dealing is by horizontal rows, so that after the square is completed, the rows which were vertical before are now horizontal. Once more the spectator is asked to state in which vertical row he sees his card. Then the magician instantly points to the chosen card.

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Solution. The magician must remember which of these columns contains the chosen card. The intersection of this column with the horizontal row known to contain the card will naturally enable the magician to point to the card instantly. ♢

The success of the trick depends, of course, on the spectator's inability to follow the procedure well enough to guess the operating principle. Unfortunately, few spectators are that dense.

A little reflection and you will see that the principle of intersecting sets is involved in this version exactly as in the older form as discussed in the example above. But the newer handling serves better to conceal the method and also adds considerably to the dramatic effect. The hands are gathered face down, beginning with the first spectator on the left and going around the table, the magician's own hand going on top of the other four. The cards are then redealt. Any hand may now be picked up and fanned. If spectator number two sees his selected card, then that card will be in the second position from the top of the fan. If the fourth spectator sees his card, it will be the fourth in the hand. In other words, the position. of the chosen card will correspond to the number of the spectator, counting from left to right around the table. The same rule applies to each of the five hands.

2 Pairs Repaired

Throw twenty cards on to a table in ten couples, and ask someone to select one couple. The cards are then taken up, and dealt out in a certain manner into four rows each containing five cards. If the rows which contain the given cards are indicated, the cards selected are known at once.

This depends on the fact that the number of *homogeneous products* of two dimensions which can be formed out of four things, say a, b, c, d, is 10 (namely: $a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd$). Hence the homogeneous products of two dimensions formed out of four things can be used to define ten things. The cards are laid down in a four by five matrix and various spectators just think of any pair of cards. After shuffling and laying the cards back down then having them just point out which rows they lie in, the magician can name the correct cards they were thinking of.

The mathematical aspects and generalizations of this trick were discussed by Bachet. We can restate this trick in terms of *Bachet's rule*. The first couple (1 and 2) are in the first row. Of the next couple (3 and 4), put one in the first row and one in the second. Of the next couple (5 and 6), put one in the first row and one in the third, and so on, as

1	2	3	5	7
4	9	10	11	13
6	12	15	16	17
8	14	18	19	20

After filling up the first row proceed similarly with the second row, and so on.

Enquire in which rows the two selected cards appear. If only one line, the ith, is mentioned as containing the cards then the required pair of cards are the ith and $(i+1)$th cards in that line. These occupy the clue squares of that line. Next, if two lines are mentioned, then proceed as follows. Let the two lines be the ith and the jth and suppose $j > i$. Then that one of the required cards which is in the jth line will be the $(j-i)$th card which is below the first of the clue squares in the ith line. The other of the required cards is in the ith line and is the $(j-i)$th card to the right of the second of the clue squares.

Bachet's rule, in this form, is applicable to a pack of $n(n+1)$ cards divided into couples, and dealt in n rows each containing $n+1$ cards; for there are $\frac{n(n+1)}{2}$ such couples, also there are $\frac{n(n+1)}{2}$ homogeneous products of two dimensions which can be formed out of n things.

Bachet gave the diagrams for the cases of 20, 30, and 42 cards. But, one can simplify this process by using sentences instead of numbers to memorize the positions. For example, "Mutus Dedit Nomen Cocis" is a classic card plot in card magic based on a mnemonic that works with a 20-card mathematical cross-reference principle trick known as the Pairs Repaired. The principle was first described in 1769 by Gilles-Edme Guyot[1]. To start with you dish out twenty cards randomly in pairs face down, and

[1] Nouvelles récréations, physiques et mathématiques, 1740, p. 28 of the Hugard translation (unpublished).

ask ten people to take a pair each, and remember them. Pick up the pairs in their order, and lay them face up on the table as

M	U	T	U	S
1	2	3	2	4
D	E	D	I	T
5	6	5	7	3
N	O	M	E	N
8	9	1	6	8
C	O	C	I	S
10	9	10	7	4

Visualising these words in your mind on the table you can take the first card of the first pair and place it on the 'M' of Mutus and the second in the pair on the 'M' in Nomen. The next pair goes in the two 'U's of Mutus. The cards of the third pair would go on the 'T' in Mutus and Dedit; and so on until all the cards are laid in their places.

The whole table consists of ten letters, each repeated so you can point out a persons cards to them just by knowing what rows they are in; if the person says their cards are in the first and last rows you can point them out in the positions of the letters 'S'.

> The number of homogeneous products of three dimensions which can be formed out of four things is 20, and of these the number consisting of products in which three things are alike and those in which three things are different is 8. This leads to a trick with 8 trios of things, which is similar to that last given–the cards being arranged in the order indicated by the sentence "Lanata levete livini novoto".
> Take a deck of twenty-four cards and place them in 3 pieces scheme "Lanata levete livini novoto". Then ask the audience to memorize the cards. Then assemble them without breaking triples. After that, you have to imagine the reduced sentence on the table and put them under this scheme.

3 The Royal Pairs

The magician removes the kings and queens from the deck. The kings are placed in one pile, the queens in another. The piles are turned face down and one placed on top of the other. A spectator *cuts* the packet of eight cards as often as he wishes. The magician holds the packet behind his back. In a moment he brings forward a pair of. cards and tosses them face up on the table. They prove to be a king and queen of the same suit. This is repeated with the other three pairs.

Method: When the two piles are formed, the magician makes sure that the order of suits is the same in each pile. Cutting will not disturb this rotation of suits. Behind his back he simply divides the packet in half, then obtains each pair by taking the top card of each half. These two cards will always be a king and queen of the same suit.

4 The Cyclic Number

This trick was published in 1942 by magician Lloyd Jones of Oakland, California. It is based on the "cyclic number" 142857. If this number is multiplied by any number from 2 through 6, the result will contain the same digits in the same cyclic order.

The spectator is handed five red cards bearing the values of 2, 3, 4, 5, and 6. The magician holds six black cards arranged so their values correspond to the digits in the number 142857. Both magician and spectator shuffle their respective packets. Actually, the magician "false shuffles" his six cards, keeping them in the original order. (An easy way to do this is to overhand shuffle the packet twice, drawing off the cards one by one with the left thumb. Done rapidly, this gives the impression of a shuffle, though all it does is reverse the order of cards twice, thus leaving them as before.)

The magician deals his cards face up in a row on the table, forming the number 142857. The spectator now draws at random one of his five cards and places it face up beneath the row. Using pencil and paper he multiplies the large number by the value of the card he selected. While he is doing this the magician assembles the six black cards, cuts them once, and leaves them on the table in a face-down pile. After the result of the multiplication is announced, the magician picks up the pile of black cards and once more deals them in a face- up row. They form a six-figure number which corresponds exactly to the result obtained by the spectator.

Method: The black cards are picked up in their original order. It is now a simple matter for the magician to determine the spot at which these cards must be cut. For example, if the spectator is multiplying the original number by 6, the result must end with 2 because 6 times 7 (the last digit in the cyclic number) is 42. So he merely cuts the packet to bring a two to the bottom. When the cards are later dealt to form a row, the two will be the last card dealt and the number will be the same as the spectator's answer.

142857 is a *Kaprekar number*[5] because $142857^2 = 20408122449$ and $122449 + 20408 = 142857$. Moreover, $\frac{1}{7} = 0.\overline{142857}$ gives rise to following,

- *Arithmetic property*: The first nine primes with complete decimal periods are 7, 17, 19, 23, 29, 47, 59, 61, and 97. These can be used to make instant multiplication disks.[3]

- *Geometric property*: Any general conic equation, $ax^2 + bxy + cy^2 + dx + ey + f = 0$ has six coefficients to determine, which means that five points determine the conic. But, in the "1/7 ellipse", $19x^2 + 36xy + 41y^2 - 333x - 531y + 1638 = 0$, the sixth point too lies on the conic, due to a symmetric relation that holds between the six points.[4]

- *Algebraic property*: If we cyclically shift the digits in the repeating part of the decimal expansions of a reduced fraction, we get the repeating part of the decimal expansion of another reduced fraction with the same denominator. In general, if a/b is a reduced fraction where $\gcd(10, b) = 1$, then the other reduced fractions whose decimal has a repeating block that is a shift of the repeating block of the decimal for a/b are precisely those fractions with denominator b and numerator a in the coset of the cyclic subgroup $\langle 10 \pmod{b} \rangle$ of $(\mathbb{Z}/(b))^\times$. Here, we have $b = 7$, $(\mathbb{Z}/(7))^\times = \{1, 2, 3, 4, 5, 6\}$ and the coset $\langle 10 \pmod 7 \rangle = \{1, 10 \mod 7, 10^2 \mod 7, 10^3 \mod 7, 10^4 \mod 7, 10^5 \mod 7\} = \{1, 3, 2, 6, 4, 5\}$. And this is the exact order in which the shift happens i.e. $\frac{3}{7} = 0.\overline{428571}$, $\frac{2}{7} = 0.\overline{285714}$, $\frac{6}{7} = 0.\overline{857142}$, $\frac{4}{7} = 0.\overline{571428}$ and $\frac{5}{7} = 0.\overline{714285}$.[6]

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