Bachet - 3

Lectures on Mathematical Card Tricks

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11 February 2017

Contents

Abstract

The earliest discussion of card magic by a mathematician seems to be in Problèmes plaisans et delectables, by Claud Gaspard Bachet, a recreational work published in France in 1612.

1 Gilbreath Principle

Have a spectator cut the deck and *riffle shuffle* the two parts together just once. Tell him/her to fan the cards and look at their faces to confirm that they are well mixed. Say, "Look near the middle of the deck and find two adjacent cards of the same color. Don't tell me the color, but cut the cards between those two, and complete the cut." The deck is now ordered as a sequence of pairs and each pair has one red and one black card.

Method: Prepare the deck ahead of time with the cards in black/white alternation. No other order is necessary. When you start this trick, you can do any false shuffle that doesn't change the card order. But if you don't have those skills, don't bother. The shuffle and cut – remember, the cut must be between two cards of the same color – destroys the alternation of red and black all right, but it leaves the cards strongly ordered. Each pair still contains both colors. If you think about it a bit, you'll see why it works. But it's not so easy to state a proof in a few words.

In 1958, the Gilbreath principle (an application of combinatorial mathematics) and its use in the trick described were first explained by Norman Gilbreath[1]. The principle can be proved informally as follows. When the deck is cut for a riffle shuffle, there are two possible situations:

Case 1. *The bottom cards of the two halves are of different colour*

After the first card falls, the bottom cards of the two halves will then be the same color, and opposite to that of the card that fell. It makes no difference, therefore, whether the next card slips past the left or right thumb; in either case, a card of opposite color must fall on the previous one. This places on the table a pair of cards that do not match. The situation is now exactly as before. The bottom cards of the halves in the hands do not match. Whichever card falls, the remaining bottom cards will both have the opposite color, and so on. The argument repeats for each pair until the deck is exhausted.

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[1] "Magnetic Colors," in a magic periodical called *The Linking Ring*, Vol. 38, No. 5, July 1958, page 60.

Case 2. *The bottom cards of the two halves are of same colour*

Now the deck is initially cut so that the two bottom cards are the same color. Either card may fall first. The previous argument now applies to all the pairs of cards that follow. One last card will remain. It must, of course, be opposite in color to the first card that fell. When the deck is cut between two cards of the same color (that is, between the ordered pairs), the top and bottom cards of the deck are brought together, and all pairs are now intact.

Another simple way to present the Gilbreath trick is to prepare the deck by reversing every other card. Someone gives the deck a thorough riffle shuffle. If top and bottom cards are facing different ways, cut the deck so they face the same way. You now can hold the deck under a table or behind your back and bring out pairs of cards, each with cards facing opposite ways.

Gilbreath later discovered that his principle is only a special case of what magicians now call the *Gilbreath general principle*. Gilbreath's general principle points up how poorly a riffle-shuffle randomizes. It applies to any repeating series of symbols and can best be explained by following example:

Example 1. Arrange a deck so that the suits repeat throughout in the same order, say spades, hearts, clubs, and diamonds. From the top of this deck deal the cards one at a time to the table to form a pile of 20 to 30 cards. (Actually it does not matter in the least how many cards are in this pile.) Riffle-shuffle the two parts of the deck together. Now, every quartet of cards, from the top down, will now contain a card of each suit.

Solution. It is necessary that one packet be reversed before the shuffle. Dealing cards to the table does this automatically. Another method is to cut off a portion of the deck, turn it over and shuffle this face-up packet into the rest of the deck, which remains face down. A third method is to take cards singly from the top of the deck and push them into the pack, inserting the first card near the bottom, the next anywhere above the previously inserted card (directly above it if you wish), the third above that, and so on until you have gone as high as you can. This is equivalent to cutting off a packet, reversing its order and riffle-shuffling. The deck's original order is destroyed, of course, but the cards remain strongly ordered in the sense that each group of four cards contains all four suits.

A trick applying the Gilbreath principle to a repeating series of length 52 is to arrange one full deck so that its cards are in the same order from top to bottom as the cards in a second deck are from bottom to top. If the two decks are riffle-shuffled into each other and then cut exactly at midpoint, each half will be a complete deck of 52 different cards!

2 The Great Discovery

This trick was published by Bob Hummer in 1939. A pack of n cards is handed to the spectator and is asked to shift the top card to the bottom of the packet, deal the next card to the table, shift the next card to the bottom, deal the next to the table, and so on, until only one card remains. It proves to be the selected card. At what position in the packet must this card originally be placed so that it will become the last card? The position will vary, of course, with the number of cards in the packet. It can be determined by experiment, but for large packets experimenting is tedious.

Fortunately, the binary method for determining the position of a card in a packet of n cards (so that it will be the last card when one follows the procedure of alternately dealing a card to the table and placing a card under the packet) published by Nathan Mendelsohn[2] provides a simple answer. Let $f(n)$ be the position of the selected card, from the top, in the original arrangement of the deck, then

$$f(n) = 2n - 2^{\lfloor \log_2 n \rfloor + 1} + 1$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

[2]Elementary Problems and Solutions: E898 (Discarding Cards), *American Mathematical Monthly*, Vol. 57, No. 7 (Aug. - Sep., 1950), pp. 488-489. http://www.jstor.org/stable/2308314

Hence, we have $f(52) = 104 - 2^{\lfloor 5.7 \rfloor + 1} + 1 = 41$. But, there is a simpler way to determine the value of $f(n)$. Express the number of cards n in the binary system, shift the first digit to the end of the number, and the resulting binary number will indicate the position that the chosen card should be in from the top of the original packet. For example, suppose an entire deck of 52 cards is used. The binary expression for 52 is 110100. We move the first digit to the end: 101001. This new number is 41, therefore the chosen card must be the 41st card from the top of the deck.

Example 2. What size packets can be used if we want the top card of the packet to be the card that remains?

Solution. The binary number for the position of the top card is 1, so we must use packets with binary numbers of $10, 100, 1000, 10000 \ldots$ (in decimal notation packets of $2, 4, 8, 16 \ldots$ cards).

Example 3. What size packets can be used if we want the bottom card of the packet to be the card that remains?

Solution. If we want the bottom card of the packet to be the remaining card, then the binary numbers of the packets must be $11, 111, 1111, 11111 \ldots$ (or $3, 7, 15, 31 \ldots$ cards).

Example 4. Is it possible for the second card from the top of a packet to be the remaining card?

Solution. No. In fact, no card at an even position from the top can ever be the remaining card. The position of the chosen card, expressed as a binary number, must end in 1 (because after the first digit, which must be 1, is moved to the end it forms a number ending in 1) . All binary numbers ending in 1 are odd numbers.

An equivalent way of calculating the position had long before been known to magicians: simply take from n the highest power of 2 that is less than n, and double the result. This gives the card's position if the first card is dealt to the table. If the first card is placed beneath the packet, 1 is added to the result. (If n is itself a power of 2, the card's position is on top of the packet if the first card goes beneath, on the bottom of the packet if the first card is dealt.)

In 1950, John Scarne published a pamphlet called Scarne's Quartette, explaining four tricks using this principle. Here, from one of Scarne's four tricks, is a simple handling that shows how the principle can be cleverly concealed.

Someone shuffles a deck and hands it to you. Fan the deck, faces toward you, and state that you will determine in advance a card that will be selected. Note the top card of the deck and write its name on a slip of paper that you put aside without letting anyone see what you have written. Assume that the card is the two of hearts.

The deck is held face down in your left hand. Ask a spectator to give you any number from 1 to 52, but preferably above 10 to make the trick more interesting. Suppose he says 23. Mentally subtract the highest power of 2 you can, in this case, 16, to get 7. Twice 7 is 14. Your task now is to get the top card, the two of hearts, to the fourteenth position in a packet of 23 cards.

This is done as follows. Count the cards singly by taking them from the toy of the deck with your right thumb. This reverses the order of the cards. After counting 14, pause and say (as though you had forgotten), "What number did you give me?" When he tells you it was 23, nod, say "Oh, yes-twenty-three," and continue counting. Now, however, you take the cards from the deck by pushing them to the right with your left thumb and sliding each card under the packet in your right hand. Thus when you have counted 23 cards, the two of hearts has subtly been placed in the fourteenth position. Your pause and question breaks the counting into two parts, and no one is likely to notice that the two counting procedures are not the same. Hand the packet of 23 cards to the spectator with the request that he deal the first card to the table, the next one to the bottom of the pile in hand, the next to the table, and so on until a single card remains. It will, of course, be the card you predicted.

There are three more ways of presenting this trick published by Sam Schwartz, Ronald Wohl and George Heubeck respectively. For details of their presentation see pp. 156–158 of [3]. The problem of determining the card's position in such tricks is a special case of a more general problem known as *Josephus problem.* Its statement is as follows:

> A group of men stand in a circle. All but one are to be executed. The executioner starts counting round and round the circle, executing every nth man, until only one man remains. The last man is given his freedom. Where should a man stand in order to escape execution?

When $n = 2$, we have the card situation.

3 Remembering the Future

This trick was invented by Stewart James of Courtright, Ontario. From a thoroughly shuffled deck you remove nine cards with values from ace (A) to 9, arranging them in sequence with the ace on top i.e. $A - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9$. Show the audience what you have done; then explain that you will cut this packet of nine cards so that no one will know what cards are at what positions. Hold the packet face down in your hands and appear to cut it randomly but actually cut it so that three cards are transferred from bottom to top. From the top down the cards will now be in the order: $7 - 8 - 9 - A - 2 - 3 - 4 - 5 - 6$.

Slowly remove one card at a time from the top of this packet, transferring these cards to the top of the deck. As you take each card, ask a spectator if he wishes to select that card. He must, of course, select one of the nine. When he says "Yes", leave the chosen card on top of the remaining cards in the packet and put the packet aside. The deck is now cut at any spot by a spectator to form two piles. Count the cards in one pile; then reduce this number to its *digital root* by adding the digits until a single digit remains. Do the same with the other pile. The two roots are now added, and if necessary the total is reduced to its digital root. The chosen card, on top of the packet placed aside, is now turned over. It has correctly predicted the outcome of the previous steps!

Since 9 is the largest digit in the decimal number system, the sum of the digits of any number will always be congruent modulo 9 to the original number i.e. the number and the sum of its digits will always leave the same remainder when divided by 9. The digits in this second number can then be added to obtain a third number congruent to the other two, and if we continue this process until only one digit remains, it will be the remainder itself. For example, 4157 has a remainder of 8 when divided by 9. Its digits total 17, also has a remainder of 8 when divided by 9. And the digits of 17 add up to 8. This last digit is called the *digital root* of the original number. It is the same as the number's remainder when divided by 9, with the exception of numbers with a remainder of 0, in which case the digital root is 9 instead of 0.

This self-working card trick also depends on the properties of digital roots. After the nine cards are properly arranged and cut, the 7 will be on top. The deck will consist of 43 cards, a number with a digital root of 7. If the spectator does not choose the 7, it is added to the deck, making a total of 44 cards. The packet now has an 8 on top, and 8 is the digital root of 44. In other words, the card selected by the spectator must necessarily correspond to the digital root of the number of cards in the deck. Cutting the deck in two parts and combining the roots of each portion as described will, of course, result in the same digit as the digital root of the entire deck.

Digital roots are often useful as negative checks in determining whether a very large number is a perfect square or cube. All square numbers have digital roots of 1, 4, 7, or 9, and the last digit of the number cannot be 2, 3, 7, or 8. A cube may end with any digit, but its digital root must be 1, 8, or 9. Most curiously of all, an even perfect number[a] (and so far no odd perfect number has been found) must end in 6 or 28 and, with the exception of 6, the smallest perfect number, have a digital root of 1.

[a] The number n is perfect if the sum of all its positive divisors except itself is equal to n. 6, 28, 496 and 8128 are the only perfect numbers below 10000. As of now, $2^{74207280} \times (2^{74207281} - 1)$ is the largest known perfect number (and is 44,677,235 digits long).

4 A Mathematical Wizard

A spectator is asked to cut the deck near the center without completing the cut, then pick up either half. He counts the cards in this half. Let us assume there are 24. The 2 and 4 are added to make 6. He looks at the sixth card from the bottom of the half-deck he is holding, then replaces the half-deck on the other half, squares the pack, and hands it to the magician. The magician starts dealing the cards from the top, spelling aloud the phrase "A M-A-T-H-E-M-A-T-I-C-A-L W-I-Z-A-R-D," one letter for each card dealt. The spelling terminates on the selected card.

Method: The described procedure always places the chosen card nineteenth from the top of the pack. Therefore any phrase of nineteen letters will spell to the chosen card. This trick is based on the fact that if you add the digits in a number and subtract the total from the original number, the result will always be a multiple of nine.

Bill Nord, the New York City amateur conjuror who invented this effect, suggested "The Magic of Manhattan," but as seen above, any phrase of nineteen letters will of course work just as well. Please note that, "wizard" (just like "magician") can be used for both males and females[3].

References

[1] Martin Gardner (1966). "Victor Eigen: Mathemagician" in *Sphere Packing, Lewis Carroll, and Reversi: Martin Gardner's New Mathematical Diversions*, pp. 106–117. The Mathematical Association of America and Cambridge University Press.

[2] Martin Gardner (1977). "Playing Cards" in *Mathematical Magic Show*, pp. 94–104. The Mathematical Association of America.

[3] Martin Gardner (1969). "Chicago Magic Convention" in *The Unexpected Hanging and Other Mathematical Diversions*, pp. 147–159. The University of Chicago Press.

[4] Martin Gardner (1961). "Digital Roots" in *Origami, Eleusis, and the Soma Cube: Martin Gardner's Mathematical Diversions*, pp. 32–38. The Mathematical Association of America and Cambridge University Press.

[5] Martin Gardner (1956). *Mathematics, Magic and Mystery*. Dover Publications.

Prepared in $\LaTeX\,2_\varepsilon$ by Gaurish Korpal

[3] In *Harry Potter and the Prisoner of Azkaban* (book), Ron uses "wizard" to describe a group of presumably mixed gender individuals. See: http://scifi.stackexchange.com/q/117706