Celebrating Uncle Paul's 103^{rd} Birthday

Student Research Seminar - 3

Gaurish Korpal*
gaurish4math.wordpress.com

March 26, 2016

Abstract

We know that the sequence of prime numbers $2, 3, 5, 7, \ldots$ is infinite and the size of its gaps between two prime numbers is not bounded. In fact we can give a sequence of k-consecutive composite numbers, $N+2, N+3, N+4, \ldots, N+(k+1)$ where $N = \prod_{p \leq k+2} p$. Bertrand conjectured that *the gap to the next prime cannot be larger than the number we start our search at.* I will pay homage to Paul Erdős (whom I will refer to as "Uncle Paul") by discussing his elegant proof of Bertrand's conjecture.

1 Introduction

Paul Erdős was one of the founder of probabilistic number theory. He pursued problems in combinatorics, graph theory, number theory, classical analysis, approximation theory, set theory, and probability theory. He holds record of publishing around 1500 articles with 507 coauthors. You can find 1265 of his papers at http://www.renyi.hu/~p_erdos/Erdos.html

He died at age of 83 and was living counterexample of G. H. Hardy's statement[1]:

> o mathematician should ever allow him to forget that mathematics, more than any other art or science, is a young man's game. & Galois died at twenty-one, Abel at twenty-seven, Ramanujan at thirty-three, Riemann at forty. There have been men who have done great work later; & [but] I do not know of a single instance of a major mathematical advance initiated by a man past fifty. & A mathematician may still be competent enough at sixty, but it is useless to expect him to have original ideas.

Why are we celebrating 103^{rd} birthday? During my 5-year stay at NISER, his 103^{rd} birthday is his only prime birthday! Of all the numbers, the primes that were Uncle Paul's "best friends".

Even at this early point in his career, Uncle Paul had definite ideas about mathematical elegance. He believed that God, whom he called the S.F. or Supreme Fascist, had a transfinite book ("transfinite" being a mathematical concept for something larger than infinity) that contained the shortest, most beautiful proof for every conceivable mathematical problem. The highest compliment he could pay to a colleague's work was to say, "That's straight from The Book."

2 Bertrand's Postulate

In 1845, *Joseph Bertrand*[2] conjectured (what he called *postulate*) that

A prime can always be found between any integer $n > 1$ and its double.

*2^{nd} year Integrated M. Sc. student at NISER, Bhubaneswar (Jatni), India

[1] *A Mathematician's Apology* (1940), available at https://www.math.ualberta.ca/mss/misc/A%20Mathematician's%20Apology.pdf

[2] Joseph Bertrand. *Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme.* Journal de l'Ecole Royale Polytechnique, Cahier 30, Vol. 18 (1845), 123-140.

Bertrand himself verified his statement for all numbers in the interval $[2, 3 \times 10^6]$.

In 1852, *Pafnuty Lvovich Chebyshev*[3] attempted to prove the Prime Number Theorem (PNT)

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln(x)} = 1$$

where $\pi(x)$ is the prime-counting function. Though he couldn't prove PNT but his estimates for $\pi(x)$ were strong enough for him to prove Bertrand's postulate.

Similar proof was given by *Edmund Georg Hermann Landau* in 1909 but it introduced a very useful idea for verification of Bertrand's Postulate for finite values of n, what we now call *Landau's Trick*[4]. *Srinivasa Ramanujan Iyengar*[5] provided a short proof in 1919 but it also used the prime-counting function, $\pi(x)$.

As a college freshman in 1932, Uncle Paul (when he was 19) made a name for himself in mathematical circles with a stunningly simple proof of Bertrand's Postulate. As for Bertrand's Postulate, no one doubted that he had found The Book proof.

3 Proof from "The Book"

I am 19 and I guess it will be nice to go through Uncle Paul's first published work on his birthday. This will be a kind of exposition in English for original paper in German[6]. The basic idea of the proof is to show that a certain central binomial coefficient needs to have a prime factor within the desired interval in order to be large enough. This is made possible by a careful analysis of the prime factorization of central binomial coefficients.

Step 1 `Let` p `denote the prime numbers then` $\prod_{p\le x} p \le 2^{2(x-1)} \quad \forall x \in \mathbb{R}_{\ge 2}$

Note that if q is the largest prime with $q \le x$, then

$$\prod_{p\le x} p = \prod_{p\le q} p \quad \text{and} \quad 2^{2(q-1)} \le 2^{2(x-1)}$$

Now we will prove given statement by induction on q.

Base case: $q = 2$ then $2 < 4$. Hence given statement is holds in this case.

Induction hypothesis: For all integers x in the set $\{2, 3, \dots, 2m\}$ we have $\prod_{p\le x} p \le 2^{2(x-1)}$

Inductive step: Consider odd primes $q = 2m + 1$, then we split the product

$$\prod_{p\le 2m+1} p = \prod_{p\le m+1} p \cdot \prod_{m+1<p\le 2m+1} p$$

Exercise. *We have*

$$\prod_{m+1<p\le 2m+1} p \le \binom{2m+1}{m} \le 2^{2m}$$

Hint: Compare the prime factors of $(2m+1)!$ and $m!(m+1)!$ for given primes. $\sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1}$ and the binomial coefficients form a sequence that is symmetric and unimodal.

$$\Rightarrow \prod_{p\le 2m+1} p \le 2^{2m} \cdot 2^{2m} = 2^{4m}$$

Hence proving the claim.

[3] P. Tchebychev. *Mémoire sur les nombres premiers* (in English: Memory on prime numbers). Journal de mathématiques pures et appliquées, Sér. 1(1852), 366-390. (Proof of the postulate: 371-382)

[4] Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (in English: Handbook of the theory of distribution of prime numbers), Leipzig and Berlin, 1909, Bd. 1, S. 92 [`https://archive.org/details/handbuchderlehre01landuoft`]

[5] Ramanujan, S. (1919), *A proof of Bertrand's postulate*, Journal of the Indian Mathematical Society 11: 181–182

[6] A nice English translation of this paper (with comments) is available on pp. 8–12 of M. Aigner and G. M. Ziegler, *Proofs from THE BOOK* (4th ed.), Springer, 2010.

Step 2 For $n \geq 3$,

$$\binom{2n}{n} > \frac{2^{2n}}{2n}$$

Observe that,

$$2n\binom{2n}{n} = \frac{2}{1} \cdot \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{5}{2} \cdots \frac{2n-2}{n-1} \cdot \frac{2n-1}{n-1} \cdot \frac{2n}{n} \cdot \frac{2n}{n} > 2^{2n}$$

Step 3 For $n \in \mathbb{Z}_{>0}$, we have $2n < 2^{6(2n)^{\frac{1}{6}}}$.

$$2n = \left((2n)^{\frac{1}{6}}\right)^6 < \left((2n)^{\frac{1}{6}} + 1\right)^6$$

Exercise. *For $n \geq 2$ we have $n+1 < 2^n$.*

Hint: Apply induction on n

$$\Rightarrow 2n < \left(2^{\left\lfloor (2n)^{\frac{1}{6}} \right\rfloor}\right)^6 < 2^{6(2n)^{\frac{1}{6}}}$$

Step 4 The largest power of primes that divide $\binom{2n}{n}$ is not larger than $2n$
We have to estimate the number of times the prime factor p occurs in $\binom{2n}{n} = \frac{(2n)!}{n!n!}$.

Exercise. *The number of times the prime factor p occurs in $n!$ is given by*

$$\sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor$$

The number of times the prime factor p occurs in $\binom{2n}{n}$ is

$$\sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

The summands vanish whenever $p^k > 2n$. Moreover, each summand is at most 1 since it is an integer and satisfies $\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor < \frac{2n}{p^k} - 2\left(\frac{n}{p^k} - 1\right) = 2$. Thus we get

$$\sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right) < \max\{\ell : p^\ell \leq 2n\}$$

Hence the largest power of p that divides $\binom{2n}{n}$ is not larger than $2n$.

Step 5 Primes with $p > \sqrt{2n}$ appear at most once in $\binom{2n}{n}$ and the primes with $\frac{2}{3}n < p \leq n$ don't appear at all in $\binom{2n}{n}$ for $n \geq 3$
This was Uncle Paul's key observation.
From Step 4 we conclude that the primes $p > \sqrt{2n}$ appear at most once in $\binom{2n}{n}$. Moreover, $3p > 2n$ implies that p and $2p$ are the only multiples of p that appear as factors in the numerator of $\frac{(2n)!}{n!n!}$, while $p \leq n$ implies that there are two p-factors in the denominator. From this we conclude that such primes don't appear at all in $\binom{2n}{n}$.

Step 6 If $\binom{2n}{n}$ didn't have any prime factors in the range $n < p \leq 2n$, then it would be too small for n large enough.
For $n \geq 5$ we have[7] $\sqrt{2n} < \frac{2}{3}n$ hence we can write

$$\prod_{p \leq 2n} p = \prod_{p \leq \sqrt{2n}} p \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \cdot \prod_{\frac{2}{3}n < p \leq n} p \cdot \prod_{n < p \leq 2n} p$$

[7] Erdős used $n \geq 3$ in [1], which I believe is wrong since $\sqrt{6} > 2$.

Exercise. *For all $n \geq 2$*
$$\binom{2n}{n} \leq \prod_{p \leq 2n} p$$

From Step 4 and Step 5 we get:

$$\binom{2n}{n} \leq (2n)^{\sqrt{2n}} \cdot \prod_{\sqrt{2n}<p\leq \frac{2}{3}n} p \cdot \prod_{n<p\leq 2n} p$$

Assume now that there is no prime factor p with $n < p \leq 2n$, so $\prod_{n<p\leq 2n} p$ vanishes. From Step 1 we get

$$\binom{2n}{n} \leq (2n)^{\sqrt{2n}} \cdot 2^{\frac{4}{3}n} \tag{1}$$

Using Step 2 in above equation

$$\frac{2^{2n}}{2n} < (2n)^{\sqrt{2n}} \cdot 2^{\frac{4}{3}n}$$
$$\Rightarrow 2^{2n} < (2n)^{3(\sqrt{2n}+1)}$$

Now Step 3 leads to

$$2^{2n} < \left(2^{6(2n)^{\frac{1}{6}}}\right)^{3(\sqrt{2n}+1)} = 2^{(18\sqrt{2n}+18)(2n)^{\frac{1}{6}}}$$

For $n \geq 50$ we have $18 < 2\sqrt{2n}$, thus

$$2^{2n} < 2^{20\sqrt{2n}(2n)^{\frac{1}{6}}} = 2^{20(2n)^{\frac{2}{3}}}$$
$$\Rightarrow (2n)^{\frac{1}{3}} < 20$$
$$\Rightarrow n < 4000$$

Hence (1) will lead to very small value of $\binom{2n}{n}$ for n large enough.[8]

Step 7 `Verification of Bertrand's postulate for` $n < 4000$
Following Landau's trick we do not need to check 4000 cases, rather it suffices to check that

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, 4001$$

is a sequence of prime numbers, where each is smaller than twice the previous one.

Exercise. *Check that every number in above sequence is a prime number.*

Hence every interval $(n, 2n]$ with $n < 4000$, contains one of these 14 primes.

References

[1] Paul Erdős. *Beweis eines Satzes von Tschebyschef* (in English: Proof of a theorem of Chebyshev). Acta Litt. Sci. (Szeged) 5 (1932), pp. 194–198.

[2] Paul Hoffman (1999). *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*. Hyperion. ISBN 978-0-7868-6362-4.

[8] We can get smaller bound on n by using stricter inequalities. For example, G. H. Hardy & E. M. Wright proved $n < 512$; see Theorem 418 on pp. 455 of *An Introduction to the Theory of Numbers* (Sixth Edition)