Celebrating 110^{th} Birthday of D. R. Kaprekar

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Abstract

The motive of this talk is to introduce students to 'Elementary Number Theory' and 'Iterations' by discussing three different works of great Indian mathematician D. R. Kaprekar.

1 Biography

Dattaraya Ramchandra Kaprekar [1]

Dattaraya Ramchandra Kaprekar (17 January 1905 – 1988) was an Indian recreational mathematician who described several classes of natural numbers. For his entire career (1930–1962) he was a schoolteacher at Nasik in Maharashtra.It required G. H. Hardy to recognize Ramanujan while Kaprekar's recognition came through Martin Gardner (he wrote about Kaprekar in his "Mathematical Games" column in March 1975 issue of "Scientific American") In his lifetime Kaprekar discovered more than 14 different types of numbers and various iterative algorithms.
In this seminar I will discuss the few of them.

2 Kaprekar Sequence

The sequence $a_1, a_2, \ldots$ is called a Kaprekar sequence, denoted by K_{a_1} , if a_1 (called *generator*) is a positive integer and $a_{k+1} = a_k + s_d(a_k)$, for $k > 1$, where $s_d(n)$ denotes the sum of the digits of n. For example, if $a_1 = 1$, we obtain the Kaprekar sequence K_1 = 1,2,4,8,16,23,28,. . . .

In 1959, Kaprekar showed that there are three types of Kaprekar sequence:

- **Type I** : Each term is not divisible by 3
- **Type II** : Each term is divisible by 3 but not by 9
- **Type III** : Each term is divisible by 9

For example: K_1 is **Type I**, K_3 & k_6 are **Type II** and K_9 is **Type III**

Determine the Kaprekar type for K_{a_1} , when $a_1 = k$, for $2 \le k \le 10$.

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Proof Outline:
As divisibility test for 3 and 9 is based on sum of digits, thus we can categorize sequences generated by sum of digits as per the divisibility of terms with respect to 3 and 9.
Let,

$$a_1 = b_0 + b_1 \times 10 + b_2 \times 10^2 + \ldots + b_n \times 10^n = \sum_{i=0}^{n} (b_i \times 10^i)$$

Also,

$$\sum_{i=0}^{n} (b_i \times 10^i) \equiv \sum_{i=0}^{n} b_i \pmod 3 \text{ and } \sum_{i=0}^{n} (b_i \times 10^i) \equiv \sum_{i=0}^{n} b_i \pmod 9$$

CASE 1: a_1 is not divisible by 3, then,

$$\Rightarrow a_1 \equiv q \pmod 3 \text{ and } \sum_{i=0}^{n} b_i \equiv q \pmod 3 \qquad (q = 1 \quad or \quad 2)$$

$$\Rightarrow a_1 + \sum_{i=0}^{n} b_i \equiv 2q \pmod 3 \equiv q' \pmod 3 \qquad (q' = 1 \quad or \quad 2)$$

Thus if generator term is NOT divisible by 3, then every term of series generated by sum of its digits is also NOT divisible by 3.

CASE 2: a_1 is divisible by 3, then,

$$\Rightarrow a_1 \equiv 0 \pmod 3 \text{ and } \sum_{i=0}^{n} b_i \equiv 0 \pmod 3$$

$$\Rightarrow a_1 + \sum_{i=0}^{n} b_i \equiv 0 \pmod 3$$

Thus if generator term is divisible by 3, then every term of series generated by sum of its digits is also divisible by 3.

CASE 3: a_1 is divisible by 9, then,

$$\Rightarrow a_1 \equiv 0 \pmod 9 \text{ and } \sum_{i=0}^{n} b_i \equiv 0 \pmod 9$$

$$\Rightarrow a_1 + \sum_{i=0}^{n} b_i \equiv 0 \pmod 9$$

Thus if generator term is divisible by 9, then every term of series generated by sum of its digits is also divisible by 9.

Kaprekar called a positive integer a *Self-Born Number* if it does not appear in a Kaprekar sequence except as the first term (or sequence *generator*. That is, a natural number n is called a self number if it cannot be written as $m + s_d(m)$, where m is a natural number less than n. For example, 1 and 3 are self-born numbers.

Kaprekar proposed following algorithm to find out if a given number is a self-born number or not (we will define n, d and c as variables related to numbers):

1. Let N be a n digit number.
2. Find d, the digital root [1] of N.
3. Compute
$$c = \begin{cases} \frac{d}{2} & \text{if } d \equiv 0 \pmod 2, \\ \frac{d+9}{2} & \text{if } d \equiv 1 \pmod 2. \end{cases}$$
4. Now examine the Kaprekar Sequence (K_{a_1}) formed by first n values of a_1 where $a_1 = N - c$, $N - (c + 9)$, $N - (c + 18)$, ...
5. If N is a term of any Kaprekar Sequence, then it is not a Self-Born Number, otherwise it is a self-born number.

This algorithm is very efficient because it reduces the number of sequences to be checked for a given number drastically. For example, if we want to check that 1,000,000 is a self born number, then by following above algorithm we need to check only seven cases instead of one million cases

[1] *digit root:* The ultimate single digit one gets by applying 'digit sum' (repeatedly if necessary)

Using above algorithm verify that 42 and 53 are self-born numbers

Above algorithm is a kind of recursive formula that generates all Self-Born Numbers.
Kaprekar said that *"The reason for a millionaire being such an important person is that 1,000,000 is a self-born number"*

Open Problem
Find a non-recursive formula that generates all Self-Born Numbers.

3 Kaprekar's Routines

Kaprekar devised three interesting procedures in 1949, called the Kaprekar's Routines . These iterative processes lead us either to a cycle or a fixed point. His sort-reverse-subtract routines are :

- Given a 2-digit natural number for which not all digits are equal, reverse the digits to get another number, subtract the smaller from the larger. Using the resulting number, repeat the process.
- Given a 3-digit natural number for which not all digits are equal, arrange its digits in descending order and in ascending order, subtract the smaller from the larger. Using the resulting number, repeat the process.
- Given a 4-digit natural number for which not all digits are equal, arrange its digits in descending order and in ascending order, subtract the smaller from the larger. Using the resulting number, repeat the process.

Note: While evaluating any of above iterative process if we get output at any stage to posses number of digits lesser than the initial number then use zero at higher value places to compensate the loss of digits and to be able to continue the algorithm application. *For Example:* we write $54 - 45 = 09$

Check Kaprekar's Routine on first three digits of your mobile number

Perform suitable Kaprekar's Routine on your birth year.

The outputs of the Kaprekar's Routine defined above will be:

- For a 2-digit number we will get a 5-cycle, namely, $09 \rightarrow 81 \rightarrow 63 \rightarrow 27 \rightarrow 45$
- For a 3-digit number we will get 495 as Kaprekar's Constant, in at most 6 iterations.
- For a 4-digit number we will get 6174 as Kaprekar's Constant, in at most 8 iterations.

The proof of all these statements involve listing and analysing of all possible cases. Recently (in 2013) Tanvir Prince [10] re-published proofs for all these statements.

Extending Kaprekar's Process

- For 5-digit numbers, the above pattern breaks down, as they may converge to 0 or one of the 10 constants 53955, 59994, 61974, 62964, 63954, 71973, 74943, 75933, 82962, 83952.
- Kaprekar's process can be generalised to n-digit numbers and bases b.
- In 1978, H. Hasse and G. D. Prichett extended Kaprekar's work to bases other than base 10.

4 Kaprekar Numbers

From ancient times, mathematicians love to classify numbers based on some special properties which they satisfy.
For Example: The Pythagoreans called a positive integer n to be a *perfect number*, if "n is equal to sum of all its positive divisors, excluding n itself". Some examples of *perfect numbers* are $6 = 1+2+3$ and $28 = 1+2+4+7+14$.

Consider an n - digit number k. Square it and add the right n digits to the left n or $n-1$ digits. If the resultant sum is k then k is called a*Kaprekar Number*

For example,

- $9^2 = 81$ and $1 + 8 = 9$.
- $45^2 = 2025$ and $25 + 20 = 45$.
- $297^2 = 88209$ and $209 + 88 = 297$.
- $2223^2 = 4941729$ and $1729 + 494 = 2223$.

$1729 = 1^3 + 12^3 = 9^3 + 10^3$ *is called Taxi-Cab number*

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