Sheaf, Cohomology and Geometry

A thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of
MASTER OF SCIENCE

by
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to the
School of Mathematical Sciences
National Institute of Science Education and Research
Bhubaneswar
26 April 2019

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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Abstract

Firstly, a sheaf-theoretic proof of de Rham cohomology being a topological invariant is presented. The de Rham cohomology of a smooth manifold is shown to be isomorphic to the Čech cohomology of that manifold with real coefficients. Then a proof of Dolbeault theorem, analogous to that of de Rham theorem, is discussed. Finally, the utility of Dolbeault-Čech isomorphism is illustrated by proving that every analytic hypersurface in $\mathbb{C}^n$ can be described as the zero-locus of an entire function.

Acknowledgment

I would first like to thank my thesis supervisor Dr. Ritwik Mukherjee. The door to Dr. Mukherjee's office was always open whenever I ran into a trouble or had a question. He consistently allowed this project to be my own work, but steered me in the right the direction whenever he thought I needed it.

The thesis has also benefited from comments and suggestions made by Dr. Manas Ranjan Sahoo during the thesis presentations. I take this opportunity to thank him.

Finally, I must express my very profound gratitude to my parents and to my friends for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of writing this thesis. This accomplishment would not have been possible without them. Thank you.

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Contents

Chapter 0

Introduction

0.1 Sheaf-theoretic de Rham isomorphism

A fundamental problem of topology is that of determining, for two spaces, whether or not they are homeomorphic. Algebraic topology originated in the attempts by mathematicians to construct suitable topological invariants. In 1895, Henri Poincaré[1] introduced a certain group, called the *fundamental group* of a topological space; which is by definition a topological invariant. Enrico Betti, on the other hand, associated with each space certain sequence of abelian groups called its *homology groups* [25, p. 1]. It was eventually proved that homeomorphic spaces had isomorphic homology groups. It was not until 1935 that another sequence of abelian groups, called *cohomology groups*, was associated with each space. The origins of cohomology groups lie in algebra rather than geometry; in a certain algebraic sense they are dual to the homology groups [25, p. 245]. There are several different ways of defining (co)homology groups, most common ones being *simplicial* and *singular* groups[2]. A third way of defining homology groups for arbitrary spaces, using the notion of open cover, is due to Eduard Čech (1932). The Čech homology theory is still not completely satisfactory [25, p. 2]. Apparently, Čech himself did not introduce *Čech cohomology*. Clifford Hugh Dowker, Samuel Eilenberg, and Norman Steenrod introduced Čech cohomology in the early 1950's [8, p. 24].

In 1920s, Élie Cartan's extensive research lead to the global study of general *differential forms* of higher degrees. É. Cartan, speculating the connections between topology and differential geometry, conjectured the *de Rham theorem* in a 1928 paper [21, p. 95]. In 1931, in his doctoral thesis, Georges de Rham[3] showed that differential forms satisfy the same axioms as *cycles* and *boundaries*, in effect proving a duality between what are now called *de Rham cohomology* and singular cohomology with real coefficients[4]. De Rham cohomology is considered to be one of the most important diffeomorphism invariant of a smooth manifold [32, p. 274].

Jean Leray, as a prisoner of war from 1940 to 1945, set himself the goal of discovering methods which could be applied to a very general class of topological space, while avoiding the use of simplicial approximation. The de Rham theorem and É. Cartan's theory of differential forms were central to Leray's thinking [19, §2]. After the war he published his results in 1945, which marked the birth of *sheaves* and *sheaf cohomology*[5]. His remarkable but rather obscure results were clarified by Émile Borel, Henri Cartan, Jean-Louis Koszul, Jean-Pierre Serre and André

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[1] Poincaré, Henri. "Analysis situs." Journal de l'École Polytechnique. 2 (1895): 1-123. `https://gallica.bnf.fr/ark:/12148/bpt6k4337198`

[2] Singular homology emerged around 1925 in the work of Oswald Veblen, James Alexander and Solomon Lefschetz, and was defined rigorously and in complete generality by Samuel Eilenberg in 1944 [8, p. 10].

[3] De Rham, Georges. "Sur l'analysis situs des variétés à n dimensions." 1931. `http://eudml.org/doc/192808`

[4] This can also be achieved directly via simplicial methods, see John Lee's *Introduction to Smooth Manifolds*, Chapter 18. In fact, this theorem has several dozens of different proofs.

[5] The word *faisceau* was introduced in the first of the announcements made by Leray in meeting of the Académie des Sciences on May 27, 1946. In 1951, John Moore fixed on "sheaf" as the English equivalent of "faisceau".

Weil in the late 1940's and early 1950's[6]. In 1952, Weil[7] found the modern proof of the de Rham theorem, this proof was a vindication of the local methods advocated by Leray [1, p. 5]. Weil's discovery provided the light which led H. Cartan to the modern formulation of *sheaf theory* [19, §2].

One can use Weil's approach, involving generalized Mayer-Vietoris principle, to study the relation between the de Rham theory to the Čech theory [1, p. 6]. However, we will follow the approach due to H. Cartan, written in the early 1950's, to give a sheaf theoretic proof of the isomorphism between de Rham and Čech cohomology with coefficients in $\mathbb{R}$ [35, p. 163]. An outline of this approach for proving de Rham cohomology to be a topological invariant can be found in the the books by Griffiths and Harris [9, p. 44] and Hirzebruch [11, §2.9–2.12].

In chapter 1 we will discuss various concepts related to differential forms and smooth manifolds needed to define de Rham cohomology. We will also develop the tools like Poincaré lemma, which will be used later to establish important sheaf theoretic results about the differential forms. In chapter 2 we will first discuss the sheaf theory necessary for defining Čech cohomology, and then prove the key results about Čech cohomology of paracompact Hausdorff spaces, like "short exact sequence of sheaves induces a long exact sequence of Čech cohomology", and "Čech cohomology vanishes on fine sheaves". Finally, in section 2.3 we will present the proof of de Rham-Čech isomorphism.

In the first section of Appendix A, to supplement the discussions in the first two chapters, we have stated few facts about paracompact spaces. In Appendix B we have discussed the theory of direct limits needed for understanding various definitions and proofs in the second chapter.

0.2 Cousin problem for analytic hypersurface in $\mathbb{C}^n$

In 1876, Karl Weierstrass asked the following three questions regarding complex valued holomorphic and meromorphic functions defined on an open subset U of $\mathbb{C}$ [33, Chapter 2]:

W1. Does there exist a holomorphic function with prescribed zeros?
W2. Is every meromorphic function on a quotient of two holomorphic functions?
W3. Does there exist a meromorphic function with prescribed poles and their principal part?

The answer to all these questions is yes. The first two questions were answered by Weierstrass himself in 1876, and the third question was answered by Gösta Mittag-Leffler during 1876-1882. The answer to the first and second question follows from the *Weierstrass factorization theorem.* Moreover, the affirmative answer to the second question is a corollary to the first one [3, Theorem VII.5.15, Corollary VII.5.20]. The answer to the third question is known as the *Mittag-Leffler theorem*, and the Weierstrass factorization theorem can be deduced from it [3, Theorem VIII.3.2, Exercise VIII.3.3].

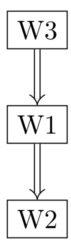


Figure 1: The relation between Weierstrass' questions identified by Mittag-Leffler.

[6]Georges Elencwajg (`https://math.stackexchange.com/users/3217/georges-elencwajg`), Why was Sheaf cohomology invented?, URL (version: 2016-05-24): `https://math.stackexchange.com/q/1798796`

[7]Weil, André. "Sur les théorèmes de de Rham." Commentarii mathematici Helvetici 26 (1952): 119-145. `http://eudml.org/doc/139040`.

The close bond between these three questions motivated other mathematicians to ask these question for complex valued holomorphic and meromorphic functions defined on open sets in $\mathbb{C}^n$. In 1883, Henri Poincaré generalized W2 by proving that every meromorphic function on $\mathbb{C}^2$ is a quotient of two holomorphic functions on $\mathbb{C}^2$ [18, Chapter 6] [2, §2]. However, there wasn't much progress made until 1895, when Pierre Cousin proved in his Ph.D. thesis that W1, W2 and W3 for product domains $X = U_1 \times U_2 \times \cdots U_n \subset \mathbb{C}^n$ are consequences of a single fundamental theorem [2, §3.1].

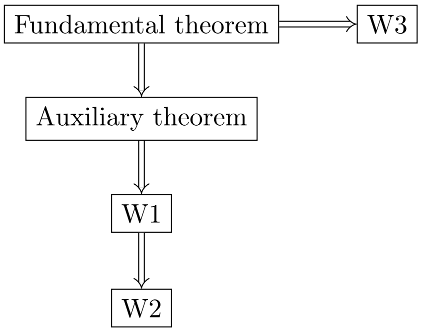


Figure 2: The relation between Weierstrass' questions for product domains identified by Cousin.

Therefore, Cousin was successful in bringing together the three problems of Weierstrass to make one coherent family. Moreover, the methods of Poincaré and Cousin exhibited what would later be called the "from local to global" problem form. However, in 1913, Thomas Hakon Grönwall and William Fogg Osgood found a counter example to W2, i.e. in the product of two ring-shaped domains there is a meromorphic function that cannot be written as the quotient of two holomorphic functions. Since W2 was an easy consequence of W1, they concluded that there was some flaw[8] in the proof of *auxiliary theorem*, which was the logarithmic variant of Cousin's fundamental theorem [2, §3.3]. Later, in 1934, Henri Cartan published a three-page note to show that the three problems had not significantly changed since Cousin, and gave the following labels [2, §3.4]:

Cousin I: Name given to Cousin's fundamental theorem. Also known as the additive problem.

Cousin II: Name given to Cousin's auxiliary theorem. Also known as the multiplicative problem.

Poincaré problem: Name given to the problem about the quotient representation of meromorphic functions.

Kyoshi Oka made a breakthrough by first solving Cousin I for bounded domains of holomorphy in 1937 and then an year later establishing that Cousin II for domains of holomorphy is a problem of purely topological nature. That is, he proved that for domains of holomorphy, the solvability of Cousin II depends only on a topological property of the zero-locus [2, §3.4.2]. To illustrate the independence of Cousin II, he also gave an example of product domain (since every product domain is a domain of holomorphy), in which Cousin I $\not\Longrightarrow$ Cousin II [15, p. 250].

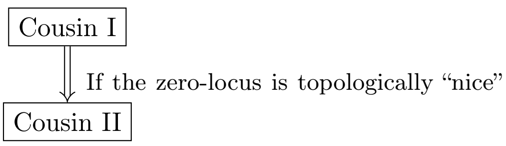


Figure 3: The relation between Cousin problems for domains of holomorphy identified by Oka.

[8] Cousin thought he had established Fundamental theorem $\Longrightarrow$ Auxiliary theorem for any product domain, but he had proved it only for those product domains in which at most one of the components is not necessarily simply connected.

In 1944, Cartan generalized the Cousin problems by recasting them in terms of ideals[9] [2, §4]. In particular, this theory is the new setting enabled the use of powerful abstract methods such as *Hilbert's Nullstellensatz* available in algebraic geometry [12, Proposition 1.1.29].

In the previous section we saw that during 1945-1951 the concept of sheaf and sheaf cohomology was developed. Fortunately, during these developments, several important questions left pending in Cartan's 1944 paper were also answered [2, §5]. From 1949 to 1953, Cartan organized various seminars which were devoted to the study of fibre-spaces, homotopy theory, cohomology theories and analytic functions in several variables. During the last three talks, the cohomology of coherent sheaves on Stein spaces was developed and Cartan proved two results concerning a coherent sheaf $\mathcal{F}$ on a Stein manifold X which were analogous to Cousin problems (called Cartan A and Cartan B) [2, §5.5]. For more details, refer to the books by Gunning and Rossi [10], Kaup and Kaup [14], Fritzsche and Grauert [6], Maurin [18], Krantz [15] and Taylor [31].

In 1952, Cartan's student Jean-Pierre Serre[10] gave the cohomological formulation of the conditions for solving the Cousin problems [2, p. 62]:

> Let X be a complex analytic variety[11], $\mathcal{O}$ be the sheaf of holomorphic complex valued functions and $\mathcal{M}$ be the sheaf of meromorphic complex valued functions on X. Then Cousin I is solvable for X if and only if $\check{\mathrm{H}}^1(X,\mathcal{O}) \to \check{\mathrm{H}}^1(X,\mathcal{M})$ is one to one and onto, and Cousin II is solvable for X if and only if $\check{\mathrm{H}}^1(X,\mathcal{O}^*) \to \check{\mathrm{H}}^1(X,\mathcal{M}^*)$ is one to one and onto. In particular, for Cousin I to be solvable, it is sufficient that $\check{\mathrm{H}}^1(X,\mathcal{O}) = 0$ and for Cousin II to be solvable, it is sufficient that $\check{\mathrm{H}}^1(X,\mathcal{O}^*) = 0$.

Pierre Dolbeault, another student of Cartan, in 1953 introduced the $\overline{\partial}$-cohomology[12] of the differential forms defined on complex analytic manifolds [7, §9.1.1]. He proved that this holomorphic analogue of de Rham cohomology defined on real manifolds is isomorphic to the sheaf cohomology of the sheaf of holomorphic differential forms [4]. Therefore, Dolbeault's theorem is a complex analogue of de Rham's theorem. Using the Dolbeault-Čech isomorphism we get that $\check{\mathrm{H}}^1(\mathbb{C}^n,\mathcal{O}) = 0$ (Theorem 4.8). Combining this with the purely topological fact that $\check{\mathrm{H}}^1(\mathbb{C}^n,\underline{\mathbb{Z}}) = 0$ (Corollary 4.2), and using the exponential sheaf sequence we can conclude that $\check{\mathrm{H}}^1(\mathbb{C}^n,\mathcal{O}^*) = 0$ (Lemma 4.3). Hence proving that both the Cousin problems are solvable for $\mathbb{C}^n$ [9, pp. 46-47].

In chapter 3 we will discuss various concepts related to complex differential forms and complex manifolds needed to define Dolbeault cohomology. We will also develop the tools like $\overline{\partial}$-Poincaré lemma, which will be used later to establish important sheaf theoretic results about the complex differential forms. In chapter 4 we will first illustrate the local to global principle by discussing the solution of Cousin problems for $\mathbb{C}$. Then we will prove Dolbeault theorem and use it to solve Cousin problem for analytic hypersurface in $\mathbb{C}^n$.

In the second section of Appendix A some fundamental results about smooth partition of unity, which will play an important role in various arguments presented in the thesis, have been stated. In Appendix C, to supplement the discussions in the third chapter, we have stated a few facts from linear algebra. In Appendix D we have discussed the function theory of several complex variables, which will be used in the third and fourth chapters.

[9]This was actually the second half of a single work. The first half was published in 1940, but the *Second World War* caused the delay in the publication of the other half.

[10]In 1953, he also proved that Poincaré's problem is solvable for Stein manifold, i.e. on a Stein manifold any meromorphic function is the quotient of two holomorphic functions.

[11]For example, complex manifold.

[12]Now called *Dolbeault cohomology*.

Chapter 1

de Rham cohomology

1.1 Differential forms on $\mathbb{R}^n$

In this section some basic definitions and facts from [24, Chapter 6] and [32, Chapter 1] will be stated. All the vector spaces are over the field $\mathbb{R}$ of real numbers.

1.1.1 Tangent space

Definition 1.1 (Tangent vector)**.** Given $p \in \mathbb{R}^n$, a *tangent vector* to $\mathbb{R}^n$ at p is a pair $(p; v)$, where $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$.

Definition 1.2 (Tangent space)**.** The set of all tangent vectors to $\mathbb{R}^n$ at p forms a vector space called *tangent space* of $\mathbb{R}^n$ at p, defined by

$$(p; v) + (p; w) = (p; v + w) \qquad \text{and} \qquad c(p; v) = (p; cv)$$

It is denoted by $T_p(\mathbb{R}^n)$.

Definition 1.3 (Germ of smooth functions)**.** Consider the set of all pairs (f, U), where U is a neighborhood of $p \in \mathbb{R}^n$ and $f : U \to \mathbb{R}$ is a smooth function. (f, U) is said to be equivalent to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. This equivalence class of (f, U) is called *germ of f at p*.

Remark 1.1. The set of all germs of smooth functions on $\mathbb{R}^n$ at p is written as $C_p^\infty(\mathbb{R}^n)$. The addition and multiplication of functions induce corresponding operations of $C_p^\infty(\mathbb{R}^n)$, making it into a ring; with scalar multiplication by real numbers $C_p^\infty(\mathbb{R}^n)$ becomes an algebra over $\mathbb{R}$.

Definition 1.4 (Derivation at a point)**.** A linear map $X_p : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

is called a *derivation at $p \in \mathbb{R}^n$* or a *point-derivation of $C_p^\infty(\mathbb{R}^n)$*.

Remark 1.2. The set of all derivations at p is denoted by $\mathcal{D}_p(\mathbb{R}^n)$. This set is a vector space.

Theorem 1. *The linear map*

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$$
$$(p; v) \mapsto D_v = \sum_{i=i}^{n} v_i \frac{\partial}{\partial x_i}\bigg|_p$$

where $(p; v) = (p; v_1, \ldots, v_n)$ and D_v is the directional derivative in the direction of v, is an isomorphism.

Remark 1.3. Under this vector space isomorphism, the standard basis $\{e_1 \dots, e_n\}$ of $T_p(\mathbb{R}^n)$ corresponds to the set $\{\partial/\partial x_1|_p, \dots, \partial/\partial x_n|_p\}$ of partial derivatives.

Definition 1.5 (Pushforward of a vector)**.** Let U be an open set in $\mathbb{R}^m$, $\alpha : U \to \mathbb{R}^n$ be a smooth function. The function f induces the linear transformation

$$\begin{aligned}\alpha_* : T_p(\mathbb{R}^m) &\to T_{\alpha(p)}(\mathbb{R}^n)\\ (p; v) &\mapsto (\alpha(p); D\alpha(p) \cdot v)\end{aligned}$$

where $D\alpha(p)$ is the total derivative of α at p. In other words, $\alpha_*(D_v)f = D_v(f \circ \alpha)$ for $f \in C^\infty_{\alpha(p)}(\mathbb{R}^n)$. Then $\alpha_*(p; v)$ is called the *pushforward of the vector v at p,*

Theorem 2. *Let U be open in $\mathbb{R}^m$, and $\alpha : U \to \mathbb{R}^n$ be a smooth map. Let V be an open set of $\mathbb{R}^n$ containing $\alpha(U)$, let $\beta : V \to \mathbb{R}^k$ be a smooth map. Then $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.*

1.1.2 Multilinear algebra

Unlike the preceding and succeeding (sub)sections, here V and W denote real vector spaces instead of open sets.

Definition 1.6 (k-tensor)**.** Let V be a vector space over $\mathbb{R}$. Let $V^k = V \times \cdots \times V$ denote the set of all k-tuples $(v_1, \dots, v_k)$ of vectors of V. A function $f : V^k \to \mathbb{R}$ is said to be a *k-tensor* if f is linear in the i^{th} variable for each i.

Remark 1.4. The set of all k-tensors on V is denoted by the symbol $\mathcal{L}^k(V)$. If $k = 1$ then $\mathcal{L}^1(V) = V^*$, the dual space of V.

Theorem 3. *Let V be a vector space of dimension n, then $\mathcal{L}^k(V)$ is a vector space of dimension n^k.*

Definition 1.7 (Tensor product)**.** Let $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^\ell(V)$, then the *tensor product* $f \otimes g \in \mathcal{L}^{k+\ell}(V)$ is defined by the equation

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) \cdot g(v_{k+1}, \dots, v_{k+\ell})$$

Definition 1.8 (Pullback of tensors)**.** Let $T : V \to W$ be a linear transformation and

$$T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$$

be the dual transformation defined for each $f \in \mathcal{L}^k(W)$ and $v_1, \dots, v_k \in V$ as

$$(T^* f)(v_1, \dots, v_k) = f(T(v_1), \dots, T(v_k))$$

Then $T^* f$ is called the *pullback of tensor* $f \in \mathcal{L}^k(W)$.

Theorem 4. *T^* is a linear transformation such that:*

1. *$T^*(f \otimes g) = T^* f \otimes T^* g$*
2. *If $S : W \to W'$ is a linear transformation, then $(S \circ T)^* f = T^*(S^* f)$.*

Definition 1.9 (Alternating k-tensor)**.** Let f be a k-tensor on V and σ be a permutation of $\{1, \cdots, k\}$. The k tensor f^σ on V is defined by the equation

$$f^\sigma(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The tensor f is said to be alternating if $f^\sigma = (\operatorname{sgn} \sigma) f$ for all permutations σ of $\{1, \cdots, k\}$.

Remark 1.5. The set of all alternating k-tensors on V is denoted by the symbol $\mathcal{A}^k(V)$. If $k = 1$ then $\mathcal{A}^1(V) = \mathcal{L}^1(V) = V^*$, the dual space of V.

Theorem 5. *Let $T : V \to W$ be a linear transformation and $T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ be the dual transformation. If f is an alternating tensor on W, then T^*f is an alternating tensor on V.*

Definition 1.10 (Alternating operator)**.** The linear transformation $A : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$ defined as

$$Af = \sum_\sigma (\operatorname{sgn}\sigma) f^\sigma$$

is called the *alternating operator.*

Remark 1.6. One can easily verify that this is a well defined linear transformation. Let τ be any permutation and $f \in \mathcal{L}^k(V)$ then

$$(Af)^\tau = \sum_\sigma (\operatorname{sgn}\sigma)(f^\sigma)^\tau = \sum_\sigma (\operatorname{sgn}\sigma) f^{\tau\circ\sigma} = (\operatorname{sgn}\tau)\sum_\sigma (\operatorname{sgn}\tau\circ\sigma) f^{\tau\circ\sigma} = (\operatorname{sgn}\tau) Af$$

hence $Af \in \mathcal{A}^k(V)$ for all $f \in \mathcal{L}^k(V)$.

Definition 1.11 (Wedge product)**.** Let $f \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^\ell(V)$, then the *wedge product* $f \wedge g \in \mathcal{A}^{k+\ell}(V)$ is defined as

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g)$$

where A is the alternating operator.

Remark 1.7. The reason for the coefficient $1/k!\ell!$ follows from the fact that $Af = k!f$ if $f \in \mathcal{A}^k(V)$.

Theorem 6. *Let f, g, h be alternating tensors on V. Then the following properties hold:*

1. *(Associative)* $f \wedge (g \wedge h) = (f \wedge g) \wedge h$
2. *(Homogeneous)* $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$ *for all* $c \in \mathbb{R}$
3. *(Distributive) If f and g have the same order, then $(f + g) \wedge h = f \wedge h + g \wedge h$ and $h \wedge (f + g) = h \wedge f + h \wedge g$*
4. *(Anti-commutative) If f and g have orders k and ℓ, respectively, then $g \wedge f = (-1)^{k\ell} f \wedge g$*
5. *Let $T : V \to W$ be a linear transformation and $T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ be the dual transformation. If f and g are alternating tensors on W, then $T^*(f \wedge g) = T^*f \wedge T^*g$*

Theorem 7. *Let V be a vector space of dimension n, with basis $\{e_1, \dots, e_n\}$, and $\{f_1, \dots, f_n\}$ be the dual basis for $V^* = \mathcal{A}^1(V)$. Then $\mathcal{A}^k(V)$ is a vector space of dimension $\binom{n}{k}$ with the set $\{f_I = f_{i_1} \wedge \ldots \wedge f_{i_k} : I = (i_1, \dots, i_k)\}$ as basis.*

Remark 1.8. If $k > \dim V$, then $\mathcal{A}^k(V) = 0$. This is because the anti-commutativity of wedge product implies that if $f \in V^*$ then $f \wedge f = 0$.

1.1.3 Differential forms

Definition 1.12 (Tensor field)**.** Let U be an open set in $\mathbb{R}^n$. A *k-tensor field* in U is a function ω assigning each $p \in U$, a k-tensor ω_p defined on the tangent space $T_p(\mathbb{R}^n)$. That is, $\omega_p \in \mathcal{L}^k(T_p(\mathbb{R}^n))$ for each $p \in U$.

Remark 1.9. Thus ω_p is a function mapping k-tuples of tangent vectors to $\mathbb{R}^n$ at p into $\mathbb{R}$. The tensor field ω is said to be of class C^r if it is of class C^r as a function of $(p, v_1, \ldots, v_k)$ for all $p \in U$ and $v_i \in T_p(\mathbb{R}^n)$.

Definition 1.13 (Differential k-form)**.** A *differential form of order* k, or *differential* k*-form* on an open subset U of $\mathbb{R}^n$ is a k-tensor field with the additional property that $\omega_p \in \mathcal{A}^k(T_p(\mathbb{R}^n))$ for all $p \in U$.

Definition 1.14 (Differential 0-form)**.** If U is open in $\mathbb{R}^n$, and if $f : U \to \mathbb{R}$ is a map of class C^r, then f is called a *differential* 0*-form* in U.

Definition 1.15 (Wedge product of 0-form and k-form)**.** The *wedge product* of a 0-form f and k-form ω on the open set U of $\mathbb{R}^n$ is defined by the rule

$$(\omega \wedge f)_p = (f \wedge \omega)_p = f(p) \cdot \omega_p$$

for all $p \in U$.

Remark 1.10. Henceforth, we restrict ourselves to differential forms of class C^∞. If U is an open set in $\mathbb{R}^n$, let $\Omega^k(U)$ denote the set of all smooth k-forms on U. The sum of two such k-forms is another k-form, and so is the product of a k-form by a scalar. Hence $\Omega^k(U)$ is the vector space of k-forms on U. Also, $\Omega^0(U) = C^\infty(U)$.

1.1.4 Exterior derivative

Definition 1.16 (Differential of a function)**.** Let U be open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a smooth real-valued function. Then the *differential of f* is defined to be the smooth 1-form *df* on U such that for any $p \in U$ and $(p; v) \in T_p(\mathbb{R}^n)$

$$(\mathrm{d}f)_p(p; v) = Df(p) \cdot v$$

where $Df(p)$ is the total derivative of f at p. In other words, $(\mathrm{d}f)_p(X_p) = X_p f$ for all derivations $X_p \in T_p(\mathbb{R}^n)$.

Remark 1.11. If x denotes the general point of $\mathbb{R}^n$, the i^{th} projection function mapping $\mathbb{R}^n$ to $\mathbb{R}$ is denoted by the symbol x_i. Then $\mathrm{d}x_i$ equals the elementary 1-from in $\mathbb{R}^n$, i.e. the set $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ is a basis of $\Omega^1(\mathbb{R}^n)$. If $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\{1, \ldots, n\}$, then

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

denotes the elementary k-forms in $\mathbb{R}^n$, i.e. the set $\{\mathrm{d}x_I : I \text{ is an ascending set of } k \text{ elements}\}$ is a basis of $\Omega^k(\mathbb{R}^n)$. The general k-form $\omega \in \Omega^k(U)$ can be written uniquely in the form

$$\omega = \sum_{[I]} a_I \, \mathrm{d}x_I$$

for some $a_I \in C^\infty(U)$.

Theorem 8. *Let U be open in $\mathbb{R}^n$ and $f \in C^\infty(U)$. Then*

$$\mathrm{d}f = \frac{\partial f}{\partial x_1} \mathrm{d}x_1 + \ldots + \frac{\partial f}{\partial x_n} \mathrm{d}x_n$$

In particular, $\mathrm{d}f = 0$ *if f is a constant function.*

Definition 1.17 (Differential of a k-form)**.** Let U be an open set in $\mathbb{R}^n$ and $\omega \in \Omega^k(U)$ such that $\omega = \sum_{[I]} f_I \, \mathrm{d}x_I$. Then for $k \geq 0$, the *differential of a k-form* ω is defined by the linear transformation

$$\begin{aligned} \mathrm{d} : \Omega^k(U) &\to \Omega^{k+1}(U) \\ \omega &\mapsto \sum_{[I]} \mathrm{d}f_I \wedge \mathrm{d}x_I \end{aligned}$$

where $\mathrm{d}f_I$ is the differential of function.

Theorem 9. *Let U be an open set in $\mathbb{R}^n$. If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^\ell(U)$ then*

1. *(Antiderivation)* $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$
2. $\mathrm{d} \circ \mathrm{d} = 0$

Definition 1.18 (Pullback of a k-form)**.** Let U be open in $\mathbb{R}^m$ and $\alpha : U \to \mathbb{R}^n$ be a smooth map. Let V be an open set in $\mathbb{R}^n$ containing $\alpha(U)$. For $k \geq 1$

$$\alpha^* : \Omega^k(V) \to \Omega^k(U)$$

is the dual transformation defined for each $\omega \in \Omega^k(V)$ and $(p; v_1), \ldots, (p; v_k) \in T_p(\mathbb{R}^m)$ as

$$(\alpha^*\omega)_p((p; v_1), \ldots, (p; v_k)) = \omega_{\alpha(p)}(\alpha_*(p; v_1), \ldots, \alpha_*(p; v_k))$$

Then the k-form $\alpha^*\omega \in \Omega^k(U)$ is called the *pullback of* $\omega \in \Omega^k(V)$.

Definition 1.19 (Pullback of a 0-form)**.** Let U be open in $\mathbb{R}^m$ and $\alpha : U \to \mathbb{R}^n$ be a smooth map. Let V be an open set in $\mathbb{R}^n$ containing $\alpha(U)$. If $f : V \to \mathbb{R}$ be a smooth map, then the pullback of $f \in \Omega^0(V)$ is the the 0-form $\alpha^* f = f \circ \alpha \in \Omega^0(U)$, i.e. $(\alpha^* f)(p) = f(\alpha(p))$ for all $p \in U$.

Theorem 10. *Let U be open in $\mathbb{R}^\ell$ and $\alpha : U \to \mathbb{R}^m$ be a smooth map. Let V be open in $\mathbb{R}^m$ which contains $\alpha(U)$ and $\beta : V \to \mathbb{R}^n$ be a smooth map. Then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$, i.e. $(\beta \circ \alpha)^*\omega = \alpha^*(\beta^*\omega)$ for all $\omega \in \Omega^k(W)$ where W is an open set in $\mathbb{R}^n$ containing $\beta(V)$.*

Theorem 11. *Let U be open in $\mathbb{R}^m$ and $\alpha : U \to \mathbb{R}^n$ be a smooth map. If ω, η and θ are differential forms defined in an open set V of $\mathbb{R}^n$ containing $\alpha(U)$, such that ω and η have same order, then*

1. *(preservation of the vector space structure)* $\alpha^*(a\omega + b\eta) = a(\alpha^*\omega) + b(\alpha^*\eta)$ *for all* $a, b \in \mathbb{R}$.
2. *(preservation of the wedge product)* $\alpha^*(\omega \wedge \theta) = \alpha^*\omega \wedge \alpha^*\theta$.
3. *(commutation with the differential)* $\alpha^*(\mathrm{d}\omega) = \mathrm{d}(\alpha^*\omega)$, *i.e. the following diagram commutes*

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{k+1}(V) \\ \Big\downarrow{\scriptstyle \alpha^*} & & \Big\downarrow{\scriptstyle \alpha^*} \\ \Omega^k(U) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{k+1}(U) \end{array}$$

1.2 Closed and exact forms on $\mathbb{R}^n$

In this section the proof of Poincaré lemma following [24, Chapter 8] will be discussed.

Definition 1.20 (Closed forms)**.** Let U be an open set in $\mathbb{R}^n$ and $\omega \in \Omega^k(U)$ for $k \geq 0$. Then ω is said to be *closed* if $\mathrm{d}\omega = 0$.

Remark 1.12. If U is an open set in $\mathbb{R}^n$, let $\mathcal{Z}^k(U)$ denote the set of all closed k-forms on U. The sum of two such k-forms is another closed k-form, and so is the product of a closed k-form by a scalar. Hence $\mathcal{Z}^k(U)$ is the vector sub-space of $\Omega^k(U)$. Also, $\mathcal{Z}^0(U)$ is the set of all locally constant[1] functions on U.

Definition 1.21 (Exact k-forms)**.** Let U be an open set in $\mathbb{R}^n$ and $\omega \in \Omega^k(U)$ for $k \geq 1$. Then ω is said to be *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.

Remark 1.13. If U is an open set in $\mathbb{R}^n$, let $\mathcal{B}^k(U)$ denote the set of all exact k-forms on U. The sum of two such k-forms is another exact k-form, and so is the product of a exact k-form by a scalar. Hence $\mathcal{B}^k(U)$ is the vector sub-space of $\Omega^k(U)$. Also, $\mathcal{B}^0(U)$ is defined to be the set consisting only zero.

Theorem 1.1. *Every exact form is closed.*

Proof. Let U be an open set in $\mathbb{R}^n$ and $\omega \in \mathcal{B}^k(U)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$. From Theorem 9 we know that $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\eta) = 0$ hence $\omega \in \mathcal{Z}^k(U)$ for all $k \geq 1$. For $k = 0$, the statement is trivially true. □

Remark 1.14. This theorem implies that $\mathcal{B}^k(U) \subseteq \mathcal{Z}^k(U)$ for all $k \geq 0$. However, the converse doesn't always hold for $k \geq 1$. For example, if $U = \mathbb{R}^2 - 0$ then the 1-form

$$\omega = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

is closed but not exact [24, Exercise 30.5, p. 261].

1.2.1 Differentiable homotopy

Definition 1.22 (Differentiable homotopy)**.** Let U and V be open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively; let $g, h : U \to V$ be smooth maps. Then g and h are said to be *differentiably homotopic* if there is a smooth map[2] $H : U \times [0,1] \to V$ such that

$$H(x,0) = g(x) \quad \text{and} \quad H(x,1) = h(x)$$

for all $x \in U$. The map H is called *differentiable homotopy* between g and h.

Lemma 1.1. *Let U be an open set in $\mathbb{R}^n$ and W be an open set in $\mathbb{R}^{n+1}$ such that $U \times [0,1] \subset W$. Let $\alpha, \beta : U \to W$ be smooth maps such that $\alpha(x) = (x,0)$ and $\beta(x) = (x,1)$. Then there is a linear transformation*

$$L : \Omega^{k+1}(W) \to \Omega^k(U)$$

defined for all $k \geq 0$, such that

$$\begin{cases} \mathrm{d}L\eta + L\,\mathrm{d}\eta = \beta^*\eta - \alpha^*\eta & \text{if } \eta \in \Omega^{k+1}(W), k \geq 0 \\ L\,\mathrm{d}\gamma = \beta^*\gamma - \alpha^*\gamma & \text{if } \gamma \in C^\infty(W) = \Omega^0(W) \end{cases}$$

where $\alpha^*, \beta^* : \Omega^k(W) \to \Omega^k(U)$ are the pullback maps defined for all $k \geq 0$.

[1] Locally constant functions are constant on any connected component of domain.

[2] This means that H is smooth in some open neighborhood of $U \times [0,1]$, like $U \times (-\epsilon, 1+\epsilon)$.

Proof. Let $x = (x_1, \dots, x_n)$ denote the general point of $\mathbb{R}^n$, and let t denote the general point of $\mathbb{R}$. Then, as in Remark 1.11, $\mathrm{d}x_1, \dots, \mathrm{d}x_n, \mathrm{d}t$ are the elementary 1-forms in $\mathbb{R}^{n+1}$. Also, for any continuous function $b : U \times [0,1] \to \mathbb{R}$ a scalar function Γb is defined on U by the formula

$$(\Gamma b)(x) = \int_{t=0}^{t=1} b(x,t)$$

Then for any $\eta \in \Omega^{k+1}(W)$

$$\eta = \sum_{[I]} a_I \,\mathrm{d}x_I + \sum_{[J]} b_J \,\mathrm{d}x_J \wedge \mathrm{d}t$$

where I is an ascending $(k+1)$-tuple and J is an ascending k-tuple from the set $\{1, \dots, n\}$, we define

$$\begin{aligned} L : \Omega^{k+1}(W) &\to \Omega^k(U) \\ \eta &\mapsto \sum_{[I]} L(a_I \,\mathrm{d}x_I) + \sum_{[J]} L(b_J \,\mathrm{d}x_J \wedge \mathrm{d}t) \end{aligned}$$

where $L(a_I \,\mathrm{d}x_I) = 0$ and $L(b_J \,\mathrm{d}x_J \wedge \mathrm{d}t) = (-1)^k (\Gamma b_J) \,\mathrm{d}x_J$.

Step 1. *L is a well defined linear transformation.*

We need to show that $L\eta \in \Omega^k(U)$. Clearly, $L\eta$ is a k-form on the subset U of $\mathbb{R}^n$. To prove that $L\eta$ is smooth, it's sufficient to show that the function Γb_J is smooth; and this result follows from Leibniz's rule [24, Theorem 39.1], since b_J is smooth.

Linearity of L follows from the uniqueness of the representation of η and linearity of the integral operator Γ.

Step 2. *$L(a \,\mathrm{d}x_I) = 0$ and $L(b \,\mathrm{d}x_J \wedge \mathrm{d}t) = (-1)^k (\Gamma b) \,\mathrm{d}x_J$ for any arbitrary $(k+1)$-tuple I and k-tuple J from the set $\{1, \dots, n\}$.*

If the indices are not distinct, then these formulae hold trivially, since $\mathrm{d}x_I = 0$ and $\mathrm{d}x_J = 0$ in that case. If the indices are distinct and in ascending order then these formulas hold by definition. Since rearranging the indices changes the values of $\mathrm{d}x_I$ and $\mathrm{d}x_J$ only by a sign, the formulae hold even in that case (the signs will cancel out due to linearity).

Step 3. *$L \,\mathrm{d}\gamma = \beta^* \gamma - \alpha^* \gamma$ if $\gamma \in C^\infty(W)$*

$$\begin{aligned} L \,\mathrm{d}\gamma &= L\left(\sum_{i=1}^n \frac{\partial \gamma}{\partial x_i} \,\mathrm{d}x_i\right) + L\left(\frac{\partial \gamma}{\partial t} \,\mathrm{d}t\right) \\ &= 0 + (-1)^0 \left(\Gamma \frac{\mathrm{d}\gamma}{\partial t}\right) \\ &= \int_{t=0}^{t=1} \frac{\partial \gamma}{\partial t}(x,t) \\ &= \gamma(x,1) - \gamma(x,0) \\ &= \gamma \circ \beta - \gamma \circ \alpha \\ &= \beta^* \gamma - \alpha^* \gamma \end{aligned}$$

Step 4. $\mathrm{d}L\eta + L \,\mathrm{d}\eta = \beta^* \eta - \alpha^* \eta$ *if* $\eta \in \Omega^{k+1}(W), k \geq 0$

Since d, L, α^* and β^* are all linear transformations, it suffices to verify the formula for the $(k+1)$-forms $\eta = a \,\mathrm{d}x_I$ and $\eta = b \,\mathrm{d}x_J \wedge \mathrm{d}t$. We will use Step 2 and Theorem 11 to simplify and compare *left hand side* (LHS) and *right hand side* (RHS) of the formula for both the cases.

Case 1. $\eta = a\,\mathrm{d}x_I$ *for any* $(k+1)$*-tuple* I *from* $\{1, \dots, n\}$
Simplify the LHS:

$$\begin{aligned}
\mathrm{d}L\eta + L\,\mathrm{d}\eta &= \mathrm{d}0 + L\,(\mathrm{d}a \wedge \mathrm{d}x_I) \\
&= L\left(\sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\,\mathrm{d}x_i \wedge \mathrm{d}x_I + \frac{\partial a}{\partial t}\,\mathrm{d}t \wedge \mathrm{d}x_I\right) \\
&= L\left(\sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\,\mathrm{d}x_i \wedge \mathrm{d}x_I\right) + L\left(\frac{\partial a}{\partial t}\,\mathrm{d}t \wedge \mathrm{d}x_I\right) \\
&= 0 + (-1)^{k+1} L\left(\frac{\partial a}{\partial t}\,\mathrm{d}x_I \wedge \mathrm{d}t\right) \\
&= (-1)^{k+1} \cdot (-1)^{k+1}\left(\Gamma \frac{\partial a}{\partial t}\right)\mathrm{d}x_I \\
&= \left(\int_{t=0}^{t=1} \frac{\partial a}{\partial t}(x,t)\right)\mathrm{d}x_I \\
&= (a(x,1) - a(x,0))\,\mathrm{d}x_I \\
&= (a \circ \beta - a \circ \alpha)\,\mathrm{d}x_I
\end{aligned}$$

Simplify the RHS:

$$\begin{aligned}
\beta^*\eta - \alpha^*\eta =& \beta^*(a\,\mathrm{d}x_I) - \alpha^*(a\,\mathrm{d}x_I) \\
=& \beta^*(a)\beta^*(\mathrm{d}x_I) - \alpha^*(a)\alpha^*(\mathrm{d}x_I) \\
=& (a \circ \beta)\beta^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{k+1}}) - (a \circ \alpha)\alpha^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{k+1}}) \\
=& (a \circ \beta)(\mathrm{d}(\beta^* x_{i_1}) \wedge \cdots \wedge \mathrm{d}(\beta^* x_{i_{k+1}})) - \\
& (a \circ \alpha)(\mathrm{d}(\alpha^* x_{i_1}) \wedge \cdots \wedge \mathrm{d}(\alpha^* x_{i_{k+1}})) \\
=& (a \circ \beta)(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{k+1}}) - (a \circ \alpha)(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{k+1}}) \\
=& (a \circ \beta - a \circ \alpha)\,\mathrm{d}x_I
\end{aligned}$$

Case 2. $\eta = b\,\mathrm{d}x_J \wedge \mathrm{d}t$ *for any* k*-tuple* J *from* $\{1, \dots, n\}$
Simplify the LHS:

$$\begin{aligned}
\mathrm{d}L\eta + L\,\mathrm{d}\eta =& \mathrm{d}\left((-1)^k (\Gamma b)\,\mathrm{d}x_J\right) + L\,(\mathrm{d}b \wedge \mathrm{d}x_J \wedge \mathrm{d}t) \\
=& \left[(-1)^k\,\mathrm{d}(\Gamma b) \wedge \mathrm{d}x_J\right] + \\
& \left[L\left(\sum_{j=1}^{n} \frac{\partial b}{\partial x_j}\,\mathrm{d}x_j \wedge dx_J \wedge dt + \frac{\partial b}{\partial t}\,\mathrm{d}t \wedge \mathrm{d}x_J \wedge \mathrm{d}t\right)\right] \\
=& \left[(-1)^k \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\Gamma b)\,\mathrm{d}x_j \wedge dx_J\right] + \left[\sum_{j=1}^{n} L\left(\frac{\partial b}{\partial x_j}\,\mathrm{d}x_j \wedge dx_J \wedge dt\right)\right] \\
=& \left[(-1)^k \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\Gamma b)\,\mathrm{d}x_j \wedge dx_J\right] + \left[\sum_{j=1}^{n} (-1)^{k+1}\left(\Gamma \frac{\partial b}{\partial x_j}\right)\mathrm{d}x_j \wedge \mathrm{d}x_J\right] \\
=& 0
\end{aligned}$$

since by Leibniz's rule [24, Theorem 39.1], $\frac{\partial}{\partial x_j}(\Gamma b) = \Gamma \frac{\partial b}{\partial x_j}$ for all j. Now we simplify the RHS:

$$\beta^*\eta - \alpha^*\eta = \beta^*(b\,\mathrm{d}x_I \wedge \mathrm{d}t) - \alpha^*(b\,\mathrm{d}x_I \wedge \mathrm{d}t)$$

$$
\begin{aligned}
&= [(\beta^* b)\, \mathrm{d}(\beta^* x_{j_1}) \wedge \cdots \wedge \mathrm{d}(\beta^* x_{j_k}) \wedge \mathrm{d}(\beta^* t)] - \\
&\quad [(\alpha^* b)\, \mathrm{d}(\alpha^* x_{j_1}) \wedge \cdots \wedge \mathrm{d}(\alpha^* x_{j_k}) \wedge \mathrm{d}(\alpha^* t)] \\
&= [(b \circ \beta)\, \mathrm{d}x_{j_1} \wedge \ldots \mathrm{d}x_{j_k} \wedge \mathrm{d}1] - [(b \circ \alpha)\, \mathrm{d}x_{j_1} \wedge \ldots \mathrm{d}x_{j_k} \wedge \mathrm{d}0] \\
&= 0 - 0 = 0
\end{aligned}
$$

This completes the proof of the lemma. □

Remark 1.15. For the special case, when $k = 0$ we have $\eta = \sum_{i=1}^{n} a_i \, \mathrm{d}x_i + b \, \mathrm{d}t$. In this case, we have $L\eta = \Gamma b$ since J is empty. Hence, just as d is in some sense a "differentiation operator", the operator L is in some sense an "integration operator", one that integrates η in the direction of the last coordinate [24, Exercise 39.4].

Proposition 1.1. *Let U and V be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $g, h : U \to V$ be smooth maps that are differentiably homotopic. Then there is a linear transformation*

$$T : \Omega^{k+1}(V) \to \Omega^k(U)$$

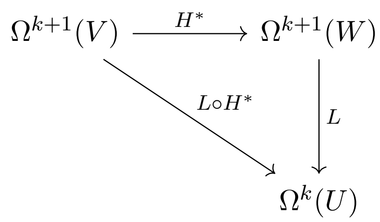
defined for all $k \geq 0$, such that

$$
\begin{cases}
\mathrm{d}T\omega + T\,\mathrm{d}\omega = h^*\omega - g^*\omega & \text{if } \omega \in \Omega^{k+1}(V), k \geq 0 \\
T\,\mathrm{d}f = h^* f - g^* f & \text{if } f \in C^\infty(V) = \Omega^0(V)
\end{cases}
$$

where $g^*, h^* : \Omega^k(V) \to \Omega^k(U)$ are the pullback maps defined for all $k \geq 0$.

Proof. The preceding lemma was a special case of this proposition since α and β were differentiably homotopic. We borrow notations from the preceding lemma.

Let $H : U \times [0,1] \to V$ be the differentiable homotopy between g and h, i.e. $H(x,0) = H(\alpha(x)) = g(x)$ and $H(x,1) = H(\beta(x)) = h(x)$. Then we have the pullback map $H^* : \Omega^k(V) \to \Omega^k(W)$ defined on an open neighborhood W of $U \times [0,1]$ and $k \geq 0$. Hence for $k \geq 0$ we have the following commutative diagram:



`Claim:` $T = L \circ H^*$

We will verify both the desired properties separately.

Step 1. $\mathrm{d}T\omega + T\,\mathrm{d}\omega = h^*\omega - g^*\omega$ *if* $\omega \in \Omega^{k+1}(V), k \geq 0$

Let $H^*\omega = \eta \in \Omega^{k+1}(W)$, then using Theorem 11, Theorem 10, and the preceding lemma

$$
\begin{aligned}
\mathrm{d}T\omega + T\,\mathrm{d}\omega &= \mathrm{d}(L(H^*\omega)) + L(H^*(\mathrm{d}\omega)) \\
&= \mathrm{d}L\eta + L\,\mathrm{d}\eta \\
&= \beta^*\eta - \alpha^*\eta \\
&= \beta^*(H^*\omega) - \alpha^*(H^*\omega) \\
&= (H \circ \beta)^*\omega - (H \circ \alpha)^*\omega \\
&= h^*\omega - g^*\omega
\end{aligned}
$$

Step 2. $T\,\mathrm{d}f = h^*f - g^*f$ *if* $f \in C^\infty(V) = \Omega^0(V)$

Let $H^*f = \gamma \in \Omega^0(W)$, then using Theorem 11, Theorem 10, and the preceding lemma

$$\begin{aligned} T\,\mathrm{d}f &= L(H^*\,\mathrm{d}f) \\ &= L\,\mathrm{d}\gamma \\ &= \beta^*\gamma - \alpha^*\gamma \\ &= \beta^*(H^*f) - \alpha^*(H^*f) \\ &= (H \circ \beta)^*f - (H \circ \alpha)^*f \\ &= h^*f - g^*f \end{aligned}$$

This completes the proof. □

1.2.2 Poincaré lemma

Definition 1.23 (Star-convex)**.** Let U be an open set in $\mathbb{R}^n$. Then U is said to be *star-convex* with respect to the point $p \in U$ is for each $x \in U$, the line segment joining x and p lies in U.

Theorem 1.2 (Poincaré lemma)**.** *Let U be a star-convex open set in $\mathbb{R}^n$. If $k \geq 1$, then every closed k-form on U is exact.*

Proof. Let $\omega \in \mathcal{Z}^k(U)$ for $k \geq 1$. We apply the preceding proposition. Let p be a point with respect to which U is star-convex. We define the maps g and h as follows:

$$\begin{aligned} g : U &\to U \\ x &\mapsto p \end{aligned} \qquad\qquad \begin{aligned} h : U &\to U \\ x &\mapsto x \end{aligned}$$

Since U is star-convex with respect to p, there always exists a straight line in U joining any point $x \in U$ with p. Hence we have the differentiable homotopy between g and h given by this straight line

$$\begin{aligned} H : U \times [0,1] &\to U \\ (x,t) &\mapsto th(x) + (1-t)g(x) \end{aligned}$$

Therefore the maps g and h are differentiably homotopic.

Now we use the previous proposition, i.e. there exists $T : \Omega^k(U) \to \Omega^{k-1}(U)$ such that $\mathrm{d}T\omega + T\,\mathrm{d}\omega = h^*\omega - g^*\omega$. Hence if $\mathrm{d}\omega = 0$ then $\mathrm{d}T\omega = \omega$ since pullback map corresponding to the identity map is the identity map i.e. $h^*\omega = \omega$ and pullback map corresponding to a constant map is the zero map i.e. $g^*\omega = 0$. Hence $\omega \in \mathcal{B}^k(U)$ for all $k \geq 1$. This completes the proof[3]. □

Remark 1.16. Being star-convex is not such a restrictive condition, since any open ball

$$B(p, \varepsilon) = \{x \in \mathbb{R}^n : ||x - p|| < \varepsilon\}$$

is star-convex with respect to p. Hence, Poincaré lemma holds for any open ball in $\mathbb{R}^n$.

[3]If we also use the second condition of the preceding proposition we get that if $\mathrm{d}f = 0$ then f is a constant map. This is Munkres' defintion of exact 0-form [24, p. 259].

1.3 Differential forms on smooth manifolds

In this section some basic definitions and facts from [32, Chapter 2, 3 and 5] and [22, §1.1, 2.1, 3.2, 3.4 and 5.1] will be stated.

Definition 1.24 (Smooth manifold)**.** A *smooth manifold* M of dimension n is a second countable Hausdorff space together with a smooth structure on it. A *smooth structure* $\mathscr{U}$ is the collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ where U_α is an open set of M and ϕ_α is a homeomorphism of U_α onto an open set of $\mathbb{R}^n$ such that

1. the open sets $\{U_\alpha\}_{\alpha \in A}$ cover M.
2. for every pair of indices $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the homeomrphisms
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$
$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$
are smooth maps between open subsets of $\mathbb{R}^n$.
3. the family $\mathscr{U}$ is maximal in the sense that it contains all possible pairs (U_α, ϕ_α) satisfying the properties 1. and 2.

Example 1.1. Following two smooth manifolds will be used throughout this thesis:

1. The Euclidean space $\mathbb{R}^n$ is a smooth manifold with single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n}$ is the identity map. In other words, $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n}) = (\mathbb{R}^n, x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$.
2. Any open subset V of a smooth manifold M is also a smooth manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})\}$ is an atlas for V, where $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$.

Theorem 12. *Every smooth manifold M is paracompact*[4].

Definition 1.25 (Smooth function on a manifold)**.** Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be a *smooth function at a point* p in M if there is a chart (U, ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is smooth at $\phi(p)$. The function f is said to be smooth on M is it is smooth at every point of M.

$$\begin{array}{ccc} (U, p) & \xrightarrow{\ \phi\ } & (\mathbb{R}^n, \phi(p)) \\ & \searrow^{f} & \downarrow{\scriptstyle f\circ\phi^{-1}} \\ & & (\mathbb{R}, f(p)) \end{array}$$

Definition 1.26 (Smooth partition of unity)**.** Let M be a smooth manifold with an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$. Then a *smooth partition of unity on* M subordinate to $\mathcal{U}$ is a family of smooth functions $\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ satisfying the following conditions

1. $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$.
2. $0 \leq \psi_\alpha(p) \leq 1$ for all $p \in M$ and $\alpha \in A$
3. the collection of supports $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha \in A}$ is locally finite.

[4]For definition and general properties of paracompact spaces, see section A.1.

4. $\sum_{\alpha \in A} \psi_\alpha(p) = 1$ for all $p \in M$

where $\text{supp}(\psi_\alpha)$ is the closure of the set of those $p \in M$ for which $\phi_\alpha(p) \neq 0$.

Theorem 13. *Any smooth manifold M with an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ admits a smooth partition of unity subordinate to $\{U_\alpha\}$.*

Remark 1.17. If $\{\psi_\alpha\}$ is a smooth partition of unity on M subordinate to $\{U_\alpha\}$, and $\{f_\alpha : U_\alpha \to \mathbb{R}\}$ is a family of smooth functions, then the function $f : M \to \mathbb{R}$ defined by $f(x) = \sum_{\alpha \in A} \phi_\alpha f_\alpha$ is smooth.

Definition 1.27 (Smooth map between smooth manifolds)**.** Let M and N be smooth manifolds of dimension m and n, respectively. A continuous map $F : M \to N$ is *smooth at a point* p if M if there are charts (V, ψ) about $F(p)$ in N and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^m$ to $\mathbb{R}^n$, is smooth at $\phi(p)$.

$$\begin{array}{ccc} (U, p) & \xrightarrow{\quad F \quad} & (V, F(p)) \\ \Big\downarrow \phi & & \Big\downarrow \psi \\ (\mathbb{R}^m, \phi(p)) & \xrightarrow{\psi \circ F \circ \phi^{-1}} & (\mathbb{R}^n, \psi(F(p))) \end{array}$$

The continuous map $F : M \to N$ is said to be *smooth* if it is smooth at every point in M.

Remark 1.18. In the definition of smooth maps between manifolds it's assumed that $F : M \to N$ is continuous to ensure that $F^{-1}(V)$ is an open set in M. Thus, smooth maps between manifolds are by definition continuous.

Theorem 14. *Let M and N be smooth manifolds of dimension m and n, respectively, and $F : M \to N$ a continuous map. The following are equivalent*

1. *The map $F : M \to N$ is smooth*

2. *There are atlases $\mathscr{U}$ for M and $\mathscr{V}$ for N such that for every chart (U, ϕ) in $\mathscr{U}$ and (V, ψ) in $\mathscr{V}$ the map*
$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^n$$
is smooth.

3. *For every chart (U, ϕ) on M and (V, ϕ) on N, the map*
$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^n$$
is smooth.

Theorem 15. *If (U, ϕ) is a chart on a smooth manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Remark 1.19. One can generalize the notation for projection maps introduced in Remark 1.11. If $\{U, \phi\}$ is a chart of a manifold, i.e. $\phi : U \to \mathbb{R}^n$, then let $r_i = x_i \circ \phi$ be the i^{th} component of ϕ and write $\phi = (r_1, \ldots, r_n)$ and $(U, \phi) = (U, r_1, \ldots, r_n)$. Thus, for $p \in U$, $(r_1(p), \ldots, r_n(p))$ is a point in $\mathbb{R}^n$. The functions $r_1, \ldots, r_n$ are called *coordinates* or *local coordinates* on U.

Theorem 16. *Let M and N be smooth manifolds of dimension m and n, respectively, and $F : M \to N$ a continuous map. The following are equivalent*

1. *The map $F : M \to N$ is smooth*

2. *The manifold N has an atlas such that for every chart $(V, \psi) = (V, s_1, \dots, s_n)$ in the atlas*[5], *the components $s_i \circ F : F^{-1}(V) \to \mathbb{R}$ of f relative to the chart are all smooth.*

3. *For every chart $(V, \psi) = (V, s_1, \dots, s_n)$ on N, the components $s_i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all smooth.*

1.3.1 Tangent space

Definition 1.28 (Germ of smooth functions)**.** Consider the set of all pairs (f, U), where U is a neighborhood of $p \in M$ and $f : U \to \mathbb{R}$ is a smooth function. Then (f, U) is said to be equivalent to (g, V) is there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. This equivalence class of (f, U) is called *germ of f* at p.

Remark 1.20. The set of all germs of smooth functions on M at p is denoted by $C_p^\infty(M)$. The addition and multiplication of functions induce corresponding operations of $C_p^\infty(M)$, making it into a ring; with scalar multiplication by real numbers $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$.

Definition 1.29 (Derivation at a point)**.** A linear map $X_p : C_p^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

is called a *derivation at $p \in M$* or a *point-derivation of $C_p^\infty(M)$*.

Definition 1.30 (Tangent vector)**.** A *tangent vector* at a point p in a manifold M is a derivation at p.

Definition 1.31 (Tangent space)**.** The tangent vectors at p form a real vector space T_pM, called the *tangent space* of M at p.

Definition 1.32 (Partial derivative)**.** Let M be a smooth manifold of dimension n, $(U, \phi) = (U, r_1, \dots, r_n)$ be a chart and $f : M \to \mathbb{R}$ be a smooth function. For $p \in U$, the *partial derivative* $\partial f / \partial r_i$ of f with respect to r_i at p is defined to be

$$\left.\frac{\partial}{\partial r_i}\right|_p f := \frac{\partial f}{\partial r_i}(p) := \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(\phi(p)) := \left.\frac{\partial}{\partial x_i}\right|_{\phi(p)} (f \circ \phi^{-1})$$

where $r_i = x_i \circ \phi$ and $\{x_1, \dots, x_n\}$ are the standard coordinates on $\mathbb{R}^n$.

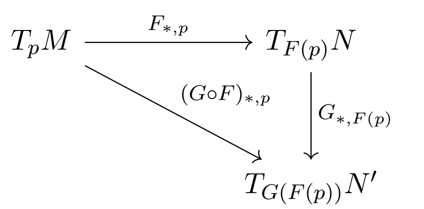
Definition 1.33 (Pushforward of a vector)**.** Let $F : M \to N$ be a smooth map between two smooth manifolds. At each point $p \in M$, the map F induces a linear map of tangent spaces

$$F_* : T_pM \to T_{F(p)}N$$

such that given $X_p \in T_pM$ we have $(F_*(X_p))f = X_p(f \circ F) \in \mathbb{R}$ for $f \in C_{F(p)}^\infty(M)$.

Remark 1.21. The pusforward map induced by an the identity map of manifolds is the identity map of vector spaces, i.e. $(\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM}$.

Theorem 17. *Let $F : M \to N$ and $G : N \to N'$ be smooth maps of manifolds, and $p \in M$, then $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$*



[5] Here $s_i = y_i \circ \psi$ if we consider the coordinates of $\mathbb{R}^n$ to be $(y_1, \dots, y_n)$ and coordinates of $\mathbb{R}^m$ to be $(x_1, \dots, x_m)$.

Theorem 18. *Let $(U, \phi) = (U, r_1, \ldots, r_n)$ be a chart about a point p in a manifold M of dimension n. Then $\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is a vector space isomorphism and T_pM has the basis*

$$\left\{ \frac{\partial}{\partial r_i}\bigg|_p, \ldots, \frac{\partial}{\partial r_n}\bigg|_p \right\}$$

where $r_i = x_i \circ \phi$ and $\{x_1, \ldots, x_n\}$ the standard coordinates of $\mathbb{R}^n$.

Remark 1.22. Hence one observes that if M is n dimensional manifold then T_pM is a vector space of dimension n over $\mathbb{R}$.

1.3.2 Cotangent bundle

Definition 1.34 (Cotangent space)**.** Let M be a smooth manifold and p a point in M. The *cotangent space* of M at point p denoted by T_p^*M is defined to be the dual space of the tangent space T_pM, i.e. the set of all linear maps from T_pM to $\mathbb{R}$.

Remark 1.23. Hence, if M is n dimensional manifold then T_p^*M is a vector space of dimension n over $\mathbb{R}$.

Definition 1.35 (Cotangent bundle)**.** The *cotangent bundle T^*M* of a manifold M is the union of the tangent spaces at all points of M

$$T^*M := \bigcup_{p \in M} T_p^*M$$

Remark 1.24. The union in the definition above is disjoint, i.e. $T^*M = \coprod_{p \in M} T_p^*M$, since for distinct points p and q in M, the cotangent spaces T_p^*M and T_q^*M are already disjoint.

Theorem 19. *Let M is a smooth manifold of dimension n, then its cotangent bundle T^*M is a smooth manifold of dimension $2n$.*

Definition 1.36 (Smooth vector bundle)**.** A *smooth vector bundle* of rank n is a triple (E, M, π) consisting of a pair of smooth manifolds E and M, and a smooth surjective map $\pi : E \to M$ satisfying the following conditions

1. for each $p \in M$, the inverse image $E_p = \pi^{-1}(p)$ is an n-dimensional vector space over $\mathbb{R}$,
2. for each $p \in M$, there is an open neighborhood U of p and a diffeomorphism $\phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that
   (a) the following diagram commutes

$$\begin{array}{ccc} U \times \mathbb{R}^n & \xrightarrow{\ \phi\ } & \pi^{-1}(U) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U & \end{array}$$

   where p_1 is the projection onto the first factor,
   (b) for each $q \in U$, the map $\phi_q : \mathbb{R}^n \to \pi^{-1}(q)$, defined by $\phi_q(x) = \phi(q, x)$, is a linear isomorphism.

Theorem 20. *The cotangent bundle T^*M with the projection map $\pi : T^*M \to M$ given by $\pi(\alpha) = p$ if $\alpha \in T_p^*M$, is a vector bundle of rank n over M.*

Definition 1.37 (Exterior power of cotangent bundle)**.** Let M be a smooth manifold. Then the k^{th} *exterior power of the cotangent bundle* $\Lambda^k(T^*M)$ is the disjoint union of all alternating k-tensors at all points of the manifold, i.e.

$$\Lambda^k(T^*M) = \bigcup_{p\in M} \mathcal{A}^k(T_pM)$$

Theorem 21. *If M is a manifold of dimension n, then the exterior power of the cotangent bundle $\Lambda^k(M)$ is a manifold of dimension $n + \binom{n}{k}$.*

Theorem 22. *The exterior power of cotangent bundle $\Lambda^k(T^*M)$ with the projection map $\pi : \Lambda^k(T^*M) \to M$ given by $\pi(\alpha) = p$ if $\alpha \in \mathcal{A}^k(T_pM)$, is a vector bundle of rank $\binom{n}{k}$ over M.*

1.3.3 Differential forms

Definition 1.38 (Smooth section)**.** A *smooth section* of a vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathbb{1}_M$.

Remark 1.25. The condition $\pi \circ s = \mathbb{1}_M$ precisely means that for each p in M, s maps p into E_p.

Definition 1.39 (Differential k-form)**.** A *differential k-form* on M is a smooth section of the vector bundle $\pi : \Lambda^k(T^*M) \to M$.

Remark 1.26. The vector space of all smooth k-forms on M is denoted by $\Omega^k(M)$. If $\omega \in \Omega^k(M)$ then $\omega : M \to \Lambda^k(T^*M)$ is a smooth map such that ω assigns each point $p \in M$ an alternating k-tensor, i.e. $\omega_p \in \mathcal{A}^k(T_pM)$ for all $p \in M$. In particular, if U is an open subset of M, then $\omega \in \Omega^k(U)$ if $\omega_p \in \mathcal{A}^k(T_pM)$ for all $p \in U$ (view U as open neighborhood of point p).

Definition 1.40 (Differential 0-form)**.** A *differential 0-form* on M is a smooth real valued function on M, i.e. $\Omega^0(M) = C^\infty(M)$.

Definition 1.41 (Wedge product of 0-form and k-form)**.** The wedge product of a 0-form $f \in C^\infty(M)$ and a k-form $\omega \in \Omega^k(M)$ is defined as the k-form $f\omega$ where

$$(\omega \wedge f)_p = (f \wedge \omega)_p = f(p) \cdot \omega_p$$

for all $p \in M$.

Definition 1.42. The wedge product extends pointwise to differential forms on a manifold, i.e. if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ then $\omega \wedge \eta \in \Omega^{k+\ell}(M)$ such that

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

at all $p \in M$.

1.3.4 Exterior derivative

Definition 1.43 (Differential of a function)**.** Let $f : M \to \mathbb{R}$ be a smooth function, its *differential* is defined to be the 1-form $\mathrm{d}f$ on M such that for any $p \in M$ and $X_p \in T_pM$

$$(\mathrm{d}f)_p(X_p) = X_pf$$

Remark 1.27. Let $(U, r_1, \ldots, r_n)$ be a coordinate chart on a smooth manifold M. Then the differentials $\{\mathrm{d}r_1, \ldots, \mathrm{d}r_n\}$ are 1-forms on U. At each point $p \in U$, the 1-forms $\{(\mathrm{d}r_1)_p, \ldots, (\mathrm{d}r_n)_p\}$

form a basis[6] of $\mathcal{A}^1(T_pM) = T_p^*M$, dual to the basis $\{\partial/\partial r_1|_p, \ldots, \partial/\partial r_n|_p\}$ for the tangent space T_pM. Hence, a 1-form on U is a linear combination $\omega = \sum_{i=1}^n a_i dr_i$ where a_i are smooth functions on U.

If $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\{1, \ldots, n\}$, then

$$\mathrm{d}r_I = \mathrm{d}r_{i_1} \wedge \cdots \wedge \mathrm{d}r_{i_k}$$

denotes the the elementary k-forms on $U \subset M$, i.e. the k-forms

$$\{(\mathrm{d}r_I)_p : I \text{ is an ascending set } k\text{-tuple}\}$$

form a basis of $\mathcal{A}^k(T_pM)$ for all $p \in U$. The general k-form $\omega \in \Omega^k(U)$ can be written uniquely in the form

$$\omega = \sum_{[I]} a_I \, \mathrm{d}r_I$$

for some $a_I \in C^\infty(U)$.

Theorem 23. *If f is a smooth function on M, then the restriction of the 1-from $\mathrm{d}f$ to U can be expressed as*

$$\mathrm{d}f = \frac{\partial f}{\partial r_1}\,\mathrm{d}r_1 + \ldots + \frac{\partial f}{\partial r_n}\,\mathrm{d}r_n$$

Theorem 24. *$\omega \in \Omega^k(M)$ if and only if on every chart $(U, \phi) = (U, r_1, \ldots, r_n)$ on M, the coefficients a_I of $\omega = \sum_{[I]} a_I \, \mathrm{d}r_I$ relative to the elementary k-forms $\{\mathrm{d}r_I\}$ are all smooth.*

Theorem 25. *Suppose ω is a smooth differential form defined on a neighborhood U of a point p in a manifold M, i.e. $\omega \in \Omega^k(U)$. Then there exists a smooth form $\widetilde{\omega}$ on M, i.e. $\widetilde{\omega} \in \Omega^k(M)$, that agrees with ω on a possible smaller neighborhood of p.*

Remark 1.28. The extension $\widetilde{\omega}$ is not unique, it depends on p and on the choice of a bump function at p. All this can be generalized to a family of differential forms, as in Remark 1.17, using smooth partitions of unity.

Definition 1.44 (Differential of a k-form)**.** Let $(U, r_1, \ldots, r_n)$ be a coordinate chart on a smooth manifold M and $\omega \in \Omega^k(U)$ is written uniquely as a linear combination

$$\omega = \sum_{[I]} a_I \, \mathrm{d}r_I, \qquad a_I \in C^\infty(U)$$

The $\mathbb{R}$-linear map $\mathrm{d}_U : \Omega^k(U) \to \Omega^{k+1}(U)$ defined as

$$\mathrm{d}_U\omega = \sum_{[I]} \mathrm{d}a_I \wedge \mathrm{d}r_I$$

is called the exterior derivative of ω on U. Let $p \in U$, then $(\mathrm{d}_U\omega)_p$ is independent of the chart containing p. Thus the *differential of a k-form* is defined by the linear operator

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$$

such that for $k \geq 0$ and $\omega \in \Omega^k(M)$ one has $(\mathrm{d}\omega)_p = (\mathrm{d}_U\omega)_p$ for all $p \in M$.

Theorem 26. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ then*

1. *(Antiderivation)* $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$

[6] In the case of $M = \mathbb{R}^n$ the expression was much more straightforward because $T_pM \cong \mathbb{R}^n$ (vector space isomorphism) for any n-dimensional manifold.

2. $\mathrm{d} \circ \mathrm{d} = 0$

Remark 1.29. Since the exterior derivative is an antiderivation, it is a local operator, i.e. for all $k \geq 0$,whenever a k-form $\omega \in \Omega^k(M)$ is such that $\omega_p = 0$ for all points p in an open set U of M, then $\mathrm{d}\omega \equiv 0$ on U. Equivalently, for all $k \geq 0$, whenever two k-forms $\omega, \eta \in \Omega^k(M)$ agree on an open set U, then $\mathrm{d}\omega \equiv \mathrm{d}\eta$ on U [24, Proposition 19.3].

Definition 1.45 (Pullback of a k-form)**.** Let $F : M \to N$ be a smooth map of manifolds. Then for $k \geq 1$

$$F^* : \Omega^k(N) \to \Omega^k(M)$$

is the pullback map defined for each $\omega \in \Omega^k(N)$ at every point $p \in M$ as

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(F_{*,p}v_1, \dots, F_{*,p}v_k)$$

where $v_i \in T_pM$. Then the k-form $F^*\omega \in \Omega^k(M)$ is called the *pullback of* $\omega \in \Omega^k(N)$.

Definition 1.46 (Pullback of a 0-form)**.** Let $F : M \to N$ be a smooth map and $f \in C^\infty(N) = \Omega^0(N)$, then the *pullback of* f is the the 0-form $F^*f = f \circ F \in \Omega^0(M)$.

Remark 1.30. Pullback of the identity map is an identity map, i.e. $(\mathbb{1}_M)^* = \mathbb{1}_{\Omega^k(M)}$.

Theorem 27. *If $F : M \to N$ and $G : N \to N'$ are smooth maps, then $(G \circ F)^* = F^* \circ G^*$.*

$$\begin{array}{ccc} \Omega^k(N') & \xrightarrow{G^*} & \Omega^k(N) \\ & \searrow^{(G\circ F)^*} & \downarrow F^* \\ & & \Omega^k(M) \end{array}$$

Theorem 28. *Let $F : M \to N$ be a smooth map. If ω, η and θ are differential forms on N, such that ω and η have same order, then*

1. *(preservation of the vector space structure) $F^*(a\omega + b\eta) = a(F^*\omega) + b(F^*\eta)$ for all $a, b \in \mathbb{R}$.*

2. *(preservation of the wedge product) $F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$.*

3. *(commutation with the differential) $F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega)$, i.e. the following diagram commutes*

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(N) \\ \downarrow F^* & & \downarrow F^* \\ \Omega^k(M) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(M) \end{array}$$

1.4 Closed and exact forms on smooth manifolds

In this section the de Rham cohomology will be defined and generalization of Poincaré lemma to smooth manifolds will be discussed following [32, §24].

Definition 1.47 (Closed forms)**.** $\omega \in \Omega^k(U)$ for $k \geq 0$ is said to be *closed* if $\mathrm{d}\omega = 0$.

Remark 1.31. We denote the set of all closed k-forms on M by $\mathcal{Z}^k(M)$. The sum of two such k-forms is another closed k-form, and so is the product of a closed k-form by a scalar. Hence $\mathcal{Z}^k(M)$ is the vector sub-space of $\Omega^k(M)$.

Definition 1.48 (Exact k-forms)**.** $\omega \in \Omega^k(U)$ for $k \geq 1$ is said to be *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.

Remark 1.32. We denote the set of all exact k-forms on M by $\mathcal{B}^k(U)$. The sum of two such k-forms is another exact k-form, and so is the product of a exact k-form by a scalar. Hence $\mathcal{B}^k(M)$ is the vector sub-space of $\Omega^k(M)$. Also, $\mathcal{B}^0(M)$ is defined to be the set consisting only zero.

Theorem 1.3. *On a smooth manifold M, every exact form is closed.*

Proof. Let $\omega \in \mathcal{B}^k(M)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(M)$. From Theorem 26 we know that $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\eta) = 0$ hence $\omega \in \mathcal{Z}^k(M)$ for all $k \geq 1$. For $k = 0$, the statement is trivially true. □

Lemma 1.2. *Let $F : M \to N$ be a smooth map of manifolds, then the pullback map F^* sends closed forms to closed forms, and sends exact forms to exact forms.*

Proof. Suppose ω is closed. From Theorem 28 we know that F^* commutes with d

$$\mathrm{d}F^*\omega = F^*\,\mathrm{d}\omega = 0$$

Hence, $F^*\omega$ is also closed. Next suppose $\theta = \mathrm{d}\eta$ is exact. Then

$$F^*\theta = F^*\,\mathrm{d}\eta = \mathrm{d}F^*\eta$$

Hence, $F^*\theta$ is exact. □

1.4.1 de Rham cohomology

Definition 1.49 (de Rham cohomology of a smooth manifold)**.** The k^{th} *de Rham cohomology group*[7] of M is the quotient group

$$H^k_{dR}(M) := \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)}$$

Remark 1.33. Hence, the de Rham cohomology of a smooth manifold measures the extent to which closed forms are not exact on that manifold.

Proposition 1.2. *If the smooth manifold M has ℓ connected components, then its de Rham cohomolgy in degree 0 is $H^0_{dR}(M) = \mathbb{R}^\ell$. An element of $H^0_{dR}(M)$ is specified by an ordered ℓ-tuple of real numbers, each real number representing a constant function on a connected component of M.*

Proof. Since there are no non-zero exact 0-forms

$$H^0_{dR}(M) = \mathcal{Z}^0(M)$$

Suppose f is a closed 0-form on M, i.e. $f \in C^\infty(M)$ such that $\mathrm{d}f = 0$. By Theorem 23 we know that on any chart $(U, r_1, \ldots, r_n)$

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial r_i}\,\mathrm{d}r_i$$

Thus $\mathrm{d}f = 0$ on U if and only if all the partial derivatives $\partial f/\partial r_i$ vanish identically on U. This is equivalent to f being locally constant on U. Hence, $\mathcal{Z}^0(M)$ is the set of all locally constant[8] functions on M. Such a function must be constant on each connected component of M. If M has ℓ connected components then a locally constant function of M can be specified by an ordered set of ℓ real numbers. Thus $\mathcal{Z}^0(M) = \mathbb{R}^\ell$. □

[7] which is really a vector space over $\mathbb{R}$

[8] Locally constant functions are constant on any connected component of domain.

Proposition 1.3. *On a smooth manifold M of dimension n, the de Rham cohomology $H^k_{dR}(M)$ vanishes for $k > n$.*

Proof. At any point $p \in M$, the tangent space T_pM is a vector space of dimension n. If $\omega \in \Omega^k(M)$, then $\omega_p \in \mathcal{A}^k(T_pM)$, the space of alternating k-linear functions on T_pM. By Remark 1.8, if $k > n$ then $\mathcal{A}^k(T_pM) = 0$. Hence for $k > n$, the only k-form on M is the zero form. □

1.4.2 Poincaré lemma for smooth manifolds

Definition 1.50 (Pullback map in cohomology)**.** Let $F : M \to N$ be a smooth map of manifolds, then its pullback F^* induces[9] a linear map of quotient spaces, denoted by $F^\#$

$$F^\# : \frac{\mathcal{Z}^k(N)}{\mathcal{B}^k(N)} \to \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)}$$
$$[\![\omega]\!] \mapsto [\![F^*(\omega)]\!]$$

This is a map in cohomology $F^\# : H^k_{dR}(N) \to H^k_{dR}(M)$ called the *pullback map in cohomology.*

Remark 1.34. From Remark 1.30 and Theorem 27 it follows that:

1. If $\mathbb{1}_M : M \to M$ is the identity map, then $\mathbb{1}^\#_M : H^k_{dR}(M) \to H^k_{dR}(M)$ is also the identity map.
2. If $F : M \to N$ and $G : N \to N'$ are smooth maps, then $(G \circ F)^\# = F^\# \circ G^\#$.

Proposition 1.4 (Diffeomorphism invariance of de Rham cohomology)**.** *Let $F : M \to N$ be a diffeomorphism of manifolds, then the pullback map in cohomology $F^\# : H^k_{dR}(N) \to H^k_{dR}(M)$ is an isomorphism.*

Proof. Since F is a diffeomorphism, $F^{-1} : N \to M$ is also a smooth map of manifolds. Therefore, using Remark 1.34 we have

$$\mathbb{1}_{H^k_{dR}(M)} = \mathbb{1}^\#_M = (F^{-1} \circ F)^\# = F^\# \circ (F^{-1})^\#$$

This implies that $(F^{-1})^\#$ is the inverse of $F^\#$, i.e. $F^\#$ is an isomorphism. □

Theorem 1.4 (Poincaré lemma for smooth manifold)**.** *Let M be a smooth manifold, then for all $p \in M$ there exists an open neighborhood U such that every closed k-form on U is exact for $k \geq 1$.*

Proof. Let (U, ϕ) be a chart on a smooth manifold M of dimension n such that $p \in U$. By Theorem 15 we know that the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism. We choose U such that $\phi(U)$ is an open ball in $\mathbb{R}^n$. Then by Theorem 1.2 every closed k-form on $\phi(U)$ is exact for $k \geq 1$, i.e. $H^k_{dR}(\phi(U)) = 0$ for $k \geq 1$. Now we can use Proposition 1.4 to conclude that $H^k_{dR}(U) = 0$ for $k \geq 1$, i.e. every closed k-form on U is exact for $k \geq 1$. □

[9] Follows from Lemma 1.2.

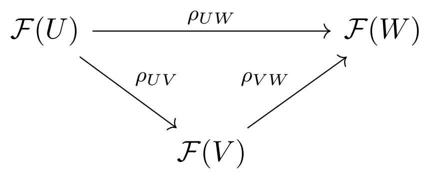
Chapter 2

Čech cohomology

2.1 Sheaf theory

Definition 2.1 (Presheaf)**.** A *presheaf*[1] $\mathcal{F}$ of abelian groups on a topological space X consists of an abelian group $\mathcal{F}(U)$ for every open subset $U \subset X$ and a group homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any two nested open subsets $V \subset U$ satisfying the following two conditions:

1. for any open subset U of X one has $\rho_{UU} = \mathbb{1}_{\mathcal{F}(U)}$
2. for open subsets $W \subset V \subset U$ one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$, i.e. the following diagram commutes



Example 2.1. Let G be a non-trivial abelian group and X be a topological space. Then the *constant presheaf* $\mathcal{G}_X$ is defined to be the collection of abelian groups $\mathcal{G}_X(U) = G$ for all non-empty subsets U of X and $\mathcal{G}_X(\emptyset) = \{0\}$, along with the group homomorphisms $\rho_{UV} = \mathbb{1}_G$ for nested open subsets $V \subset U$. In particular, for $G = \mathbb{R}$ we get the constant presheaf $\mathbb{R}$ which is the collection of constant real valued functions $\mathbb{R}(U)$ for every open subset U of X and restriction maps ρ_{UV} for nested open subsets $V \subset U$.

Example 2.2. Let X be a topological space. For each open subset U of X we define $\mathcal{F}(U)$ to be the set of (continuous/differentiable) real valued functions[2], and ρ_{UV} to be the natural restriction map for the nested open subsets $V \subset U$. Then $\mathcal{F}$ is a presheaf of (continuous/differentiable) real valued functions.

Definition 2.2 (Sheaf)**.** A presheaf $\mathcal{F}$ on a topologial space X is called a *sheaf* if for every collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X with $U = \cup_{\alpha \in A} U_\alpha$ the following conditions are satisfied

1. (Uniqueness) If $f, g \in \mathcal{F}(U)$ and $\rho_{UU_\alpha}(f) = \rho_{UU_\alpha}(g)$ for all $\alpha \in A$, then $f = g$.
2. (Gluing) If for all $\alpha \in A$ we have $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $\rho_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$ for any $\alpha, \beta \in A$, then there exists a $f \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}(f) = f_\alpha$ for all $\alpha \in A$ (this f is unique by previous axiom).

[1]Presheaves and sheaves are typically denoted by calligraphic letters, $\mathcal{F}$ being particularly common, presumably for the French word for sheaves, *faisceaux*.

[2]Note that there exists only one function from an empty set to any other set, hence $\mathcal{F}(\emptyset)$ is singleton.

Example 2.3. It is easy to observe that the gluing axiom doesn't hold for the constant presheaf $\mathbb{R}$ on X if X is disconnected. We therefore define a *constant sheaf* $\underline{\mathbb{R}}$ on X to be the collection of *locally constant* real valued functions $\underline{\mathbb{R}}(U)$ corresponding to every open subset $U \subset X$ and restriction maps ρ_{UV} for nested open subsets $V \subset U$.

In general, given a non-trivial abelian group G, the *constant sheaf* $\underline{G}$ on X is defined by endowing G with the discrete topology and assigning each open set U of X the set $\underline{G}(U)$ of all continuous functions $f : U \to G$ along with the restriction maps ϕ_{UV} for nested open sets $V \subset U$.

Example 2.4. If one has a presheaf of functions (or forms) on X which is defined by some property which is a local property[3], then the presheaf is also a sheaf. This is because the agreement of functions (or forms) on the overlap intersections automatically gives a well defined unique function (or form) on the open set U, and one must only check that it satisfies the property [20, p. 272].

In particular, if X is a smooth manifold then Ω^q is the sheaf of smooth q-forms on X such that for every open subset U of X we have the abelian group $\Omega^q(U)$ of smooth q-forms on U (smooth sections of a exterior power of cotangent bundle, i.e. smooth maps of manifolds) along with the natural restriction maps as the group homomorphisms ρ_{UV} for nested open subsets $V \subset U$ [37, Example II.1.9].

Remark 2.1. When defining presheaf, many authors like Liu [17, §2.2.1] and Miranda [20, §IX.1], additionally require $\mathcal{F}(\emptyset) = 0$, i.e. the trivial group with one element. This is a necessary condition for the sheaf to be well defined, but this follows from our sheaf axioms. Namely, note that the empty set is covered by the empty open covering, and hence the collection $f_i \in \mathcal{F}(U_i)$ from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in[4]. In other words, we don't require this condition while defining presheaf (see [37, §II.1] or [1, §II.10]) since from the definition of sheaf one can deduce that that $\mathcal{F}(\emptyset)$ is equal to a final object, which in the case of a sheaf of sets is a singleton.

Remark 2.2. There is another equivalent way of defining sheaf $\mathcal{F}$ (of abelian groups) over X as a triple (F, π, X) which satisfies certain axioms [11, §2.1]. For a discussion on the equivalence of both these definitions see [35, §5.7]. However, the defintion that we have adopted is useful since for many important sheaves, particularly those that arise in algebraic geometry, the sheaf space F is obscure, and its topology complicated [13, Remark 2.6].

Remark 2.3. The definition of sheaf can be generalized to any category like groups, rings, modules, and algebras instead of abelian groups.

2.1.1 Stalks

Definition 2.3 (Stalk)**.** Let $\mathcal{F}$ be a sheaf on X, and let $x \in X$. Then the *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

where the direct limit[5] is indexed over all the open subsets containing x, with order relation induced by reverse inclusion, i.e. $U < V$ if $V \subset U$. Also, the image of $f \in \mathcal{F}(U)$ in $\mathcal{F}_x$ under

[3] A property $\mathcal{P}$ is said to be *local* if whenever $\{U_\alpha\}_{\alpha \in A}$ is an open cover of an open set U, then the property holds on U if and only if it holds for each U_α. In other words, a *local property* $\mathcal{P}$ of functions is the one which is initially defined at points, i.e. a function f defined in a neighborhood of a point $p \in X$ has property $\mathcal{P}$ at p if and only if some condition holds at the point p. For example, the preoperties like continuity and differentiability.

[4] The Stacks project, Tag 006U: `https://stacks.math.columbia.edu/tag/006U`

[5] For the definition of direct limit see Appendix B. To get the direct system $\{\mathcal{F}(U), \rho_{UV}\}$, the "reverse inclusion" is defined to be the order relation for the directed set.

the group homomorphism induced[6] by the inclusion map $\mathcal{F}(U) \hookrightarrow \coprod_{U \ni x} \mathcal{F}(U)$ is denoted by f_x, i.e $[\![f]\!] = f_x$.

Remark 2.4. This definition of stalks also holds for presheaves, which leads to the useful tool of *sheafification*, i.e. finding sheaf associated to a given presheaf. This technique of sheafification is very useful but we won't need it in this thesis. For more details, see the books by Hirzebruch [11, §2] and Liu [17, §2.2.1].

Lemma 2.1. *Let $\mathcal{F}$ be a sheaf of abelian groups on X and $f, g \in \mathcal{F}(X)$ be such that $f_x = g_x$ for every $x \in X$. Then $f = g$.*

Proof. Without loss of generality, we may assume $g = 0$. Then $f_x = 0$ implies that f_x and 0 belong to same equivalence class, i.e. for every $x \in X$ there exists an open neighborhood U_x of x such that $\rho_{XU_x}(f) = 0$. As $\{U_x\}_{x\in X}$ covers X , we have $f = 0$ by the uniqueness condition of sheaf. □

2.1.2 Sheaf maps

Definition 2.4 (Map of sheaves). Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on a topological space X. A maps of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X is given by a collection of group homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any open subset U of X, which commute with the group homomorphisms ρ for the two sheaves, i.e. for $V \subset U$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \Big\downarrow{\rho^{\mathcal{F}}_{UV}} & & \Big\downarrow{\rho^{\mathcal{G}}_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

Example 2.5. The identity map $\mathbb{1}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is always a sheaf map, and the composition of sheaf maps is a sheaf map.

Example 2.6. As seen above, for the sheaf of functions (or forms) the natural restriction map is the group homomorphism ρ_{UV} for nested open subsets $V \subset U$. Also, from Remark 1.29 we know that the exterior derivative is a local operator, hence it commutes with restriction. Therefore, $\mathrm{d} : \Omega^q \to \Omega^{q+1}$ is a map of sheaves, where Ω^q and Ω^{q+1} are sheaves of smooth q-forms and $q+1$-forms, respectively, defined on a smooth manifold X for $q \geq 0$. In other words, Remark 1.29 implies that the following diagram commutes for $V \subset U$

$$\begin{array}{ccc} \Omega^q(U) & \xrightarrow{\mathrm{d}_U} & \Omega^{q+1}(U) \\ \Big\downarrow{\rho_{UV}} & & \Big\downarrow{\rho_{UV}} \\ \Omega^q(V) & \xrightarrow{\mathrm{d}_V} & \Omega^{q+1}(V) \end{array}$$

where by abuse of notation we use the same symbol for restriction maps of both sheaves.

Definition 2.5 (Associated presheaf). Given a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ between two sheaves of abelian groups on X, one constructs the *associated presheaves* $\ker(\phi), \mathrm{im}(\phi)$, and $\mathrm{coker}(\phi)$ which are defined in the obvious way[7], i.e. $\ker(\phi)(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ with group homomorphism ρ inherited from $\mathcal{F}$.

[6] As defined in the universal property of direct limit, see Theorem B.1.

[7] Let U be an open subset and $f \in \ker(\phi_U)$, then for $V \subset U$ we have $\rho^{\mathcal{F}}_{UV}(f) \in \ker(\phi_V)$ since $\phi_V \circ \rho^{\mathcal{F}}_{UV} = \rho^{\mathcal{G}}_{UV} \circ \phi_U$.

Proposition 2.1. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf map between two sheaves of abelian groups on X, then* $\ker(\phi)$ *is a sheaf.*

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be a collection of open sets of X, and $U = \cup_{\alpha\in A} U_\alpha$ be their union. It suffices to show that if for all $\alpha \in A$ we have $f_\alpha \in \ker(\phi_{U_\alpha})$ such that $\rho^{\mathcal{F}}_{U_\alpha, U_\alpha\cap U_\beta}(f_\alpha) = \rho^{\mathcal{F}}_{U_\beta, U_\alpha\cap U_\beta}(f_\beta)$ for any $\alpha, \beta \in A$, then there exists a unique $f \in \ker(\phi_U)$ such that $\rho^{\mathcal{F}}_{UU_\alpha}(f) = f_\alpha$ for all $\alpha \in A$.

Since $\mathcal{F}$ is a sheaf, there exists a unique element $f \in \mathcal{F}(U)$ such that $\rho^{\mathcal{F}}_{UU\alpha}(f) = f_\alpha$ for all $\alpha \in A$. We just need to show that $f \in \ker(\phi_U)$, i.e. $\phi_U(f) = 0$ in $\mathcal{G}(U)$.

Let $g_\alpha = \rho^{\mathcal{G}}_{UU\alpha}(\phi_U(f))$, then by the commutativity of ϕ with ρ, we have that

$$g_\alpha = \rho^{\mathcal{G}}_{UU\alpha}(\phi_U(f)) = \phi_{U_\alpha}(\rho^{\mathcal{F}}_{UU_\alpha}(f)) = \phi_{U_\alpha}(f_\alpha) = 0$$

since $f_\alpha \in \ker(\phi_{U_\alpha})$. Now using the uniqueness axiom for the sheaf $\mathcal{G}$ we conclude that $\phi_U(f) = 0$, since $\rho^{\mathcal{G}}_{UU\alpha}(\phi_U(f)) = 0$ for all $\alpha \in A$. □

Example 2.7. Let X be a smooth manifold and $\mathrm{d} : \Omega^q \to \Omega^{q+1}$ be the exterior derivative. Then $\ker(\mathrm{d}) = \mathscr{Z}^q$ is the sheaf of closed q-forms on X.

Remark 2.5. There is an important subtlety here. The associated presheaves $\mathrm{im}(\phi)$ and $\mathrm{coker}(\phi)$ need not be sheaves in general. Also, in general, quotient of sheaves need not be a sheaf. In order to define the cokernel, image and quotient sheaf one need to use sheafification, see [12, Definition B.0.26] and [9, pp. 36-37].

Definition 2.6 (Injective map of sheaves)**.** A map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is called injective if for every open subset U of X, ϕ_U is an injective group homomorphism.

Definition 2.7 (Surjective map of sheaves)**.** A map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is called surjective if for every $x \in X$ the induced map of stalks[8] $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a surjective group homomorphism.

Remark 2.6. In other words, ϕ is surjective if for every point $x \in X$, every open set U containing x and every $g \in \mathcal{G}(U)$, there is an open subset $V \subset U$ containing x such that $\phi_V(f) = \rho^{\mathcal{G}}_{UV}(g)$ for some $f \in \mathcal{F}(V)$.

Proposition 2.2. *The sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for every $x \in X$.*

Proof. ($\Rightarrow$) This is trivial.

($\Leftarrow$) Let U be any open subset of X, it suffices to show that $\ker(\phi_U) = \{0_{\mathcal{F}(U)}\}$. Let $f \in \mathcal{F}(U)$ such that $\phi_U(f) = 0_{\mathcal{G}(U)}$. Then for every $x \in U$, $\phi_x(f_x) = [\![\phi_U(f)]\!] = 0_{G_x}$. Since ϕ_x is injective, we have $f_x = 0_{\mathcal{F}_x}$ for every $x \in U$. By Lemma 2.1 we conclude that $f = 0_{\mathcal{F}(U)}$, hence completing the proof. □

Remark 2.7. Analogous statement is not true for the surjective map of sheaves, see [17, Example 2.2.11]

Proposition 2.3. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be an injective map of sheaves. Then ϕ is surjective if and only if $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets $U \subset X$.*

Proof. ($\Rightarrow$) Let U be any open subset of X, and $g \in \mathcal{G}(U)$. We need to show that there exists a $f \in \mathcal{F}(U)$ such that $\phi_U(f) = g$. Since ϕ_x is surjective for every $x \in X$, for every $g_x \in \mathcal{G}_x$ there exists an open neighborhood V of x and $f \in \mathcal{F}(V)$ such that $\phi_x(f_x) = [\![\phi_V(f)]\!] = g_x$. Therefore, we can find an open covering $\{U_\alpha\}_{\alpha\in A}$ of U such that each U_α is an open neighborhood of

[8]The map of sheaves is a map of direct systems $\phi : \{(\mathcal{F}(U), \rho^{\mathcal{F}}_{UV})\} \to \{(\mathcal{G}(U), \rho^{\mathcal{G}}_{UV})\}$, and the map of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is the direct limit of the homomorphisms ϕ_U.

$x \in U$ such that $\phi_x(f_x) = [\![\phi_{U_\alpha}(f_\alpha)]\!] = g_x$ for some $f_\alpha \in \mathcal{F}(U_\alpha)$. In other words, there exist $f_\alpha \in \mathcal{F}(U_\alpha)$ such that

$$\phi_{U_\alpha}(f_\alpha) = \rho^{\mathcal{G}}_{UU_\alpha}(g) \tag{2.1}$$

for all $\alpha \in A$. In particular, for $f_\alpha \in \mathcal{F}(U_\alpha)$ and $f_\beta \in \mathcal{F}(U_\beta)$ we have

$$\phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{F}}_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha)) = \rho^{\mathcal{G}}_{U, U_\alpha \cap U_\beta}(g) \quad \text{and} \quad \phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{F}}_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)) = \rho^{\mathcal{G}}_{U, U_\alpha \cap U_\beta}(g)$$

Since ϕ is injective, the map $\phi_{U_\alpha \cap U_\beta} : \mathcal{F}(U_\alpha \cap U_\beta) \to \mathcal{G}(U_\alpha \cap U_\beta)$ is injective for all $\alpha, \beta \in A$. Hence we have

$$\rho^{\mathcal{F}}_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \rho^{\mathcal{F}}_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$$

for all $\alpha, \beta \in A$. Now by the gluing axiom of the sheaf $\mathcal{F}$, there exists a $f \in \mathcal{F}(U)$ such that $\rho^{\mathcal{F}}_{UU_\alpha}(f) = f_\alpha$ for all $\alpha \in A$. Using this in (2.1) we get

$$\rho^{\mathcal{G}}_{UU_\alpha}(g) = \phi_{U_\alpha}(\rho^{\mathcal{F}}_{UU_\alpha}(f)) = \rho^{\mathcal{G}}_{UU_\alpha}(\phi_U(f))$$

for all $\alpha \in A$. By the uniqueness axiom of the sheaf $\mathcal{G}$, we conclude that $g = \phi_U(f)$. Hence completing the proof.

($\Leftarrow$) This is trivial. □

2.1.3 Exact sequence of sheaves

Definition 2.8 (Exact sequence of sheaves)**.** A sequence of sheaves $\mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is said to be exact if $\mathcal{F}'_x \xrightarrow{\phi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{F}''_x$ is an exact sequence of abelian groups for every $x \in X$.

Example 2.8. By Theorem 1.3, Theorem 1.4 and Proposition 1.2 we know that for every point x in a smooth manifold X there exists an open subset U containing x such that

$$0 \longrightarrow \underline{\mathbb{R}}(U) \hookrightarrow \Omega^0(U) \xrightarrow{\mathrm{d}_U} \Omega^1(U) \xrightarrow{\mathrm{d}_U} \Omega^2(U) \xrightarrow{\mathrm{d}_U} \cdots$$

is an exact sequence of abelian groups. In other words, for all $x \in X$ we have a long exact sequence at the level of stalks

$$0 \longrightarrow \underline{\mathbb{R}}_x \hookrightarrow \Omega^0_x \xrightarrow{\mathrm{d}_x} \Omega^1_x \xrightarrow{\mathrm{d}_x} \Omega^2_x \xrightarrow{\mathrm{d}_x} \cdots$$

Therefore, by Poincaré lemma, the sequence of sheaves of differential forms on a smooth manifold

$$0 \longrightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \xrightarrow{\mathrm{d}} \cdots$$

is exact.

Lemma 2.2. *If* $0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ *is an exact sequence of sheaves over* X*, then the induced sequence of abelian groups for any open set* $U \subset X$

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\phi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U)$$

is also exact.

Proof. For all $x \in X$ we have an exact sequence of stalks

$$0 \longrightarrow \mathcal{F}'_x \xrightarrow{\phi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{F}''_x$$

Using Proposition 2.2 we conclude that ϕ_U is injective. Hence we just need to show that $\operatorname{im}(\phi_U) = \ker(\psi_U)$.

$\boxed{\ker(\psi_U) \subseteq \operatorname{im}(\phi_U)}$ Let $f \in \ker(\psi_U)$, then for all $x \in U$ we have $f_x \in \ker(\psi_x)$ since $\psi_x(f_x) = [\![\psi_U(f)]\!]$. Since the sequence of stalks is exact at $\mathcal{F}_x$, $f_x = \phi_x(g_x)$ for some $g_x \in \mathcal{F}'_x$. Therefore, we can find an open covering $\{U_\alpha\}_{\alpha \in A}$ of U such that each U_α is an open neighborhood of $x \in U$ such that $\phi_x(g_x) = [\![\phi_{U_\alpha}(g_\alpha)]\!] = f_x$ for some $g_\alpha \in \mathcal{F}'(U_\alpha)$. In other words, there exist $g_\alpha \in \mathcal{F}'(U_\alpha)$ such that

$$\phi_{U_\alpha}(g_\alpha) = \rho^{\mathcal{F}}_{UU_\alpha}(f) \tag{2.2}$$

for all $\alpha \in A$. In particular, for $g_\alpha \in \mathcal{F}'(U_\alpha)$ and $g_\beta \in \mathcal{F}'(U_\beta)$ we have

$$\phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{F}'}_{U_\alpha, U_\alpha \cap U_\beta}(g_\alpha)) = \rho^{\mathcal{F}}_{U, U_\alpha \cap U_\beta}(f) \quad \text{and} \quad \phi_{U_\alpha \cap U_\beta}(\rho^{\mathcal{F}'}_{U_\beta, U_\alpha \cap U_\beta}(g_\beta)) = \rho^{\mathcal{F}}_{U, U_\alpha \cap U_\beta}(f)$$

Since ϕ is injective, the map $\phi_{U_\alpha \cap U_\beta} : \mathcal{F}(U_\alpha \cap U_\beta) \to \mathcal{G}(U_\alpha \cap U_\beta)$ is injective for all $\alpha, \beta \in A$. Hence we have

$$\rho^{\mathcal{F}'}_{U_\alpha, U_\alpha \cap U_\beta}(g_\alpha) = \rho^{\mathcal{F}'}_{U_\beta, U_\alpha \cap U_\beta}(g_\beta)$$

for all $\alpha, \beta \in A$. Now by the gluing axiom of the sheaf $\mathcal{F}'$, there exists a $g \in \mathcal{F}'(U)$ such that $\rho^{\mathcal{F}'}_{UU_\alpha}(g) = g_\alpha$ for all $\alpha \in A$. Using this in (2.2) we get

$$\rho^{\mathcal{F}}_{UU_\alpha}(f) = \phi_{U_\alpha}(\rho^{\mathcal{F}'}_{UU_\alpha}(g)) = \rho^{\mathcal{F}}_{UU_\alpha}(\phi_U(g))$$

for all $\alpha \in A$. By the uniqueness axiom of the sheaf $\mathcal{F}$, we conclude that $f = \phi_U(g)$.

$\boxed{\operatorname{im}(\phi_U) \subseteq \ker(\psi_U)}$ Let $f \in \operatorname{im}(\phi_U)$, i.e. there exists $g \in \mathcal{F}'(U)$ such that $\phi_U(g) = f$. Then for all $x \in U$ we have $f_x \in \operatorname{im} \phi_x$ since $\phi_x(g_x) = [\![\psi_U(f)]\!] = f_x$. Since the sequence of stalks is exact at $\mathcal{F}_x$, $\psi_x(f_x) = 0_{\mathcal{F}''_x}$ for all $x \in X$. Since $\psi_x(f_x) = [\![\psi_U(f)]\!]$, by Lemma 2.1 we conclude that $\psi_U(f) = 0$. □

Remark 2.8. In general, given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\ \phi\ } \mathcal{F} \xrightarrow{\ \psi\ } \mathcal{F}'' \longrightarrow 0$$

Then the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{F}'(X) \xrightarrow{\ \phi_X\ } \mathcal{F}(X) \xrightarrow{\ \psi_X\ } \mathcal{F}''(X) \longrightarrow 0$$

is always exact at $\mathcal{F}'(X)$ and $\mathcal{F}(X)$ but not necessarily at $\mathcal{F}''(X)$, see [37, §II.3] and [27, §4.1].

2.2 Čech cohomology of sheaves

Definition 2.9 (Čech cochain)**.** Let $\mathcal{F}$ be sheaf of abelian groups on a topologial space X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X, and fix an integer $k \geq 0$. A Čech k-cochain for the sheaf $\mathcal{F}$ over the open cover $\mathcal{U}$ is an element of $\prod_{(i_0, i_1, \ldots, i_k)} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k})$ where Cartesian product is take over all collections of $k+1$ indices $(i_0, \ldots, i_k)$ from I.

Remark 2.9. To simplify the notation, we will write

$$U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k} := U_{i_0, i_1, \ldots, i_k} \quad \text{and} \quad \mathcal{F}(U_{i_0, i_1, \ldots, i_k}) = \{f_{i_0, i_1, \ldots, i_k}\}$$

Hence a Čech k-cochain is a tuple of the form $(f_{i_0, i_1, \ldots, i_k})$. The abelian group of Čech k-cochains for $\mathcal{F}$ over $\mathcal{U}$ is denoted by $\check{C}^k(\mathcal{U}, \mathcal{F})$; thus

$$\check{C}^k(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, i_1, \ldots, i_k)} \mathcal{F}(U_{i_0, i_1, \ldots, i_k})$$

Definition 2.10 (Coboundary operator)**.** The coboundary operator is defined as

$$\delta : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{C}}^{k+1}(\mathcal{U}, \mathcal{F})$$
$$(f_{i_0,i_1,\ldots,i_k}) \mapsto (g_{i_0,i_1,\ldots,i_{k+1}})$$

where

$$g_{i_0,i_1,\ldots,i_{k+1}} = \sum_{\ell=0}^{k+1}(-1)^\ell \rho(f_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}})$$

and $\rho : \mathcal{F}(U_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}}) \to \mathcal{F}(U_{i_0,i_1,\ldots,i_{k+1}})$ is the group homomorphism for the sheaf $\mathcal{F}$ corresponding to the nested open subsets $U_{i_0,i_1,\ldots,i_{k+1}} \subset U_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}}$.

Remark 2.10. To simplify the notations above, we wrote

$$U_{i_0,i_1,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k} := U_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_k} \quad \text{and} \quad \mathcal{F}(U_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_k}) = \{f_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_k}\}$$

Definition 2.11 (Čech cocycle)**.** A Čech k-cochain $f = (f_{i_0,i_1,\ldots,i_k})$ with $\delta(f) = 0$ is called Čech k-cocycle.

Remark 2.11. The abelian group of k-cocycles is denoted by $\check{Z}^k(\mathcal{U}, \mathcal{F})$, i.e. kernel of δ at the k^{th} level.

Proposition 2.4. *Let* $f = (f_{i_0,\ldots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$*, then*

1. $f_{i_0,\ldots,i_n} = 0$ *if any two indices are equal.*
2. $f_{\sigma(i_0),\sigma(i_1),\ldots,\sigma(i_k)} = \mathrm{sgn}(\sigma) f_{i_0,i_1,\ldots,i_k}$ *if* σ *is a permutation of* $\{i_0, \ldots, i_k\}$

Proof. We will check just for the simplest case, $k = 1$. Let $f = (f_{i_0 i_1})$ and $\delta(f) = (g_{i_0 i_1 i_2}) = 0$. For any $i \in I$ we have

$$0 = g_{i,i,i} = \rho_{U_{i,i}U_{i,i,i}}(f_{i,i}) - \rho_{U_{i,i}U_{i,i,i}}(f_{i,i}) + \rho_{U_{i,i}U_{i,i,i}}(f_{i,i})$$

This implies that $f_{i,i} = 0$ by the uniqueness axiom of sheaf. On the other hand, applied to (i, j, i) instead, it says

$$0 = g_{i,j,i} = \rho_{U_{j,i}U_{i,j,i}}(f_{j,i}) - \rho_{U_{i,i}U_{i,j,i}}(f_{i,i}) + \rho_{U_{i,j}U_{i,j,i}}(f_{i,j})$$

This implies that

$$\rho_{U_{j,i}U_{i,j,i}}(f_{j,i}) + \rho_{U_{i,j}U_{i,j,i}}(f_{i,j}) = 0 \quad \text{for all } i \in I$$

But the $\{U_{i,j,i}\}_{i\in I}$ is an open cover of $U_{i,j}$, and hence indeed $f_{i,j} = -f_{i,j}$ due to the uniqueness axiom of the sheaf $\mathcal{F}$. □

Definition 2.12 (Čech coboundary)**.** A Čech k-cochain $f = (f_{i_0,i_1,\ldots,i_k})$ which is the image of δ, i.e. there exists $(k-1)$-cochain $g = (g_{i_0,i_1,\ldots,i_{k-1}})$ such that $\delta(g) = f$, is called Čech k-coboundary.

Remark 2.12. The abelian group of k-coboundaries is denoted by $\check{B}^k(\mathcal{U}, \mathcal{F})$, i.e. image of δ at the $(k-1)^{th}$ level. Also, we define $\check{B}^0(\mathcal{U}, \mathcal{F}) = 0$ for any sheaf $\mathcal{F}$ and open cover $\mathcal{U}$.

Lemma 2.3. $\delta \circ \delta = 0$

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be the open cover. We will check it just for the simplest case

$$\begin{aligned} \check{\mathrm{C}}^0(\mathcal{U},\mathcal{F}) &\xrightarrow{\ \delta\ } \check{\mathrm{C}}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\ \delta\ } \check{\mathrm{C}}^2(\mathcal{U},\mathcal{F}) \\ (f_\alpha) &\longmapsto (g_{\alpha\beta}) \longmapsto (h_{\alpha\beta\gamma}) \end{aligned}$$

By the definition of coboundary operator, for $i_0=\alpha$ and $i_1=\beta$, we have

$$\begin{aligned} g_{\alpha\beta} &= (-1)^0\rho_{U_\beta U_{\alpha\beta}}(f_\beta)+(-1)^1\rho_{U_\alpha U_{\alpha\beta}}(f_\alpha) \\ &= \rho_{U_\beta U_{\alpha\beta}}(f_\beta)-\rho_{U_\alpha U_{\alpha\beta}}(f_\alpha) \end{aligned} \tag{2.3}$$

Also for $i_0=\alpha, i_1=\beta$ and $i_2=\gamma$, we have

$$\begin{aligned} h_{\alpha\beta\gamma} &= (-1)^0\rho_{U_{\beta\gamma}U_{\alpha\beta\gamma}}(g_{\beta\gamma})+(-1)^1\rho_{U_{\alpha\gamma}U_{\alpha\beta\gamma}}(g_{\alpha\gamma})+(-1)^2\rho_{U_{\alpha\beta}U_{\alpha\beta\gamma}}(g_{\alpha\beta}) \\ &= \rho_{U_{\beta\gamma}U_{\alpha\beta\gamma}}(g_{\beta\gamma})-\rho_{U_{\alpha\gamma}U_{\alpha\beta\gamma}}(g_{\alpha\gamma})+\rho_{U_{\alpha\beta}U_{\alpha\beta\gamma}}(g_{\alpha\beta}) \end{aligned} \tag{2.4}$$

Using (2.3) in (2.4) we get

$$\begin{aligned} h_{\alpha\beta\gamma} &= \rho_{U_{\beta\gamma}U_{\alpha\beta\gamma}}\left(\rho_{U_\gamma U_{\beta\gamma}}(f_\gamma)-\rho_{U_\beta U_{\beta\gamma}}(f_\beta)\right)-\rho_{U_{\alpha\gamma}U_{\alpha\beta\gamma}}\left(\rho_{U_\gamma U_{\alpha\gamma}}(f_\gamma)-\rho_{U_\alpha U_{\alpha\gamma}}(f_\alpha)\right) \\ &\quad +\rho_{U_{\alpha\beta}U_{\alpha\beta\gamma}}\left(\rho_{U_\beta U_{\alpha\beta}}(f_\beta)-\rho_{U_\alpha U_{\alpha\beta}}(f_\alpha)\right) \\ &= \rho_{U_\gamma U_{\alpha\beta\gamma}}(f_\gamma)-\rho_{U_\beta U_{\alpha\beta\gamma}}(f_\beta)-\rho_{U_\gamma U_{\alpha\beta\gamma}}(f_\gamma)+\rho_{U_\alpha U_{\alpha\beta\gamma}}(f_\alpha)+\rho_{U_\beta U_{\alpha\beta\gamma}}(f_\beta)-\rho_{U_\alpha U_{\alpha\beta\gamma}}(f_\alpha) \\ &= 0 \end{aligned}$$

Hence completing the verification. ☐

Proposition 2.5. *Every k-coboundary is a k-cocycle.*

Proof. Let $f=(f_{i_0,i_1,\dots,i_k})\in \check{B}^k(\mathcal{U},\mathcal{F})$ such that $f=\delta(g)$ for some $g=(g_{i_0,i_1,\dots,i_{k-1}})\in \check{\mathrm{C}}^{k-1}(\mathcal{U},\mathcal{F})$. From Lemma 2.3 we know that $\delta(f)=\delta(\delta(g))=0$ hence $f\in\check{Z}^k(\mathcal{U},\mathcal{F})$ for all $k\geq 1$. For $k=0$, the statement is trivially true. ☐

Definition 2.13 (Čech cohomology with respect to a cover)**.** The k^{th} *Čech cohomology group* of $\mathcal{F}$ with respect to the open cover $\mathcal{U}$ is the quotient group

$$\check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}) := \frac{\check{Z}^k(\mathcal{U},\mathcal{F})}{\check{B}^k(\mathcal{U},\mathcal{F})}$$

Remark 2.13. Hence, the Čech cohomology with respect to a cover measures the extent to which cocycles are not coboundaries for a given open cover.

Lemma 2.4. *For any sheaf $\mathcal{F}$ and open covering $\mathcal{U}$ of X, we have $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F})\cong\mathcal{F}(X)$.*

Proof. Since $\check{B}^0(\mathcal{U},\mathcal{F})=0$, we just need to show that $\check{Z}^0(\mathcal{U},\mathcal{F})\cong\mathcal{F}(X)$. Consider the following group homomorphism

$$\begin{aligned} \psi : \mathcal{F}(X) &\to \check{\mathrm{C}}^0(\mathcal{U},\mathcal{F}) \\ f &\mapsto (f_i)=(\rho_{XU_i}(f)) \end{aligned}$$

Then $\delta((f_i))=(g_{ij})$, where $g_{ij}=\rho_{U_jU_{ij}}(f_j)-\rho_{U_iU_{ij}}(f_i)$; this is zero for every i and j since $\rho_{U_iU_{ij}}(\rho_{XU_i}(f))=\rho_{U_jU_{ij}}(\rho_{XU_j}(f))$. Hence ψ maps $\mathcal{F}(X)$ to $\check{Z}^0(\mathcal{U},\mathcal{F})$. This map is injective and surjective by the uniqueness and gluing axioms of the sheaf $\mathcal{F}$, respectively. ☐

Definition 2.14 (Refining map)**.** Let $\mathcal{U}=\{U_i\}_{i\in I}$ and $\mathcal{V}=\{V_j\}_{j\in J}$ be two open coverings of X such that $\mathcal{V}$ is a refinement[9] of $\mathcal{U}$. Then the map $r:J\to I$ such that $V_j\subset U_{r(j)}$ for every $j\in J$, is called the *refining map* for the coverings.

[9]For its definition see section A.1.

Remark 2.14. The refining map is not unique. Also, the set of all open covers is a *directed set*[10] where the ordering is done via refinement, i.e. $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. The upper bound of $\mathcal{U}$ and $\mathcal{V}$ is given by $\mathcal{W} = \{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$ [25, §73, Example 2].

Lemma 2.5. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ be two open coverings of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ along with the refining map $r : J \to I$. The induced map at the level of cochains is given by*

$$\begin{aligned}\widetilde{r} : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{C}}^k(\mathcal{V}, \mathcal{F})\\ (f_{i_0,\dots,i_k}) &\mapsto (g_{j_0,\dots,j_k})\end{aligned}$$

where

$$g_{j_0,\dots,j_k} = \rho(f_{r(j_0),\dots,r(j_k)})$$

and $\rho : \mathcal{F}(U_{r(j_0),\dots,r(j_k)}) \to \mathcal{F}(V_{j_0,\dots,j_k})$ is the group homomorphism for the sheaf $\mathcal{F}$ corresponding to the nested open subsets $V_{j_0,\dots,j_k} \subset U_{r(j_0),\dots,r(j_k)}$. This map sends cocycles to cocycles and coboundaries to coboundaries.

Proof. We will check it just for the simplest case. We have the map

$$\begin{aligned}\widetilde{r} : \check{\mathrm{C}}^0(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{C}}^0(\mathcal{V}, \mathcal{F})\\ (f_{i_0}) &\mapsto \left(\rho_{U_{r(j_0)}V_{j_0}}(f_{r(j_0)})\right)\end{aligned}$$

Let $\delta((f_{i_0})) = 0$, then $\rho_{U_{i_1}U_{i_0 i_1}}(f_{i_1}) = \rho_{U_{i_0}U_{i_0 i_1}}(f_{i_0})$ for every pair of indices $i_0, i_1 \in I$. Next we compute $\delta\left(\left(\rho_{U_{r(j_0)}V_{j_0}}(f_{r(j_0)})\right)\right) = (g_{j_0,j_1})$

$$\begin{aligned}g_{j_0,j_1} &= \rho_{V_{j_1}V_{j_0 j_1}}\left(\rho_{U_{r(j_1)}V_{j_1}}(f_{r(j_1)})\right) - \rho_{V_{j_0}V_{j_0 j_1}}\left(\rho_{U_{r(j_0)}V_{j_0}}(f_{r(j_0)})\right)\\ &= \rho_{U_{r(j_1)}V_{j_0 j_1}}(f_{r(j_1)}) - \rho_{U_{r(j_0)}V_{j_0 j_1}}(f_{r(j_0)})\end{aligned}$$

But, we have

$$\rho_{U_{r(j_1)}U_{r(j_0)r(j_1)}}(f_{r(j_1)}) = \rho_{U_{r(j_0)}U_{r(j_0)r(j_1)}}(f_{r(j_0)})$$

and $V_{j_0,j_1} \subset U_{r(j_0)r(j_1)}$. Therefore $g_{j_0,j_1} = 0$, and $\widetilde{r}$ maps cocycle to cocycle. Since 0 is the only coboundary in this case, it also maps coboundary to coboundary. □

Lemma 2.6. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ be two open coverings of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ along with the refining map $r : J \to I$. The induced map at the level of cohomology*[11] *is given by*

$$\begin{aligned}H_r : \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{H}}^k(\mathcal{V}, \mathcal{F})\\ [\![(f_{i_0,\dots,i_k})]\!] &\mapsto [\![(g_{j_0,\dots,j_k})]\!]\end{aligned}$$

for $(f_{i_0,\dots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, where

$$g_{j_0,\dots,j_k} = \rho(f_{r(j_0),\dots,r(j_k)})$$

and $\rho : \mathcal{F}(U_{r(j_0),\dots,r(j_k)}) \to \mathcal{F}(V_{j_0,\dots,j_k})$ is the group homomorphism for the sheaf $\mathcal{F}$ corresponding to the nested open subsets $V_{j_0,\dots,j_k} \subset U_{r(j_0),\dots,r(j_k)}$. This map is independent of the refining map r and depends only on the two coverings $\mathcal{U}$ and $\mathcal{V}$.

[10] For its definition see Appendix B.
[11] This map is well defined by the previous lemma.

Proof. Suppose the $r, r' : J \to I$ are two distinct refining maps for the refinement $\mathcal{V}$ of $\mathcal{U}$.

`Claim:` $H_r = H_{r'}$

If $k = 0$, then $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X) \cong \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$. Therefore $H_r = \mathbb{1}_{\mathcal{F}(X)} = H_{r'}$. Let's assume that $k \geq 1$, and fix a cohomology class $f \in \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F})$ represented by $(f_{i_0,i_1,\dots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, i.e. $f = [\![(f_{i_0,i_1,\dots,i_k})]\!]$. Then we have

$$H_r(f) = [\![(g_{j_0,j_1,\dots,j_k})]\!] \quad \text{and} \quad H_{r'}(f) = [\![(g'_{j_0,j_1,\dots,j_k})]\!]$$

where

$$g_{j_0,j_1,\dots,j_k} = \rho_\alpha(f_{r(j_0),\dots,r(j_k)}) \quad \text{and} \quad g'_{j_0,j_1,\dots,j_k} = \rho_\beta(f_{r'(j_0),\dots,r'(j_k)})$$

where ρ_α and ρ_β are the appropriate group homomorphism for the sheaf $\mathcal{F}$. To prove our claim, it suffices to show that $(g_{j_0,j_1,\dots,j_k} - g'_{j_0,j_1,\dots,j_k}) \in \check{B}^k(\mathcal{V}, \mathcal{F})$.

`Claim:` $\delta(h) = (g'_{j_0,j_1,\dots,j_k} - g_{j_0,j_1,\dots,j_k})$ where $h = (h_{j_0,j_1,\dots,j_{k-1}}) \in \check{\mathrm{C}}^{k-1}(\mathcal{V}, \mathcal{F})$ is such that[12]

$$h_{j_0,j_1,\dots,j_{k-1}} = \sum_{\ell=0}^{k-1} (-1)^\ell \rho\left(f_{r(j_0),\dots,r(j_\ell),r'(j_\ell),\dots,r'(j_{k-1})}\right)$$

The claim follows from the fact that $(f_{i_0,i_1,\dots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$ for all indices $(i_0, \dots, i_k)$.

We will check the claim just for the simplest case, when $k = 1$. In this case we have $f = [\![(f_{i_0,i_1})]\!]$, since $(f_{i_0,i_1}) \in \check{Z}^1(\mathcal{U}, \mathcal{F})$ we have $\delta((f_{i_0,i_1})) = 0$, that is

$$\rho_{U_{i_1,i_2} U_{i_0 i_1 i_2}}(f_{i_1,i_2}) - \rho_{U_{i_0,i_2} U_{i_0 i_1 i_2}}(f_{i_0,i_2}) + \rho_{U_{i_0,i_1} U_{i_0 i_1 i_2}}(f_{i_0,i_1}) = 0 \tag{2.5}$$

for any triplet of indices $i_0, i_1, i_2 \in I$. Also,

$$H_r(f) = [\![(g_{j_0,j_1})]\!] \quad \text{and} \quad H_{r'}(f) = [\![(g'_{j_0,j_1})]\!]$$

where

$$g_{j_0,j_1} = \rho_{U_{r(j_0),r(j_1)} V_{j_0,j_1}}(f_{r(j_0),r(j_1)}) \quad \text{and} \quad g'_{j_0,j_1} = \rho_{U_{r'(j_0),r'(j_1)} V_{j_0,j_1}}(f_{r'(j_0),r'(j_1)})$$

From this we get

$$g'_{j_0,j_1} - g_{j_0,j_1} = \rho_{U_{r'(j_0),r'(j_1)} V_{j_0,j_1}}(f_{r'(j_0),r'(j_1)}) - \rho_{U_{r(j_0),r(j_1)} V_{j_0,j_1}}(f_{r(j_0),r(j_1)}) \tag{2.6}$$

We have $h = (h_{j_0}) = \left(\rho_{U_{r(j_0) r'(j_0)} V_{j_0}}\left(f_{r(j_0),r'(j_0)}\right)\right)$. Let $\delta(h) = (h'_{j_0 j_1})$, then

$$\begin{aligned} h'_{j_0 j_1} &= \rho_{V_{j_1} V_{j_0 j_1}}(h_{j_1}) - \rho_{V_{j_0} V_{j_0 j_1}}(h_{j_0}) \\ &= \rho_{V_{j_1} V_{j_0 j_1}}\left(\rho_{U_{r(j_1) r'(j_1)} V_{j_1}}\left(f_{r(j_1),r'(j_1)}\right)\right) - \rho_{V_{j_0} V_{j_0 j_1}}\left(\rho_{U_{r(j_0) r'(j_0)} V_{j_0}}\left(f_{r(j_0),r'(j_0)}\right)\right) \\ &= \rho_{U_{r(j_1) r'(j_1)} V_{j_0 j_1}}\left(f_{r(j_1),r'(j_1)}\right) - \rho_{U_{r(j_0) r'(j_0)} V_{j_0 j_1}}\left(f_{r(j_0),r'(j_0)}\right) \end{aligned} \tag{2.7}$$

To simplify the notations, we rename indices as $r(j_0) = i_0, r(j_1) = i_1, r'(j_0) = i_2$ and $r'(j_1) = i_3$. Since $V_{j_0 j_1} \subset U_{i_0 i_1 i_2}$ and $V_{j_0,j_1} \subset U_{i_1,i_2,i_3}$ from (2.5) we get

$$\begin{aligned} \rho_{U_{i_1 i_2} V_{j_0 j_1}}\left(f_{i_1,i_2}\right) - \rho_{U_{i_0 i_2} V_{j_0 j_1}}\left(f_{i_1,i_2}\right) + \rho_{U_{i_0 i_1} V_{j_0 j_1}}\left(f_{i_0,i_1}\right) = 0 \\ \rho_{U_{i_2 i_3} V_{j_0 j_1}}\left(f_{i_2,i_3}\right) - \rho_{U_{i_1 i_3} V_{j_0 j_1}}\left(f_{i_1,i_3}\right) + \rho_{U_{i_1 i_2} V_{j_0 j_1}}\left(f_{i_1,i_2}\right) = 0 \end{aligned} \tag{2.8}$$

[12] For a more general argument see [35, §5.33, equation (11)] and [11, Lemma 2.6.1].

We will use (2.8) to convert (2.7) to (2.6). Hence we have

$$\begin{aligned}
h'_{j_0j_1} &= \rho_{U_{i_1i_3}V_{j_0j_1}}(f_{i_1,i_3}) - \rho_{U_{i_0i_2}V_{j_0j_1}}(f_{i_0,i_2}) \\
&= \Big(\rho_{U_{i_1i_2}V_{j_0j_1}}(f_{i_1,i_2}) - \rho_{U_{i_0i_2}V_{j_0j_1}}(f_{i_0,i_2}) + \rho_{U_{i_0i_1}V_{j_0j_1}}(f_{i_0,i_1})\Big) \\
&\quad - \Big(\rho_{U_{i_2i_3}V_{j_0j_1}}(f_{i_2,i_3}) - \rho_{U_{i_1i_3}V_{j_0j_1}}(f_{i_1,i_3}) + \rho_{U_{i_1i_2}V_{j_0j_1}}(f_{i_1,i_2})\Big) \\
&\quad + \rho_{U_{i_2,i_3}V_{j_0,j_1}}(f_{i_2,i_3}) - \rho_{U_{i_0,i_1}V_{j_0,j_1}}(f_{i_0,i_1}) \\
&= \rho_{U_{i_2,i_3}V_{j_0,j_1}}(f_{i_2,i_3}) - \rho_{U_{i_0,i_1}V_{j_0,j_1}}(f_{i_0,i_1}) \\
&= g'_{j_0,j_1} - g_{j_0,j_1}
\end{aligned}$$

Therefore these two cocycles differ by a coboundary. Hence completing the proof. □

Remark 2.15. We will therefore denote this refining map on the cohomology level by $H_{\mathcal{UV}}$ for $\mathcal{U} < \mathcal{V}$. Hence, $\{\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}), H_{\mathcal{UV}}\}$ is a *direct system*[13]. We have $H_{\mathcal{UU}} = \mathbb{1}_{\check{\mathrm{H}}^k(\mathcal{U},\mathcal{F})}$ since we can choose refining map r to be identity, and $H_{\mathcal{UW}} = H_{\mathcal{VW}} \circ H_{\mathcal{UV}}$ for $\mathcal{U} < \mathcal{V} < \mathcal{W}$ since composition of two refining maps is again a refining map.

Definition 2.15 (Čech cohomology)**.** Let $\mathcal{F}$ be a sheaf of abelian groups on X and $k \geq 0$ be an integer. Then the k^{th} *Čech cohomology group* of $\mathcal{F}$ on X is the group

$$\check{\mathrm{H}}^k(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F})$$

where the direct limit[14] is indexed over all the open covers of X with order relation induced by refinement, i.e. $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Proposition 2.6. *For any sheaf $\mathcal{F}$ of X, we have $\check{\mathrm{H}}^0(X, \mathcal{F}) \cong \mathcal{F}(X)$.*

Proof. By Lemma 2.4 we know that at the $\check{\mathrm{H}}^0$ level all the groups are isomorphic to $\mathcal{F}(X)$. Since all the maps $H_{\mathcal{UV}}$ are compatible isomorphisms, using Proposition B.1 we conclude that the direct limit is also isomorphic to $\mathcal{F}(X)$. □

Remark 2.16. What we have defined here is not the true definition of either Čech or sheaf cohomology [20, §IX.3] [9, pp. 38-40]. Čech cohomology can be defined either using the concept of *nerve* [25, §73][21, §3.4(a)], or presheaf[15] [1, §10]. One can prove equivalence of both these definitions using the constant presheaf $\underline{G}$ [35, §5.33]. Also note that Čech cohomology of the cover $\mathcal{U}$ is a purely combinatorial object [1, Theorem 8.9].

Sheaf cohomology can be defined either using resolution of sheaf [37, Definition 3.10] [27, Definition 4.2.11] or axiomatically [35, §5.18]. The definition of Čech cohomology agrees with that of sheaf cohomology for smooth manifolds since Čech cohomology is isomorphic to sheaf cohomology for any sheaf on a paracompact Hausdorff space [35, §5.33]. This is all we need to obtain the desired proof, hence our definition of Čech cohomology of sheaves serves the purpose.

Remark 2.17. Another way of defining Čech cohomology groups with coefficients in sheaves is via sheafification. First step is to define the cohomology groups $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F})$ on an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X with coefficients in a presheaf $\mathcal{F}$. Then the cohomology groups $\check{\mathrm{H}}^k(\mathcal{U}, \widetilde{\mathcal{F}})$ of $\mathcal{U}$ with coefficients in a sheaf $\widetilde{\mathcal{F}}$ are defined to be the cohomology groups of $\mathcal{U}$ with coefficients in the canonical presheaf $\mathcal{F}$ of $\widetilde{\mathcal{F}}$. Finally, the cohomology groups $\check{\mathrm{H}}^k(X, \mathcal{F})$ and $\check{\mathrm{H}}^k(X, \widetilde{\mathcal{F}})$ are defined as the direct limit of all groups $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F})$ and $\check{\mathrm{H}}^k(\mathcal{U}, \widetilde{\mathcal{F}})$, respectively, as $\mathcal{U}$ runs through all open coverings of X (directed by refinement) [11, §2.6].

[13] For its definition see Appendix B.

[14] For the definition of direct limit see Appendix B. To get the direct system $\{\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}), H_{\mathcal{UV}}\}$, the "refinement" is defined to be the order relation for the directed set.

[15] For a discussion on the motivation behind this definition see [13, §2] and [8, §10.2].

2.2.1 Induced map of cohomology

Definition 2.16 (Induced map of cochains)**.** If $\phi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves on X, then the *induced map on cochains* is defined as

$$\begin{aligned} \phi_* : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{G}) \\ (f_{i_0,i_1,\ldots,i_k}) &\mapsto (\phi_{U_{i_0,\ldots,i_k}}(f_{i_0,i_1,\ldots,i_k})) \end{aligned}$$

for any open covering $\mathcal{U}$ of X.

Proposition 2.7. *The coboundary operator commutes with the induced map of cochains. That is, the following diagram commutes*

$$\begin{array}{ccc} \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) & \xrightarrow{\delta} & \check{\mathrm{C}}^{k+1}(\mathcal{U}, \mathcal{F}) \\ \big\downarrow{\scriptstyle \phi_*} & & \big\downarrow{\scriptstyle \phi_*} \\ \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{G}) & \xrightarrow{\delta} & \check{\mathrm{C}}^{k+1}(\mathcal{U}, \mathcal{G}) \end{array}$$

Proof. The coboundary operator δ acts on each element via the group homomorphism ρ of the sheaf, and the induced map ϕ_* acts on each element via the group homomorphism $\phi_{U_{i_0,\ldots,i_k}}$ of the sheaf map. By Definition 2.4, we know that the group homomorphism of the sheaf and the group homomorphism of the sheaf map commute. □

Corollary 2.1. *The induced map of cochains sends cocycles to cocycles, and coboundaries to cobundaries.*

Proof. Let f be a cocycle, i.e. $\delta(f) = 0$. From the previous proposition we know that $\delta(\phi_*(f)) = \phi_*(\delta(f)) = 0$. Hence $\phi_*(f)$ is also a cocycle. Next, let g be a coboundary, i.e. $g = \delta(h)$. From the previous proposition we know that $\phi_*(g) = \phi_*(\delta(h)) = \delta(\phi_*(h))$. Hence $\phi_*(g)$ is also a coboundary. □

Proposition 2.8. *If* $0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ *is an exact sequence of sheaves over* X*, then the induced sequence of cochains for any open cover* $\mathcal{U}$ *of* X

$$0 \longrightarrow \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}'')$$

is also exact.

Proof. We can re-write the desired exact sequence of abelian groups as

$$0 \longrightarrow \prod_{(i_0,i_1,\ldots,i_k)} \mathcal{F}'(U_{i_0,i_1,\ldots,i_k}) \xrightarrow{\phi_*} \prod_{(i_0,i_1,\ldots,i_k)} \mathcal{F}(U_{i_0,i_1,\ldots,i_k}) \xrightarrow{\psi_*} \prod_{(i_0,i_1,\ldots,i_k)} \mathcal{F}''(U_{i_0,i_1,\ldots,i_k})$$

The exactness of the above sequence follows from Lemma 2.2, since

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\phi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U)$$

is an exact sequence of abelian groups for all open sets U of X. □

Definition 2.17 (Induced map of cohomology)**.** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves on X, then the *induced*[16] *map of cohomology* is defined as

$$\begin{aligned} \Phi : \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{G}) \\ [\![f]\!] &\mapsto [\![\phi_*(f)]\!] \end{aligned}$$

for $f \in \check{Z}^k(\mathcal{U}, \mathcal{F})$.

[16]It's well defined because of Corollary 2.1.

Lemma 2.7. *The refining maps at the level of cohomology commute with any induced map of cohomology. That is, the following diagram commutes*

$$\begin{array}{ccc} \check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}) & \xrightarrow{\Phi} & \check{\mathrm{H}}^k(\mathcal{U},\mathcal{G}) \\ \big\downarrow{\scriptstyle H_{\mathcal{U}\mathcal{V}}} & & \big\downarrow{\scriptstyle H_{\mathcal{U}\mathcal{V}}} \\ \check{\mathrm{H}}^k(\mathcal{V},\mathcal{F}) & \xrightarrow{\Phi} & \check{\mathrm{H}}^k(\mathcal{V},\mathcal{G}) \end{array}$$

Proof. The refining map $H_{\mathcal{U}\mathcal{V}}$ acts on each element via the group homomorphism ρ of the sheaf, and the induced map Φ acts on each element via the group homomorphism $\phi_{U_{i_0,\dots,i_k}}$ of the sheaf map. By Definition 2.4, we know that the group homomorphism of the sheaf and the group homomorphism of the sheaf map commute. □

Remark 2.18. This lemma implies that Φ is a map of direct systems $\{\check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}), H^{\mathcal{F}}_{\mathcal{U}\mathcal{V}}\}$ and $\{\check{\mathrm{H}}^k(\mathcal{U},\mathcal{G}), H^{\mathcal{G}}_{\mathcal{U}\mathcal{V}}\}$. Hence $\phi : \mathcal{F} \to \mathcal{G}$ in fact induces the homomorphism at the level of Čech cohomology of X

$$\underrightarrow{\Phi} : \check{\mathrm{H}}^k(X,\mathcal{F}) \to \check{\mathrm{H}}^k(X,\mathcal{G})$$

2.2.2 Long exact sequence of cohomology

In this subsection, proof of the fact that a short exact sequence of sheaves on paracompact Hausdorff space induces a long exact sequence of Čech cohomology will be presented following Serre [30, §I.3] and Warner [35, §5.33].

Theorem 2.1. *Let X be a paracompact Hausdorff space and*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves on X. Then there are connecting homomorphisms Δ : $\check{\mathrm{H}}^k(X,\mathcal{F}'') \to \check{\mathrm{H}}^{k+1}(X,\mathcal{F}')$ for every $k \geq 0$ such that we have a long exact sequence of Čech cohomology groups

$$\cdots \xrightarrow{\underrightarrow{\Phi}} \check{\mathrm{H}}^k(X,\mathcal{F}) \xrightarrow{\underrightarrow{\Psi}} \check{\mathrm{H}}^k(X,\mathcal{F}'') \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(X,\mathcal{F}') \xrightarrow{\underrightarrow{\Phi}} \check{\mathrm{H}}^{k+1}(X,\mathcal{F}) \xrightarrow{\underrightarrow{\Psi}} \cdots$$

Proof. Given to us is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

Then by Proposition 2.8, for any open cover $\mathcal{U}$ of X,

$$0 \longrightarrow \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}') \xrightarrow{\phi_*} \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}) \xrightarrow{\psi_*} \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'')$$

is an exact sequence. However, if we replace $\check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'')$ by $\operatorname{im}\psi_*$, we get a short exact sequence of abelian groups:

$$0 \longrightarrow \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}') \xrightarrow{\phi_*} \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}) \xrightarrow{\psi_*} \operatorname{im}\psi_* \longrightarrow 0$$

To explicitly show the dependence of $\operatorname{im}\psi_*$ on $\mathcal{U}$ and k, let's write $I^k(\mathcal{U},\mathcal{F}'') = \operatorname{im}\psi_*$. Hence we have the following short exact sequence of cochain complexes[17]

$$0 \longrightarrow \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}') \xrightarrow{\phi_*} \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}) \xrightarrow{\psi_*} I^k(\mathcal{U},\mathcal{F}'') \longrightarrow 0$$

[17] All these are chain complexes since $\delta \circ \delta = 0$.

Then by the *zig-zag lemma*[18] we get a long exact sequence in cohomology with respect to open cover $\mathcal{U}$

$$\cdots \xrightarrow{\Phi} \check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}) \xrightarrow{\Psi} \mathcal{I}^k(\mathcal{U},\mathcal{F}'') \xrightarrow{\partial} \check{\mathrm{H}}^{k+1}(\mathcal{U},\mathcal{F}') \xrightarrow{\Phi} \check{\mathrm{H}}^{k+1}(\mathcal{U},\mathcal{F}) \xrightarrow{\Psi} \cdots$$

where ∂ is the connecting homomorphism induced by the coboundary operator δ and

$$\mathcal{I}^k(\mathcal{U},\mathcal{F}'') = \frac{\ker\{\delta : I^k(\mathcal{U},\mathcal{F}'') \to I^{k+1}(\mathcal{U},\mathcal{F}'')\}}{\mathrm{im}\{\delta : I^{k-1}(\mathcal{U},\mathcal{F}'') \to I^k(\mathcal{U},\mathcal{F}'')\}}$$

Since direct limit is an exact functor[19], we get the following long exact sequence in Čech cohomology

$$\cdots \xrightarrow{\underrightarrow{\Phi}} \check{\mathrm{H}}^k(X,\mathcal{F}) \xrightarrow{\underrightarrow{\Psi}} \mathcal{I}^k(X,\mathcal{F}'') \xrightarrow{\partial} \check{\mathrm{H}}^{k+1}(X,\mathcal{F}') \xrightarrow{\underrightarrow{\Phi}} \check{\mathrm{H}}^{k+1}(X,\mathcal{F}) \xrightarrow{\underrightarrow{\Psi}} \cdots$$

where we have[20]

$$\mathcal{I}^k(X,\mathcal{F}'') = \varinjlim_{\mathcal{U}} \mathcal{I}^k(\mathcal{U},\mathcal{F}'')$$

Now to obtain the desired long exact sequence of Čech cohomology, it's sufficient to show that $\boxed{\mathcal{I}^k(X,\mathcal{F}'') \cong \check{\mathrm{H}}^k(X,\mathcal{F}'')}$. Then the map $\Delta : \check{\mathrm{H}}^k(X,\mathcal{F}'') \to \check{\mathrm{H}}^{k+1}(X,\mathcal{F}')$ can be defined as the composition of the inverse of this isomorphism with $\partial : \mathcal{I}^k(X,\mathcal{F}'') \to \check{\mathrm{H}}^{k+1}(X,\mathcal{F}')$.

We observe that the inclusion map $I^k(\mathcal{U},\mathcal{F}'') \hookrightarrow \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'')$ induces a group homomorphism at the level of cohomology with respect to the cover (quotient group), which on passing through the limit induces a map at the level of Čech cohomology. Consider the quotient group

$$Q^k(\mathcal{U},\mathcal{F}'') := \frac{\check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'')}{I^k(\mathcal{U},\mathcal{F}'')}$$

Then we have the following short exact sequence of cochain complexes

$$0 \longrightarrow I^k(\mathcal{U},\mathcal{F}'') \hookrightarrow \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'') \longrightarrow Q^k(\mathcal{U},\mathcal{F}'') \longrightarrow 0$$

Then by the zig-zag lemma we get a long exact sequence in cohomology with respect to open cover $\mathcal{U}$

$$\cdots \longrightarrow \check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}'') \longrightarrow \mathcal{Q}^k(\mathcal{U},\mathcal{F}'') \xrightarrow{\partial} \mathcal{I}^{k+1}(\mathcal{U},\mathcal{F}'') \longrightarrow \check{\mathrm{H}}^{k+1}(\mathcal{U},\mathcal{F}'') \longrightarrow \cdots$$

where ∂ is the connecting homomorphism induced by the coboundary operator δ and

$$\mathcal{Q}^k(\mathcal{U},\mathcal{F}'') = \frac{\ker\{\delta : Q^k(\mathcal{U},\mathcal{F}'') \to Q^{k+1}(\mathcal{U},\mathcal{F}'')\}}{\mathrm{im}\{\delta : Q^{k-1}(\mathcal{U},\mathcal{F}'') \to Q^k(\mathcal{U},\mathcal{F}'')\}}$$

Since direct limit is an exact functor, we get the following long exact sequence in Čech cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^k(X,\mathcal{F}'') \longrightarrow \mathcal{Q}^k(X,\mathcal{F}'') \xrightarrow{\partial} \mathcal{I}^{k+1}(X,\mathcal{F}'') \longrightarrow \check{\mathrm{H}}^{k+1}(X,\mathcal{F}'') \longrightarrow \cdots$$

[18]For proof see [25, Lemma 24.1] and [32, Theorem 25.6].

[19]For proof see Theorem B.2.

[20]One needs to repeat the calculations done in Lemma 2.6 to conclude that $\{\mathcal{I}^k(\mathcal{U},\mathcal{F}''), H_{\mathcal{U}\mathcal{V}}\}$ is a direct system. Here also the indexing set is directed by refinement, i.e. $\mathcal{U} < \mathcal{V}$ is $\mathcal{V}$ is a refinement of $\mathcal{U}$.

where we have

$$\mathcal{Q}^k(X,\mathcal{F}'') = \varinjlim_{\mathcal{U}} \mathcal{Q}^k(\mathcal{U},\mathcal{F}'')$$

Now to obtain the desired isomorphism, it's sufficient to show that $\boxed{\mathcal{Q}^k(X,\mathcal{F}'')=0}$. To prove this, we will use the fact that X is a paracompact Hausdorff space and ψ is surjective.

`Claim:` *Let $\mathcal{U} = \{U_i\}_{i\in A}$ be an open cover of X, and $f = (f_{i_0,\dots,i_k})$ be an element of $\check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'')$. Then there exists a refinement $\mathcal{V} = \{V_j\}_{j\in B}$ along with a refining map $r : B \to A$ such that $V_j \subset U_{r(j)}$ and $\widetilde{r}(f) \in I^k(\mathcal{V},\mathcal{F}'')$, where $\widetilde{r}$ is the map defined in Lemma 2.5. Therefore $\mathcal{Q}^k(X,\mathcal{F}'') = 0$.*

Since X is paracompact, without loss of generality, assume $\mathcal{U}$ to be locally finite. Also, by *shrinking lemma* (Theorem A.1) there exists a locally finite open covering $\mathcal{W} = \{W_i\}_{i\in A}$ of X such that $\overline{W_i} \subset U_i$ for each $i \in A$. For every $x \in X$, choose an open neighborhood V_x of x such that

1. If $x \in U_i$ then $V_x \subset U_i$ for all such i's. If $x \in W_i$ then $V_x \subset W_i$ for all such i's.
2. If $V_x \cap W_i \neq \emptyset$ then $V_x \subset U_i$ for all such i's.
3. If $x \in U_{i_0,i_1,\dots,i_k}$ then there exists a $h \in \mathcal{F}(V_x)$ such that
$$\psi_{V_x}(h) = \rho^{\mathcal{F}''}_{U_{i_0,\dots,i_k},V_x}(f_{i_0,\dots,i_k})$$
where by the first condition $V_x \subset U_{i_0,\dots,i_k}$.

The first condition can be satisfied because $\mathcal{U}$ and $\mathcal{W}$ are *point finite*[21]. Given the first condition, the second condition will be satisfied because $\overline{W_i} \subset U_i$. The third condition will be satisfied because $\mathcal{U}$ is point finite and ψ is a surjective map of sheaves, i.e. there are only finitely many $U_{i_0,\dots,i_k}$ which contain x and for every open set $U_{i_0,\dots,i_k}$ containing x and every $f_{i_0,\dots,i_k} \in \mathcal{F}''(U_{i_0,\dots,i_k})$, there is an open subset $V_x \subset U_{i_0,\dots,i_k}$ containing x such that $\psi_{V_x}(h) = \rho^{\mathcal{F}''}_{U_{i_0,\dots,i_k},V_x}(f_{i_0,\dots,i_k})$ for some $h \in \mathcal{F}(V_x)$ (Remark 2.6).

Choose a map $r : X \to A$ such that $x \in W_{r(x)}$. Then by the first condition, $V_x \subset W_{r(x)} \subset U_{r(x)}$ and $\mathcal{V} = \{V_x\}_{x\in X}$ is a refinement of $\mathcal{U}$. Now consider the map

$$\begin{aligned}\widetilde{r} : \check{\mathrm{C}}^k(\mathcal{U},\mathcal{F}'') &\to \check{\mathrm{C}}^k(\mathcal{V},\mathcal{F}'')\\ f = (f_{i_0,\dots,i_k}) &\mapsto g = (g_{x_0,\dots,x_k})\end{aligned}$$

where

$$g_{x_0,\dots,x_k} = \rho(f_{r(x_0),\dots,r(x_k)})$$

and $\rho : \mathcal{F}''(U_{r(x_0),\dots,r(x_k)}) \to \mathcal{F}''(V_{x_0,\dots,x_k})$ is the group homomorphism for the sheaf $\mathcal{F}''$ corresponding to the nested open subsets $V_{x_0,\dots,x_k} \subset U_{r(x_0),\dots,r(x_k)}$. It remains to show that $\widetilde{r}(f) \in I^k(\mathcal{V},\mathcal{F}'') = \psi_*(\check{\mathrm{C}}^k(\mathcal{V},\mathcal{F}''))$, i.e. there exists $h \in \mathcal{F}(V_{x_0,x_1,\dots,x_k})$ such that

$$\rho(f_{r(x_0),\dots,r(x_k)}) = \psi_{V_{x_0,x_1,\dots,x_k}}(h) \tag{2.9}$$

If $V_{x_0,\dots,x_k} = \emptyset$ then there is nothing to prove. If not, then we have $V_{x_0} \cap V_{x_\ell} \neq \emptyset$ for all $0 \le \ell \le k$. Since $V_{x_\ell} \subset W_{r(x_\ell)}$ we have $V_{x_0} \cap W_{r(x_\ell)} \neq \emptyset$ for all $0 \le \ell \le k$, then by the second condition we have $V_{x_0} \subset U_{r(x_\ell)}$ for all $0 \le \ell \le k$. Hence, $x_0 \in U_{r(x_0),\dots,r(x_k)}$ and we can use the third condition to conclude that there exists $h' \in \mathcal{F}(V_{x_0})$ such that

$$\psi_{V_{x_0}}(h') = \rho^{\mathcal{F}''}_{U_{r(x_0),\dots,r(x_k)},V_{x_0}}(f_{r(x_0),\dots,r(x_k)})$$

[21] An open cover $\mathcal{U} = \{U_i\}_{i\in A}$ of X is *point finite* if each point of X is contained in U_i for only finitely many $i \in A$. Every locally finite cover is point finite, but the converse is not true. For example, $\{1/n\}_{n\in\mathbb{N}}$ is a point finite cover of $\mathbb{R}$, but is not locally finite at 0.

Now let $h = \rho^{\mathcal{F}''}_{V_{x_0}, V_{x_0,x_1,\ldots,x_k}}(h')$ and use the fact that ψ commutes with ρ to get (2.9). Hence completing the proof. □

Remark 2.19. By Theorem 12 we know that manifolds are paracompact. Hence the above theorem can be applied to the sheaf of differential forms. In particular, by Example 2.7 and Example 2.8, we have the short exact sequence of sheaves on a smooth manifold M

$$0 \longrightarrow \mathcal{Z}^q \hookrightarrow \Omega^q \xrightarrow{\mathrm{d}} \mathcal{Z}^{q+1} \longrightarrow 0$$

This induces the following long exact sequence

$$\cdots \longrightarrow \check{\mathrm{H}}^k(M, \Omega^q) \longrightarrow \check{\mathrm{H}}^k(M, \mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M, \mathcal{Z}^q) \longrightarrow \check{\mathrm{H}}^{k+1}(M, \Omega^q) \longrightarrow \cdots$$

2.2.3 Fine sheaves

In this subsection, the condition under which $\check{\mathrm{H}}^k(X, \mathcal{F})$ vanishes for all $k \geq 1$ will be discussed following Hirzebruch [11, §2.11] and Warner [35, §5.10, 5.33].

Definition 2.18 (Sheaf partition of unity)**.** Let $\mathcal{F}$ be a sheaf of abelian groups over a paracompact Hausdorff space X. Given a locally finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, the *partition of unity of* $\mathcal{F}$ subordinate to the cover $\mathcal{U}$ is a family of sheaf maps $\{\eta_i : \mathcal{F} \to \mathcal{F}\}$ such that

1. $\mathrm{supp}(\eta_i) \subset U_i$ for each U_i
2. $\sum_{i\in I} \eta_i = \mathbb{1}_{\mathcal{F}}$ (the sum can be formed because $\mathcal{U}$ is locally finite)

where $\mathrm{supp}(\eta_i)$ is the closure of the set of those $x \in X$ for which $(\eta_i)_x : \mathcal{F}_x \to \mathcal{F}_x$ is not a zero map.

Definition 2.19 (Fine sheaf)**.** A sheaf of abelian groups $\mathcal{F}$ over a paracompact Hausdorff space X is *fine* if for any locally finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X there exists a partition of unity of $\mathcal{F}$ subordinate to the covering $\mathcal{U}$.

Example 2.9. Since the multiplication by a continuous or differentiable globally defined function defines a sheaf map in a natural way. From Theorem A.2 we conclude that the sheaf of continuous functions on a paracompact Hausdorff space is a fine sheaf. Also, by Theorem 13, the sheaf Ω^q of smooth q-forms on a smooth manifold M is a fine sheaf [37, Example II.3.4].

Theorem 2.2. *Let $\mathcal{F}$ be a fine sheaf over a paracompact Hausdorff space X. Then $\check{\mathrm{H}}^k(X, \mathcal{F})$ vanishes for $k \geq 1$.*

Proof. Since X is paracompact, every open cover of X has a locally finite refinement, it suffices to prove that $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) = 0$ for all $k \geq 1$ if $\mathcal{U} = \{U_i\}_{i\in I}$ is any locally finite open cover of X. For $k \geq 1$, we define the homomorphism

$$\begin{aligned} \lambda_k : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{C}}^{k-1}(\mathcal{U}, \mathcal{F}) \\ (f_{i_0,i_1,\ldots,i_k}) &\mapsto (h_{i_0,i_1,\ldots,i_{k-1}}) \end{aligned}$$

where

$$h_{i_0,i_1,\ldots,i_{k-1}} = \sum_{i\in I} \eta_i \left(f_{i,i_0,\ldots,i_{k-1}}\right)$$

and $\{\eta_i : \mathcal{F} \to \mathcal{F}\}_{i\in I}$ is a partition of unity of $\mathcal{F}$ subordinate to the covering $\mathcal{U}$. Also, let $\delta_k : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{C}}^{k+1}(\mathcal{U}, \mathcal{F})$ be the coboundary operator as in Definition 2.10. Then from Proposition 2.4 it follows that for $f = (f_{i_0,\ldots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$ we have

$$\delta_{k-1}\left(\lambda_k(f)\right) = f \quad \text{for } k \geq 1$$

Therefore, $f \in \check{B}^k(\mathcal{U}, \mathcal{F})$ and $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) = 0$ for all $k \geq 1$.

We will check the claim just for the simplest case, when $k = 1$. For $f = (f_{i_0 i_1}) \in \check{Z}^1(\mathcal{U}, \mathcal{F})$ and $\delta(f) = (g_{i_0 i_1 i_2}) = 0$ we have [9, pp. 42]

$$\begin{aligned}
\delta_0\left(\lambda_1\left((f_{i_0 i_1})\right)\right) &= \delta_0\left(\left(\sum_{i \in I} \eta_i(f_{i i_0})\right)\right) \\
&= \left(\rho_{U_{i_1} U_{i_0 i_1}}\left(\sum_{i \in I} \eta_i\left(f_{i i_1}\right)\right) - \rho_{U_{i_0} U_{i_0 i_1}}\left(\sum_{i \in I} \eta_i\left(f_{i i_0}\right)\right)\right) \\
&= \left(\sum_{i \in I} \eta_i\left(\rho_{U_{i i_1} U_{i i_1 i_0}}\left(f_{i i_1}\right)\right) - \sum_{i \in I} \eta_i\left(\rho_{U_{i i_0} U_{i i_1 i_0}}(f_{i i_0})\right)\right) \\
&= \left(\sum_{i \in I} \eta_i\left(\rho_{U_{i i_1} U_{i i_1 i_0}}\left(f_{i i_1}\right) - \rho_{U_{i i_0} U_{i i_1 i_0}}(f_{i i_0})\right)\right) \\
&= \left(\sum_{i \in I} \eta_i\left(\rho_{U_{i_1 i_0} U_{i i_1 i_0}}(f_{i_0 i_1})\right)\right) \\
&= \left(\rho_{U_{i_1 i_0} U_{i_1 i_0}}\left(\sum_{i \in I} \eta_i(f_{i_0 i_1})\right)\right) \\
&= (f_{i_0 i_1})
\end{aligned}$$

since sheaf map η_i commutes with ρ, ρ_{UU} is identity, $\{\eta_i\}$ is partition of unity and by Proposition 2.4 we have

$$\begin{aligned}
0 = g_{i i_1 i_0} &= \rho_{U_{i_1 i_0} U_{i i_1 i_0}}(f_{i_1 i_0}) - \rho_{U_{i i_0} U_{i i_1 i_0}}(f_{i i_0}) + \rho_{U_{i i_1} U_{i i_1 i_0}}(f_{i i_1}) \\
\rho_{U_{i_1 i_0} U_{i i_1 i_0}}(f_{i_0 i_1}) &= -\rho_{U_{i i_0} U_{i i_1 i_0}}(f_{i i_0}) + \rho_{U_{i i_1} U_{i i_1 i_0}}(f_{i i_1})
\end{aligned}$$

□

Remark 2.20. We can apply this theorem to the the sheaf of smooth q-forms on a smooth manifold M, hence $\check{\mathrm{H}}^k(M, \Omega^q) = 0$ for all $k \geq 1$.

2.3 de Rham-Čech isomorphism

Theorem 2.3. *Let M be a smooth manifold. Then for each $k \geq 0$ there exists a group isomorphism*

$$H^k_{dR}(M) \cong \check{\mathrm{H}}^k(M, \underline{\mathbb{R}})$$

Proof. For $k = 0$, from Proposition 1.2 and Proposition 2.6, we know that both $H^0_{dR}(M)$ and $\check{\mathrm{H}}^0(M, \underline{\mathbb{R}})$ are isomorphic to the group of locally constant real valued functions on M. That is

$$\boxed{H^0_{dR}(M) \cong \check{\mathrm{H}}^0(M, \underline{\mathbb{R}})}$$

Now let's restrict our attention to $k \geq 1$. From Example 2.8 we know that the Poincaré lemma implies the existence of the following long exact sequence of sheaves of differential forms

$$0 \longrightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \xrightarrow{\mathrm{d}} \cdots$$

Then, as noted in Remark 2.19, we get a family of short exact sequence of sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathbb{R}} & \hookrightarrow & \Omega^0 & \xrightarrow{\mathrm{d}} & \mathcal{Z}^1 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{Z}^1 & \hookrightarrow & \Omega^1 & \xrightarrow{\mathrm{d}} & \mathcal{Z}^2 & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & \mathcal{Z}^q & \hookrightarrow & \Omega^q & \xrightarrow{\mathrm{d}} & \mathcal{Z}^{q+1} & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Since a smooth manifold is a paracompact Hausdorff space, we can apply Theorem 2.1 to get a family of long exact sequence of Čech cohomology

$$\begin{array}{l}
\cdots \longrightarrow \check{\mathrm{H}}^k(M,\Omega^0) \longrightarrow \check{\mathrm{H}}^k(M,\mathcal{Z}^1) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M,\underline{\mathbb{R}}) \longrightarrow \check{\mathrm{H}}^{k+1}(M,\Omega^0) \longrightarrow \cdots \\
\cdots \longrightarrow \check{\mathrm{H}}^k(M,\Omega^1) \longrightarrow \check{\mathrm{H}}^k(M,\mathcal{Z}^2) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M,\mathcal{Z}^1) \longrightarrow \check{\mathrm{H}}^{k+1}(M,\Omega^1) \longrightarrow \cdots \\
\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\cdots \longrightarrow \check{\mathrm{H}}^k(M,\Omega^q) \longrightarrow \check{\mathrm{H}}^k(M,\mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M,\mathcal{Z}^q) \longrightarrow \check{\mathrm{H}}^{k+1}(M,\Omega^q) \longrightarrow \cdots \\
\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots
\end{array}$$

Now let's study one of these long exact sequence of Čech cohomology. By Proposition 2.6 we have $\check{\mathrm{H}}^0(M,\Omega^q) \cong \Omega^q(M)$ and $\check{\mathrm{H}}^0(M,\mathcal{Z}^q) \cong \mathcal{Z}^q(M)$. Also by Remark 2.20 we have $\check{\mathrm{H}}^k(M,\Omega^q) = 0$ for all $k \geq 1$ and $q \geq 0$. Hence for any $q \geq 0$ we get the exact sequence

$$\begin{array}{l}
0 \longrightarrow \mathcal{Z}^q(M) \hookrightarrow \Omega^q(M) \xrightarrow{\mathrm{d}} \mathcal{Z}^{q+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M,\mathcal{Z}^q) \longrightarrow 0 \longrightarrow \check{\mathrm{H}}^1(M,\mathcal{Z}^{q+1}) \\
\hfill \big\downarrow \Delta \\
\cdots \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^3(M,\mathcal{Z}^q) \xleftarrow{\Delta} \check{\mathrm{H}}^2(M,\mathcal{Z}^{q+1}) \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^2(M,\mathcal{Z}^q)
\end{array}$$

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^q(M) \hookrightarrow \Omega^q(M) \xrightarrow{\mathrm{d}} \mathcal{Z}^{q+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M,\mathcal{Z}^q) \longrightarrow 0$$

Since this sequence is exact, the map $\Delta : \mathcal{Z}^{q+1}(M) \to \check{\mathrm{H}}^1(M,\mathcal{Z}^q)$ is a surjective group homomorphism and $\mathrm{im}\{\mathrm{d} : \Omega^q(M) \to \mathcal{Z}^{q+1}(M)\} = \ker(\Delta)$. Hence by the *first isomorphism theorem* we get

$$\check{\mathrm{H}}^1(M,\mathcal{Z}^q) \cong \frac{\mathcal{Z}^{q+1}(M)}{\ker(\Delta)} \quad \text{for all } q \geq 0$$

Since $\mathrm{im}\{\mathrm{d} : \Omega^q(M) \to \mathcal{Z}^{q+1}(M)\} = \mathrm{im}\{\mathrm{d} : \Omega^q(M) \to \Omega^{q+1}(M)\} = \mathcal{B}^{q+1}(M)$, we get

$$\check{\mathrm{H}}^1(M,\mathcal{Z}^q) \cong H^{q+1}_{dR}(M) \quad \text{for all } q \geq 0 \tag{2.10}$$

Note that $\mathcal{Z}^0 = \underline{\mathbb{R}}$, hence from (2.10) we get

$$\boxed{\check{\mathrm{H}}^1(M,\underline{\mathbb{R}}) \cong H^1_{dR}(M)}$$

Next we consider the remaining parts of the long exact sequence, i.e. for $k \geq 1$ and $q \geq 0$ we have

$$0 \longrightarrow \check{\mathrm{H}}^k(M,\mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M,\mathcal{Z}^q) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{\mathrm{H}}^{k+1}(M, \mathcal{Z}^q) \cong \check{\mathrm{H}}^k(M, \mathcal{Z}^{q+1}) \quad \text{for all } k \geq 1, q \geq 0 \tag{2.11}$$

Again substituting $\mathcal{Z}^0 = \underline{\mathbb{R}}$ and restricting our attention to $k \geq 2$, we apply (2.11) recursively to get

$$\begin{aligned}\check{\mathrm{H}}^k(M, \underline{\mathbb{R}}) &\cong \check{\mathrm{H}}^{k-1}\left(M, \mathcal{Z}^1\right) \\ &\cong \check{\mathrm{H}}^{k-2}\left(M, \mathcal{Z}^2\right) \\ &\vdots \\ &\cong \check{\mathrm{H}}^1\left(M, \mathcal{Z}^{k-1}\right)\end{aligned}$$

Then using (2.10) we get

$$\boxed{\check{\mathrm{H}}^k(M, \underline{\mathbb{R}}) \cong H^k_{dR}(M)} \quad \text{for all } k \geq 2$$

Hence completing the proof. □

Remark 2.21. One can use Weil's method involving generalized Mayer-Vietoris principle for the Čech-de Rham complex to directly show the isomorphism between Čech cohomology with values in $\mathbb{R}$ and de Rham cohomology of smooth manifold M, without using sheaf theory. There are two versions of the proof depending on the definition of Čech cohomology used, see [21, Theorem 3.19] if defined using nerve and [1, Proposition 10.6] if defined using presheaf.

Remark 2.22. In Theorem 4.7 we will see that de Rham-Čech isomorphism in fact implies that de Rham cohomology is a topological invariant.

Chapter 3

Dolbeault cohomology

3.1 Differential forms on $\mathbb{C}^n$

This section generalizes the concepts discussed in section 1.1 and section 1.3, following the discussion from [12, §1.3] and [37, §I.3].

3.1.1 Tangent space

Definition 3.1 (Real tangent space)**.** Let $U \subset \mathbb{C}^n$ be an open subset. In particular, we can consider $U \subset \mathbb{R}^{2n}$, to be a smooth manifold of dimension $2n$. Then for $w \in U$ we define the *real tangent space* of U at the point w as the real vector space of $\mathbb{R}$-linear derivations on the ring of real-valued smooth functions in a neighborhood of w, i.e.

$$T_{w,\mathbb{R}}U = \{X_w : C_w^\infty(U) \to \mathbb{R} \mid X_w(fg) = X_w(f)g(w) + f(w)X_w(g)\}$$

Remark 3.1. If we write the standard coordinates on $\mathbb{C}^n$ as $z_j = x_j + iy_j$, then a canonical basis of $T_{w,\mathbb{R}}U$ is given by the tangent vectors

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_w, \dots, \left.\frac{\partial}{\partial x_n}\right|_w, \left.\frac{\partial}{\partial y_1}\right|_w, \dots, \left.\frac{\partial}{\partial y_n}\right|_w \right\}$$

Clearly, $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}U) = 2n$ as seen in the case of smooth manifolds.

Definition 3.2 (Complexified tangent space)**.** Let $U \subset \mathbb{C}^n$ be an open subset. Then we define the *complexified tangent space* of U at the point w to be the complexification[1] of real tangent space of U at w

$$T_{w,\mathbb{C}}U = T_{w,\mathbb{R}}U \otimes_{\mathbb{R}} \mathbb{C}$$

Remark 3.2. We can also use the canonical basis of real tangent space to define its complexification [28, p. 379]. We can view $T_{w,\mathbb{C}}U$ as the complex vector space of $\mathbb{C}$-linear derivations in the ring of complex-valued smooth functions[2] in a neighborhood of w, i.e. $T_{w,\mathbb{C}}U$ also has the same basis

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_w, \dots, \left.\frac{\partial}{\partial x_n}\right|_w, \left.\frac{\partial}{\partial y_1}\right|_w, \dots, \left.\frac{\partial}{\partial y_n}\right|_w \right\}$$

Hence, as expected, we have $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}U) = \dim_{\mathbb{C}}(T_{w,\mathbb{C}}U)$.

[1] For the definition, see Definition C.2.

[2] That is, they posses partial derivatives of all orders with respect to the $2n$ real coordinates in $\mathbb{C}^n$.

Definition 3.3 (Complex structure for $T_{w,\mathbb{R}}U$)**.** Each real tangent space $T_{w,\mathbb{R}}U$ admits a natural complex structure[3] defined on the basis as

$$\begin{aligned} J : T_{w,\mathbb{R}}U &\to T_{w,\mathbb{R}}U \\ \left.\frac{\partial}{\partial x_j}\right|_w &\mapsto \left.\frac{\partial}{\partial y_j}\right|_w \\ \left.\frac{\partial}{\partial y_j}\right|_w &\mapsto -\left.\frac{\partial}{\partial x_j}\right|_w \end{aligned}$$

Remark 3.3. We will regard this J as a vector bundle endomorphism of the smooth vector bundle $T_{\mathbb{R}}U$ over U.

Proposition 3.1. *The complexified tangent bundle $T_{\mathbb{C}}U = T_{\mathbb{R}}U \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles*

$$T_{\mathbb{C}}U = (T_{\mathbb{R}}U)^{1,0} \oplus (T_{\mathbb{R}}U)^{0,1}$$

such that the $\mathbb{C}$-linear extension $\tilde{J} = J \otimes \mathbb{1}_{\mathbb{C}}$ satisfies

$$\tilde{J}|_{(T_{\mathbb{R}}U)^{1,0}} = i \cdot \mathbb{1}_{T_{\mathbb{C}}U} \quad \text{and} \quad \tilde{J}|_{(T_{\mathbb{R}}U)^{0,1}} = -i \cdot \mathbb{1}_{T_{\mathbb{C}}U}$$

Proof. Fix a point $w \in U$, and substitute $V = T_{w,\mathbb{R}}U$ and $V_{\mathbb{C}} = T_{w,\mathbb{C}}U$ in the proof of Proposition C.7. □

Remark 3.4. As seen in the proof of Proposition C.7, we can write

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - iJ\left(\frac{\partial}{\partial x_j}\right)\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_j} + iJ\left(\frac{\partial}{\partial x_j}\right)\right) \\ \frac{\partial}{\partial y_j} &= \frac{1}{2}\left(\frac{\partial}{\partial y_j} - iJ\left(\frac{\partial}{\partial y_j}\right)\right) + \frac{1}{2}\left(\frac{\partial}{\partial y_j} + iJ\left(\frac{\partial}{\partial y_j}\right)\right) \end{aligned}$$

where

$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} - iJ\left(\frac{\partial}{\partial x_j}\right)\right), \frac{1}{2}\left(\frac{\partial}{\partial y_j} - iJ\left(\frac{\partial}{\partial y_j}\right)\right) \in (T_{\mathbb{R}}U)^{1,0}$$

and

$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} + iJ\left(\frac{\partial}{\partial x_j}\right)\right), \frac{1}{2}\left(\frac{\partial}{\partial y_j} + iJ\left(\frac{\partial}{\partial y_j}\right)\right) \in (T_{\mathbb{R}}U)^{0,1}$$

Next, use the definition of J to get:

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \\ \frac{\partial}{\partial y_j} &= \frac{i}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) - \frac{i}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \end{aligned}$$

Definition 3.4 (Complex partial derivative)**.** Based on the discussion above, we define the operators:

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for $j = 1, \ldots, n$.

[3] For definition, see Definition C.3.

Remark 3.5. Hence we can say that $\left\{ \frac{\partial}{\partial z_1}\big|_w, \dots, \frac{\partial}{\partial z_n}\big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}U)^{1,0}$ and $\left\{ \frac{\partial}{\partial \overline{z}_1}\big|_w, \dots, \frac{\partial}{\partial \overline{z}_n}\big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}U)^{0,1}$. Therefore, the following forms a basis of $T_{w,\mathbb{C}}U$

$$\left\{ \frac{\partial}{\partial z_1}\bigg|_w, \dots, \frac{\partial}{\partial z_n}\bigg|_w, \frac{\partial}{\partial \overline{z}_1}\bigg|_w, \dots, \frac{\partial}{\partial \overline{z}_n}\bigg|_w \right\}$$

Proposition 3.2. *Let $f : U \to V$ be a holomorphic map between open subsets $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^n$. The $\mathbb{C}$-linear extension of the pushforward map*[4] *$f_* : T_{w,\mathbb{R}}U \to T_{f(w),\mathbb{R}}V$ respects the above decomposition, i.e. $\tilde{f}_*\left((T_{w,\mathbb{R}}U)^{1,0}\right) \subset (T_{w,\mathbb{R}}V)^{1,0}$ and $\tilde{f}_*\left((T_{w,\mathbb{R}}U)^{0,1}\right) \subset (T_{f(w),\mathbb{R}}V)^{0,1}$.*

Proof. Follows directly from the Remark D.5. □

3.1.2 Cotangent space

Definition 3.5 (Real cotangent space)**.** Let $U \subset \mathbb{C}^n$ be an open subset. In particular, we can consider $U \subset \mathbb{R}^{2n}$, to be a smooth manifold of dimension $2n$. Then for $w \in U$ we define the *real cotangent space* of U at the point w as dual space of the real vector space $T_{w,\mathbb{R}}U$, i.e.

$$T^*_{w,\mathbb{R}}U = \mathrm{Hom}_{\mathbb{R}}(T_{w,\mathbb{R}}U, \mathbb{R})$$

Remark 3.6. If we write the standard coordinates on $\mathbb{C}^n$ as $z_j = x_j + iy_j$, then a canonical basis of $T^*_{w,\mathbb{R}}U$ is given by the cotangent vectors

$$\left\{ \mathrm{d}x_1\big|_w, \cdots, \mathrm{d}x_n\big|_w, \mathrm{d}y_1\big|_w, \cdots, \mathrm{d}y_n\big|_w \right\}$$

Clearly, $\dim_{\mathbb{R}}(T^*_{w,\mathbb{R}}U) = 2n$ as seen in the case of smooth manifolds.

Definition 3.6 (Complexified cotangent space)**.** Let $U \subset \mathbb{C}^n$ be an open subset. Then we defined the *complexified cotangent space* of U at the point w to be the complexification of real cotangent space

$$T^*_{w,\mathbb{C}}U = T^*_{w,\mathbb{R}}U \otimes_{\mathbb{R}} \mathbb{C}$$

Remark 3.7. We can also use the canonical basis of real cotangent space to define its complexification [28, p. 379]. We can view $T^*_{w,\mathbb{C}}U$ as the complex vector space with the basis

$$\left\{ \mathrm{d}x_1\big|_w, \cdots, \mathrm{d}x_n\big|_w, \mathrm{d}y_1\big|_w, \cdots, \mathrm{d}y_n\big|_w \right\}$$

Hence, as expected, we have $\dim_{\mathbb{R}}(T^*_{w,\mathbb{R}}U) = \dim_{\mathbb{C}}(T^*_{w,\mathbb{C}}U)$.

Remark 3.8. As in Proposition C.8, we get the complex structure $\mathcal{J}$ on $T^*_{w,\mathbb{R}}U$ from the complex structure J on $T_{w,\mathbb{R}}U$. We will regard this $\mathcal{J}$ as a vector bundle endomorphism of the smooth vector bundle $T^*_{\mathbb{R}}U$ over U.

Proposition 3.3. *The complexified cotangent bundle $T^*_{\mathbb{C}}U = T^*_{\mathbb{R}}U \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles*

$$T^*_{\mathbb{C}}U = (T^*_{\mathbb{R}}U)^{1,0} \oplus (T^*_{\mathbb{R}}U)^{0,1}$$

such that the $\mathbb{C}$-linear extension $\tilde{\mathcal{J}} = \mathcal{J} \otimes \mathbb{1}_{\mathbb{C}}$ satisfies

$$\tilde{\mathcal{J}}|_{(T^*_{\mathbb{R}}U)^{1,0}} = i \cdot \mathbb{1}_{T^*_{\mathbb{C}}U} \quad \textit{and} \quad \tilde{\mathcal{J}}|_{(T^*_{\mathbb{R}}U)^{0,1}} = -i \cdot \mathbb{1}_{T^*_{\mathbb{C}}U}$$

Proof. Fix a point $w \in U$, and substitute $V = T_{w,\mathbb{R}}U$ and $V_{\mathbb{C}} = T_{w,\mathbb{C}}U$ in the proof of Proposition C.8. □

[4] It was defined in the first chapter, see Definition 1.5.

Remark 3.9. From Corollary C.2, we have $T^*_{w,\mathbb{C}}U = \left(T^*_{w,\mathbb{R}}U\right)_{\mathbb{C}} \cong (T_{w,\mathbb{C}}U)^*$. Hence we can obtain another basis for $T^*_{w,\mathbb{C}}U$ by defining the dual basis of $(T_{w,\mathbb{R}}U)^{1,0}$ and $(T_{w,\mathbb{R}}U)^{0,1}$. Observe that:

$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(\mathrm{d}x_k + i\,\mathrm{d}y_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$
$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(\mathrm{d}x_k - i\,\mathrm{d}y_k) = 0$$

and

$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)(\mathrm{d}x_k + i\,\mathrm{d}y_k) = 0$$
$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)(\mathrm{d}x_k - i\,\mathrm{d}y_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Definition 3.7 (Complex differential)**.** Based on the discussion above, we define the differentials:

$$\mathrm{d}z_j := \mathrm{d}x_j + i\,\mathrm{d}y_j \quad \text{and} \quad \mathrm{d}\overline{z}_j := \mathrm{d}x_j - i\,\mathrm{d}y_j$$

for $j = 1, \ldots, n$.

Remark 3.10. Hence we can say that $\left\{\mathrm{d}z_1\big|_w, \ldots, \mathrm{d}z_n\big|_w\right\}$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}U)^{1,0}$ and $\left\{\mathrm{d}\overline{z}_1\big|_w, \ldots, \mathrm{d}\overline{z}_n\big|_w\right\}$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}U)^{0,1}$. Therefore, the following forms a basis of $T^*_{w,\mathbb{C}}U$

$$\left\{\mathrm{d}z_1\big|_w, \ldots, \mathrm{d}z_n\big|_w, \mathrm{d}\overline{z}_1\big|_w, \ldots, \mathrm{d}\overline{z}_n\big|_w\right\}$$

3.1.3 Differential forms

Definition 3.8 (Differential (p,q)-form)**.** Let $U \subset \mathbb{C}^n$ be an open subset. Over U one has the complex vector bundle[5] of rank $\binom{n}{p}\binom{n}{q}$ defined as

$$\bigwedge^{p,q} T^*_{\mathbb{R}}U := \bigwedge^{p}\left((T^*_{\mathbb{R}}U)^{1,0}\right) \otimes_{\mathbb{C}} \bigwedge^{q}\left((T^*_{\mathbb{R}}U)^{0,1}\right)$$

whose fiber is $\bigwedge^{p,q} T^*_{w,\mathbb{R}}U$. The smooth sections of this vector bundle are called the *differential forms of type* (p,q) on U. The space of all smooth differential forms of type (p,q) on U is denoted by $\Omega^{p,q}(U)$.

Remark 3.11. Any (p,q)-form $\omega \in \Omega^{p,q}(U)$ can be written uniquely as

$$\omega = \sum_{|\alpha|=p, |\beta|=q} f_{\alpha\beta}\,\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_q)$ are multi-indices with $1 \leq \alpha_j, \beta_k \leq n$; $\mathrm{d}z_\alpha = \mathrm{d}z_{\alpha_1} \wedge \ldots \wedge \mathrm{d}z_{\alpha_p}$ and $\mathrm{d}\overline{z}_\beta = \mathrm{d}\overline{z}_{\beta_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{\beta_q}$; and $f_{\alpha\beta}$ is a complex-valued smooth function on U, i.e. $f_{\alpha\beta} \in C^\infty(U)$. In particular, $\Omega^{0,0}(U) = C^\infty(U)$.

Remark 3.12. Let $\Omega^k_{\mathbb{C}}(U)$ be the space of sections of vector bundle $\bigwedge^k T^*_{\mathbb{C}}U$. Any element $\omega \in \Omega^1_{\mathbb{C}}(U)$ can thus be written in a unique manner in the form

$$\omega = \sum_{j=1}^{n} f_j\,\mathrm{d}z_j + \sum_{k=1}^{n} f_k\,\mathrm{d}\overline{z}_k$$

Moreover, if $\omega \in \Omega^r_{\mathbb{C}}(U)$ and $\eta \in \Omega^s_{\mathbb{C}}(U)$ then $\omega \wedge \eta = (-1)^{rs}\eta \wedge \omega \in \Omega^{r+s}_{\mathbb{C}}(U)$.

[5]That is, in the definition of smooth vector bundle, replace $\mathbb{R}$ by $\mathbb{C}$. This will be discussed in detail later, see Remark 3.27.

Remark 3.13. By Remark C.12 we have

$$\bigwedge^k T^*_{\mathbb{C}}U \cong \bigoplus_{p+q=k} \bigwedge^{p,q} T^*_{\mathbb{R}}U \quad \Longrightarrow \Omega^k_{\mathbb{C}}(U) \cong \bigoplus_{p+q=k} \Omega^{p,q}(U)$$

Thus we have natural projection operators $\bigwedge^k T^*_{\mathbb{C}}U \to \bigwedge^{p,q} T^*_{\mathbb{R}}U$ and $\Omega^k_{\mathbb{C}}(U) \to \Omega^{p,q}(U)$, denoted by $\Pi^{p,q}$ for $p+q=k$.

3.1.4 Exterior derivative

Definition 3.9 (Differential of a (p,q)-form)**.** Let $U \subset \mathbb{C}^n$ be an open subset, and $\mathrm{d} : \Omega^k_{\mathbb{C}}(U) \to \Omega^{k+1}_{\mathbb{C}}(U)$ be the complex linear extension of the usual exterior differential[6]. Then

$$\partial : \Omega^{p,q}(U) \to \Omega^{p+1,q}(U) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(U) \to \Omega^{p,q+1}(U)$$

are defined as $\partial := \Pi^{p+1,q} \circ \mathrm{d}$ and $\overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}$.

Remark 3.14. For any $f \in \Omega^0_{\mathbb{C}}(U) = C^\infty(U)$ one has

$$\mathrm{d}f = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j}\,\mathrm{d}y_j = \sum_{j=1}^n \frac{\partial f}{\partial z_j}\,\mathrm{d}z_j + \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}\,\mathrm{d}\overline{z}_j = \partial f + \overline{\partial} f$$

Since $\{\mathrm{d}\overline{z}_j\}$ are linearly independent, by Theorem D.2, f is holomorphic if and only if $\overline{\partial} f = 0$.

Lemma 3.1. *For the differential operators ∂ and $\overline{\partial}$ one has:*

1. $\mathrm{d} = \partial + \overline{\partial}$
2. $\partial^2 = \overline{\partial}^{\,2} = 0$ *and* $\partial\overline{\partial} = -\overline{\partial}\partial$
3. *They satisfy the Leibniz's rule, i.e.*

$$\begin{aligned}\partial(\omega \wedge \eta) &= \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta \\ \overline{\partial}(\omega \wedge \eta) &= \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta\end{aligned}$$

for $\omega \in \Omega^{p,q}(U)$ *and* $\eta \in \Omega^{r,s}(U)$.

Proof. We will use the properties of d studied earlier in Theorem 26.

1. This follows from the local description of ∂ and $\overline{\partial}$. Given $\omega = \sum_{\alpha,\beta} f_{\alpha\beta}\,\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^{p,q}(U)$, we have

$$\begin{aligned}\partial\omega &= \sum_{j=1}^n \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial z_j}\,\mathrm{d}z_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \\ \overline{\partial}\omega &= \sum_{j=1}^n \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \overline{z}_j}\,\mathrm{d}\overline{z}_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta\end{aligned}$$

2. Recall that $\mathrm{d}^2 = 0$ since the second order partial derivatives commute. Since $\mathrm{d} = \partial + \overline{\partial}$, we have

$$\begin{aligned}\mathrm{d}^2 &= \mathrm{d} \circ \mathrm{d} \\ &= \mathrm{d} \circ \partial + \mathrm{d} \circ \overline{\partial}\end{aligned}$$

[6] This was defined in the first chapter, see Definition 1.17.

$$= \partial \circ \partial + \overline{\partial} \circ \partial + \partial \circ \overline{\partial} + \overline{\partial} \circ \overline{\partial}$$
$$= \partial^2 + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\partial}^2$$

Moreover, each operator projects to a different summand of $\Omega_{\mathbb{C}}^{p+q+2}(U)$, we obtain

$$\partial^2 = \overline{\partial}\partial + \partial\overline{\partial} = \overline{\partial}^2 = 0$$

Therefore, $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$.

3. Recall that for $\omega \in \Omega_{\mathbb{C}}^{p+q}(U)$ and $\eta \in \Omega_{\mathbb{C}}^{r+s}(U)$ we have

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \mathrm{d}\eta \in \Omega_{\mathbb{C}}^{p+q+r+s+1}(U)$$

Since $\partial := \Pi^{p+r+1,q+s} \circ \mathrm{d}$, taking the $(p+r+1, q+s)$-parts on both sides one obtains

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$

Similarly, taking the $(p+r, q+s+1)$-parts one obtains

$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$$

□

Remark 3.15. As noted in Remark C.13, $\Omega_{\mathbb{C}}^{k}(U)$ does not reflect the complex structure J, whereas its decomposition into subspaces $\Omega^{p,q}(U)$ does.

3.2 $\overline{\partial}$-closed and exact forms on $\mathbb{C}^n$

In this section the proof of $\overline{\partial}$-Poincaré lemma will be discussed, following [10, §I.D] and [31, §1.4, 10.1].

Definition 3.10 ($\overline{\partial}$-closed forms)**.** Let $U \subset \mathbb{C}^n$ be an open subset. Then a differential form $\omega \in \Omega^{p,q}(U)$ is called $\overline{\partial}$-closed if $\overline{\partial}\omega = 0$.

Remark 3.16. If U is an open set in $\mathbb{C}^n$, let $\mathcal{Z}^{p,q}(U)$ denote the set of all $\overline{\partial}$-closed (p,q)-forms on U. The sum of two such (p,q)-forms is another $\overline{\partial}$-closed (p,q)-form, and so is the product of a $\overline{\partial}$-closed (p,q)-form by a scalar. Hence $\mathcal{Z}^{p,q}(U)$ is the vector sub-space of $\Omega^{p,q}(U)$. Also, from Theorem D.2 it follows that $\mathcal{Z}^{p,0}(U)$ is the space of $(p,0)$-forms whose coefficients are complex-valued holomorphic functions in U. In particular, note that $\mathcal{Z}^{0,0}(U) = \mathcal{O}(U)$, the space of complex-valued functions holomorphic in U.

Definition 3.11 ($\overline{\partial}$-exact forms)**.** Let $U \subset \mathbb{C}^n$ be an open subset. Then a differential form $\omega \in \Omega^{p,q}(U)$, for $q > 0$, is called $\overline{\partial}$-exact if $\omega = \overline{\partial}\eta$ for some differential form $\eta \in \Omega^{p,q-1}(U)$.

Remark 3.17. If U is an open set in $\mathbb{C}^n$, let $\mathcal{B}^{p,q}(U)$ denote the set of all $\overline{\partial}$-exact (p,q)-forms on U. The sum of two such (p,q)-forms is another $\overline{\partial}$-exact (p,q)-form, and so is the product of a $\overline{\partial}$-exact (p,q)-form by a scalar. Hence $\mathcal{B}^{p,q}(U)$ is the vector sub-space of $\Omega^{p,q}(U)$. Moreover, the trivial form $\omega \equiv 0$ is the only $(p,0)$-form which is $\overline{\partial}$-exact for any value of $p = 0, 1, \ldots, n$. That is, $\mathcal{B}^{p,0}(U)$ consists only of zero.

Theorem 3.1. *Every $\overline{\partial}$-exact form is $\overline{\partial}$-closed.*

Proof. Let U be an open set in $\mathbb{C}^n$ and $\omega \in \mathcal{B}^{p,q}(U)$ such that $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{p,q-1}(U)$. From Lemma 3.1 we know that $\overline{\partial}\omega = \overline{\partial}(\overline{\partial}\eta) = 0$ hence $\omega \in \mathcal{Z}^{p,q}(U)$ for all $q \geq 1$. For $q = 0$, the statement is trivially true. □

Remark 3.18. This theorem implies that $\mathcal{B}^{p,q}(U) \subset \mathcal{Z}^{p,q}(U)$ for all $q \geq 1$. However, the converse doesn't always hold. For example, if $U = \mathbb{C}^2 \setminus \{0\}$, then the $(0,1)$-form

$$\omega = \begin{cases} \overline{\partial}\left(\dfrac{\overline{z}_2}{z_1 r^2}\right) & \text{when } z_1 \neq 0 \\ -\overline{\partial}\left(\dfrac{\overline{z}_1}{z_2 r^2}\right) & \text{when } z_2 \neq 0 \end{cases}$$

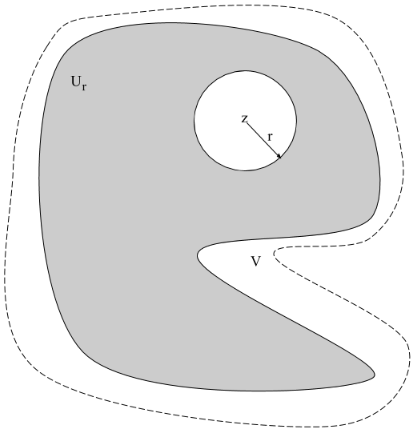
where $(z_1, z_2) \in U$ and $r^2 = \left|z_1^2\right| + \left|z_2^2\right|$, is $\overline{\partial}$-closed but not $\overline{\partial}$-exact [10, pp. 30–31].

3.2.1 Cauchy integral formula

Proposition 3.4 (Generalized Cauchy integral formula)**.** *Let U be a region*[7] *in $\mathbb{C}$ bounded by a simple closed rectifiable curve*[8] *γ, and f be complex-valued smooth function in some open neighborhood V of $\overline{U}$. Then for any point $z \in U$,*

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma} f(w)\frac{\mathrm{d}w}{w-z} + \frac{1}{2\pi i}\iint\limits_{U} \frac{\partial f(w)}{\partial \overline{w}}\frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z}$$

Proof. For any point $z \in U$ select a disc $\Delta(z;r)$ with closure contained in U. Let γ_r be the boundary of the $\Delta(z;r)$, a circle of radius r centered at z. Furthermore, let $U_r = U \setminus \overline{\Delta}(z;r)$ and observe that this is an open region bounded by $\gamma - \gamma_r$.



Now note that as a function of w, for a fixed z,

$$\frac{\partial f(w)}{\partial \overline{w}}\frac{\mathrm{d}\overline{w} \wedge \mathrm{d}w}{w-z} = \frac{\partial}{\partial \overline{w}}\left(\frac{f(w)}{w-z}\right)\mathrm{d}\overline{w} \wedge \mathrm{d}w = \mathrm{d}\left(f(w)\frac{\mathrm{d}w}{w-z}\right)$$

whenever the functions involved are well defined[9]. Therefore, by the Stokes theorem[10] in the plane we get

$$\iint\limits_{U_r} \frac{\partial f(w)}{\partial \overline{w}}\frac{\mathrm{d}\overline{w} \wedge \mathrm{d}w}{w-z} = \iint\limits_{U_r} \mathrm{d}\left(f(w)\frac{\mathrm{d}w}{w-z}\right) = \int\limits_{\gamma} f(w)\frac{\mathrm{d}w}{w-z} - \int\limits_{\gamma_r} f(w)\frac{\mathrm{d}w}{w-z} \tag{3.1}$$

[7] A region is an open connected subset of the complex plane [3, p. 40].

[8] A rectifiable curve is a curve having finite length. In other words, the measure (for example, arc length or distance) between any two points of this curve is finite. For more details, see [3, p. 62].

[9] Note the abuse of notations. Here $f(w)$ is a function of w and $\overline{w}$ which are linearly independent "variables". The better notation would have been $f(w, \overline{w})$ just like we have $f(x, y)$ in $\mathbb{R}^2$. Hence $\partial/\partial\overline{w}$ treats w as a constant. Moreover, the differential is well defined whenever $w \neq z$, which will hold when we apply the Stokes theorem.

[10] This is the standard Stokes theorem expressed in the complex notation [15, Theorem 1.1.1]: *Let $U \subset \mathbb{C}^n$ be a bounded open set with rectifiable boundary and $\omega \in \Omega^{p,q}(U)$ with $p + q = 2n$. Then*

$$\int_{\partial U} \omega = \int_U \mathrm{d}\omega = \int_U \partial\omega + \overline{\partial}\omega$$

Note that the integral of $(w-z)^{-1}\,\mathrm{d}\overline{w}\wedge\mathrm{d}w$ exists on a bounded region, as seen by integrating it using polar coordinates centered at z. That is, substituting $w = z + Re^{i\theta}$ and

$$\begin{aligned}\mathrm{d}\overline{w}\wedge\mathrm{d}w &= (\mathrm{d}x + i\,\mathrm{d}y)\wedge(\mathrm{d}x - i\,\mathrm{d}y)\\ &= -2i\,\mathrm{d}x\wedge\mathrm{d}y\\ &= -2i(\cos\theta\,\mathrm{d}R - R\sin\theta\,\mathrm{d}\theta)\wedge(\sin\theta\,\mathrm{d}R + R\cos\theta\,\mathrm{d}\theta)\\ &= 2iR\,\mathrm{d}\theta\wedge\mathrm{d}R\end{aligned}$$

for $w = x+iy$, $x = R\cos\theta$, and $y = R\sin\theta$. We get

$$\iint\limits_{U_r}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{w-z} = 2i\iint\limits_{U_r} e^{-i\theta}\,\mathrm{d}\theta\,\mathrm{d}R$$

Therefore, as $r\to 0$, the surface integral over U_r converges to the surface integral over U

$$\lim_{r\to 0}\iint\limits_{U_r}\frac{\partial f(w)}{\partial\overline{w}}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{w-z} = \iint\limits_{U}\frac{\partial f(w)}{\partial\overline{w}}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{w-z} \tag{3.2}$$

Moreover, since γ_r is defined by $w = z + re^{it}$ with $0\le t\le 2\pi$, we have

$$\lim_{r\to 0}\int\limits_{\gamma_r} f(w)\frac{\mathrm{d}w}{w-z} = \lim_{r\to 0}\int_{t=0}^{2\pi} f\left(z+re^{it}\right)i\,\mathrm{d}t = if(z)\int_{t=0}^{2\pi}\mathrm{d}t = 2\pi i f(z) \tag{3.3}$$

Letting $r\to 0$ in (3.1), and using (3.2) and (3.3) we get

$$\begin{aligned}\iint\limits_{U}\frac{\partial f(w)}{\partial\overline{w}}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{w-z} &= \int\limits_{\gamma} f(w)\frac{\mathrm{d}w}{w-z} - 2\pi i f(z)\\ \Longrightarrow f(z) &= \frac{1}{2\pi i}\int\limits_{\gamma} f(w)\frac{\mathrm{d}w}{w-z} + \frac{1}{2\pi i}\iint\limits_{U}\frac{\partial f(w)}{\partial\overline{w}}\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{w-z}\end{aligned}$$

Hence completing the proof. □

Remark 3.19. If f is holomorphic then $\frac{\partial f(w)}{\partial\overline{w}} = 0$ and we get the familiar Cauchy integral formula [3, Theorem IV.5.4]:

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma} f(w)\frac{\mathrm{d}w}{w-z}$$

Corollary 3.1. *Let U be a region in $\mathbb{C}$ bounded by a simple closed rectifiable curve γ, and f be complex-valued smooth function in some open neighborhood V of $\overline{U}$. Then for any point $z\in U$,*

$$f(z) = -\frac{1}{2\pi i}\int\limits_{\gamma} f(w)\frac{\mathrm{d}\overline{w}}{\overline{w}-\overline{z}} + \frac{1}{2\pi i}\iint\limits_{U}\frac{\partial f(w)}{\partial w}\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{\overline{w}-\overline{z}}$$

Proof. Note that as a function of w, for a fixed z,

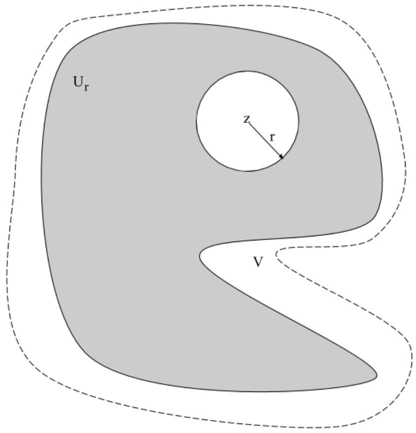
$$\frac{\partial f(w)}{\partial w}\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{\overline{w}-\overline{z}} = \frac{\partial}{\partial w}\left(\frac{f(w)}{\overline{w}-\overline{z}}\right)\mathrm{d}w\wedge\mathrm{d}\overline{w} = \mathrm{d}\left(f(w)\frac{\mathrm{d}\overline{w}}{\overline{w}-\overline{z}}\right)$$

whenever the functions involved are well defined. Now repeat the steps performed in the proof of previous result. □

Proposition 3.5. *Let U be an open subset of $\mathbb{C}$ bounded by a simple closed rectifiable curve γ, and f be complex-valued smooth function in an open neighborhood V of $\overline{U}$. Then there exists a complex-valued smooth function $g \in C^{\infty}(U)$ such that*

$$\frac{\partial g(z)}{\partial \overline{z}} = f(z)$$

Proof. For any point $z \in U$ select a disc $\Delta(z; r)$ with closure contained in U. Let γ_r be the boundary of the $\Delta(z; r)$, a circle of radius r centered at z. Furthermore, let $U_r = U \setminus \overline{\Delta}(z; r)$ and observe that this is an open region bounded by $\gamma - \gamma_r$.



Now note that as a function of w, for a fixed z,

$$\mathrm{d} \log |w - z|^2 = \mathrm{d}(\log(w - z) + \log(\overline{w} - \overline{z})) = \frac{\mathrm{d}w}{w - z} + \frac{\mathrm{d}\overline{w}}{\overline{w} - \overline{z}}$$

whenever the functions involved are well defined[11]. Therefore, by the Stokes theorem in the plane we get

$$\begin{aligned} \int_{\gamma} f(w) \log |w - z|^2 \, \mathrm{d}\overline{w} - \int_{\gamma_r} f(w) \log |w - z|^2 \, \mathrm{d}\overline{w} &= \iint_{U_r} \mathrm{d}\left(f(w) \log |w - z|^2 \, \mathrm{d}\overline{w}\right) \\ &= \iint_{U_r} \frac{\partial f(w)}{\partial w} \log |w - z|^2 \, \mathrm{d}w \wedge \mathrm{d}\overline{w} + \iint_{U_r} f(w) \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z} \end{aligned} \tag{3.4}$$

Observe that, as $r \to 0$, the surface integral over U_r converges to the surface integral over U

$$\lim_{r \to 0} \iint_{U_r} \frac{\partial f(w)}{\partial w} \log |w - z|^2 \, \mathrm{d}w \wedge \mathrm{d}\overline{w} = \iint_{U} \frac{\partial f(w)}{\partial w} \log |w - z|^2 \, \mathrm{d}w \wedge \mathrm{d}\overline{w} \tag{3.5}$$

and

$$\lim_{r \to 0} \iint_{U_r} f(w) \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z} = \iint_{U} f(w) \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z} \tag{3.6}$$

[11] Note that $\partial / \partial w$ and $\partial / \partial \overline{w}$ treat $\overline{w}$ and w as constants, respectively. Also recall that we can define the logarithm in every simply connected open set not containing 0 [3, Corollary IV.6.17]. In every of these open sets we can compute the differentials. It turns out that on the overlaps these differentials agree because different branches of the logarithm differ locally by a constant which is killed by taking a derivative [3, Corollary III.2.21]. Therefore, even though logarithm is not a globally defined function, its derivative is defined and smooth everywhere in $\mathbb{C} \setminus \{0\}$.

Moreover, since γ_r is defined by $w = z + re^{it}$ with $0 \le t \le 2\pi$, we have

$$\begin{aligned}\lim_{r\to 0}\int\limits_{\gamma_r} f(w)\log|w-z|^2\,\mathrm{d}\overline{w} &= \lim_{r\to 0}\int_{t=0}^{2\pi} f\left(z+re^{it}\right)(-2r)(\log r)ie^{-it}\,\mathrm{d}t \\ &\le \lim_{r\to 0}\int_{t=0}^{2\pi}\left|f\left(z+re^{it}\right)(-2r)(\log r)ie^{-it}\,\mathrm{d}t\right| \\ &\le \lim_{r\to 0} 2Mr(\log r)\int_{t=0}^{2\pi}\mathrm{d}t \\ &= 4\pi M\lim_{r\to 0} r\log r = 0\end{aligned} \tag{3.7}$$

where $M = \sup_{z\in U}|f(z)|$ and $|ie^{-it}| = 1$. Letting $r \to 0$ in (3.4), and using (3.5), (3.6) and (3.7) we get

$$\int\limits_{\gamma} f(w)\log|w-z|^2\,\mathrm{d}\overline{w} = \iint\limits_{U}\frac{\partial f(w)}{\partial w}\log|w-z|^2\,\mathrm{d}w\wedge\mathrm{d}\overline{w} + \iint\limits_{U} f(w)\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{w-z} \tag{3.8}$$

Next, we apply the operator $\partial/\partial\overline{z}$ to each integral in (3.8). We can use Leibniz's differentiation under the integral sign[12] for the integrals where the integrand obtained after differentiation is still integrable. Hence we have

$$\frac{\partial}{\partial\overline{z}}\int\limits_{\gamma} f(w)\log|w-z|^2\,\mathrm{d}\overline{w} = \int\limits_{\gamma}\frac{\partial\log|w-z|^2}{\partial\overline{z}}f(w)\,\mathrm{d}\overline{w} = -\int\limits_{\gamma} f(w)\frac{\mathrm{d}\overline{w}}{\overline{w}-\overline{z}}$$

$$\frac{\partial}{\partial\overline{z}}\iint\limits_{U}\frac{\partial f(w)}{\partial w}\log|w-z|^2\,\mathrm{d}w\wedge\mathrm{d}\overline{w} = \iint\limits_{U}\frac{\partial\log|w-z|^2}{\partial\overline{z}}\frac{\partial f(w)}{\partial w}\,\mathrm{d}w\wedge\mathrm{d}\overline{w} = -\iint\limits_{U}\frac{\partial f(w)}{\partial w}\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{\overline{w}-\overline{z}}$$

Hence by applying $\partial/\partial\overline{z}$ to (3.8), we get:

$$-\int\limits_{\gamma} f(w)\frac{\mathrm{d}\overline{w}}{\overline{w}-\overline{z}} = -\iint\limits_{U}\frac{\partial f(w)}{\partial w}\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{\overline{w}-\overline{z}} + \frac{\partial}{\partial\overline{z}}\iint\limits_{U} f(w)\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{w-z}$$

$$\Rightarrow \frac{\partial}{\partial\overline{z}}\iint\limits_{U} f(w)\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{w-z} = -\int\limits_{\gamma} f(w)\frac{\mathrm{d}\overline{w}}{\overline{w}-\overline{z}} + \iint\limits_{U}\frac{\partial f(w)}{\partial w}\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{\overline{w}-\overline{z}} = 2\pi i f(z) \quad \text{(Corollary 3.1)}$$

Therefore, we have

$$\boxed{g(z) = \frac{1}{2\pi i}\iint\limits_{U} f(w)\frac{\mathrm{d}w\wedge\mathrm{d}\overline{w}}{w-z}} \quad\Longrightarrow\quad \frac{\partial g(z)}{\partial\overline{z}} = f(z)$$

Observe that from (3.8) it follows that $g \in C^1(U)$ since

$$g(z) = \frac{1}{2\pi i}\int\limits_{\gamma} f(w)\log|w-z|^2\,\mathrm{d}\overline{w} - \frac{1}{2\pi i}\iint\limits_{U}\frac{\partial f(w)}{\partial w}\log|w-z|^2\,\mathrm{d}w\wedge\mathrm{d}\overline{w}$$

and the differential equation shows that $\partial g/\partial\overline{z} \in C^\infty(U)$. In particular, $g \in C^\infty(U)$, as desired. □

Corollary 3.2. *Let V be an open neighborhood of the closure of a disc $\Delta \subset \overline{\Delta} \subset V \subset \mathbb{C}$. For $f \in C^\infty(V)$, the function*

$$g(z) := \frac{1}{2\pi i}\iint\limits_{\Delta}\frac{f(w)}{w-z}\,\mathrm{d}w\wedge\mathrm{d}\overline{w}$$

satisfies $\partial g(z)/\partial\overline{z} = f(z)$ for $z \in \Delta$.

Corollary 3.3. *Let $f \in C^\infty(V)$ on an open set V of $\mathbb{C}$. Then, locally*[13] *on this open set, there*

[12]The proof of this result is an application of Dominated Convergence Theorem [5, Theorem 2.27].

[13]Here "locally" means that for any point $z \in V$ there is some open neighborhood U of z where $\partial g/\partial\overline{z} = f$.

exists a complex-valued smooth function g such that $\partial g/\partial \overline{z} = f$.

Corollary 3.4. *If $f \in C^\infty(V)$, for an open set $V \subset \mathbb{C}$ containing a compact set K, then there exists an open set U, with $K \subset U \subset V$, and a $g \in C^\infty(U)$, such that $\partial g/\partial \overline{z} = f$ in U.*

Remark 3.20. We can prove the above three corollaries directly: Huybrechts [12, Proposition 1.3.7] and Kaup and Kaup [14, Lemma 61.6] prove Corollary 3.2 using Lemma A.3, Proposition D.4 and Stokes theorem; Voisin [34, Theorem 1.28] proves Corollary 3.3 by assuming that f has a compact support since we want to prove a local statement and using Stokes theorem; and Taylor [31, Proposition 1.4.2] proves Corollary 3.4 by using Lemma A.3 and the generalized Cauchy integral formula. The proof discussed here is by Gunning and Rossi [10, Lemma I.D.2].

Theorem 3.2. *If U is any open subset of $\mathbb{C}$ and $f \in C^\infty(U)$, then there exists $g \in C^\infty(U)$ such that $\partial g/\partial \overline{z} = f$.*

Proof. From Lemma A.2 we know that there exists a sequence $\{K_n\}$ of compact subsets of U such that

1. $K_n \subset \operatorname{int}(K_{n+1})$ for each n;
2. $\bigcup_{n\in\mathbb{N}} \operatorname{int}(K_n) = U$; and
3. each bounded component of the complement of K_n meets the complement of U.

First we will prove by induction that there exists a sequence of complex-valued smooth functions $\{g_n\}$ satisfying $\partial g_n/\partial \overline{z} = f$ on an open neighborhood of K_n, such that

$$|g_n(z) - g_{n-1}(z)| < \frac{1}{2^{n-1}} \quad \text{for all } z \in K_{n-1} \text{ if } n > 1$$

For the base case we get g_1 by Corollary 3.4. Next, as the induction hypothesis, assume that there exist complex-valued smooth functions $\{g_1, \dots, g_m\}$ satisfying the desired conditions. We again apply Corollary 3.4 to get a function h which is smooth in an open neighborhood of K_{m+1} and satisfies $\partial h/\partial \overline{z} = f$ on this neighborhood. Since $K_m \subset \operatorname{int}(K_{m+1})$, on an open neighborhood of K_m we have

$$\frac{\partial (h - g_m)}{\partial \overline{z}} = 0$$

So, by Theorem D.2, $h - g_m$ is holomorphic on this neighborhood of K_m. By Runge's theorem [3, Theorem VIII.1.7], we can choose a rational function r, with poles in $\mathbb{C} \setminus U$, such that

$$|h(z) - g_m(z) - r(z)| < \frac{1}{2^m} \quad \text{for all } z \in K_m$$

If we set $\boxed{g_{m+1} = h - r}$, then $\partial g_{m+1}/\partial \overline{z} = f$ on an open neighborhood on K_{m+1} and

$$|g_{m+1}(z) - g_m(z)| < \frac{1}{2^m} \quad \text{for all } z \in K_m$$

By induction, a sequence $\{g_n\}$ with the required properties exists.

Next, we note that[14] the sequence $\{g_n\}$ of complex-valued smooth functions converges uniformly on each compact set K_n to a function g defined on U. Moreover, $g_n - g_m$ is holomorphic on an open neighborhood of K_m for each $n > m$. Thus for each fixed m, $\{g_n - g_m\}$ is a sequence

[14] Recall the following three facts from real analysis: (1). If a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}^n$ satisfies $\sum_{n\geq 1} |x_{n+1} - x_n| < \infty$, then it is Cauchy.; (2). A sequence $\{f_n\}$ converges uniformly if and only if $\{f_n\}$ is uniformly Cauchy; (3). A sequence of functions $\{f_n\}$ from a set A to a metric space X is said to be uniformly Cauchy if for all $\varepsilon > 0$, there exists $N > 0$ such that for all $a \in A$ we have $|f_n(a) - f_m(a)| < \varepsilon$ whenever $m, n > N$.

of complex-valued holomorphic functions on an open neighborhood of K_m which is uniformly convergent on K_m. Therefore, by Morera's theorem [3, Exercise IV.5.8], the limit function $g-g_m$ is holomorphic on $\mathrm{int}(K_m)$. Hence, g is smooth on $\mathrm{int}(K_m)$. Since this is true for each m and $\bigcup_m \mathrm{int}(K_m) = U$, we conclude that g is a complex-valued smooth function on the whole of U. Clearly, $\partial g/\partial \overline{z} = f$ in U. □

Remark 3.21. In particular, if U is simply connected and $f : U \to \mathbb{C}$ is holomorphic, then f has a primitive in U [3, Corollary IV.6.16].

3.2.2 $\overline{\partial}$-Poincaré lemma

Lemma 3.2. *Let $\overline{\Delta} \subset \mathbb{C}^n$ be a compact polydisc*[15]*, and $\omega \in \Omega^{p,q}(V)$ for some open neighborhood V of $\overline{\Delta}$. If $q > 0$ and $\overline{\partial}\omega = 0$, then there is $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega = \overline{\partial}\eta$.*

Proof. Consider the following explicit representation of $\omega \in \Omega^{p,q}(V)$

$$\omega = \sum_{|\alpha|=p, |\beta|=q} f_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

Let ℓ be the least integer such that the expression for ω involves no conjugate differential $\mathrm{d}\overline{z}_j$ with $j > \ell$; i.e. ω can be written in terms of the conjugate differentials $\mathrm{d}\overline{z}_1, \dots, \mathrm{d}\overline{z}_\ell$ and the differentials $\mathrm{d}z_1, \dots, \mathrm{d}z_n$. We will proceed by induction on ℓ.

For the base case there is nothing to prove since for $\ell = 0$ we have $\omega = 0$ because by hypothesis $q > 0$. Next, as the induction hypothesis, assume that for $0 < \ell < k$, every $\overline{\partial}$-closed (p, q)-form in an open neighborhood of $\overline{\Delta}$ is $\overline{\partial}$-exact on Δ. In general, for the induction step, we write

$$\omega = \mathrm{d}\overline{z}_k \wedge \theta + \xi$$

where θ and ξ involve only the conjugate differentials $\mathrm{d}\overline{z}_1, \dots, \mathrm{d}\overline{z}_{k-1}$. Since ω is $\overline{\partial}$-closed, we have

$$\begin{aligned} 0 = \overline{\partial}\omega &= \overline{\partial}(\mathrm{d}\overline{z}_k \wedge \theta) + \overline{\partial}\xi \\ &= \big(\overline{\partial}(\mathrm{d}\overline{z}_k) \wedge \theta + (-1)^{0+1}\, \mathrm{d}\overline{z}_k \wedge \overline{\partial}\theta\big) + \overline{\partial}\xi \\ &= \big(-\mathrm{d}\overline{z}_k \wedge \overline{\partial}\theta\big) + \overline{\partial}\xi \end{aligned}$$

It follows, by Theorem D.2, that the coefficients of the forms θ and ξ are holomorphic in $z_{k+1}, \dots, z_n$ since the partial derivatives $\partial/\partial\overline{z}_{k+1}, \dots, \partial/\partial\overline{z}_n$ for any such coefficient are all zero. Consider the following explicit representation of θ

$$\theta = \sum_{\substack{|\alpha|=p \\ \beta_j \in \{1,\dots,k-1\}}} g_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

Observe that any coefficient $g_{\alpha\beta}$ of θ is a complex-valued smooth function of the variable z_k in an open neighborhood of $\overline{\Delta}_k$, where the original polydisc has the product decomposition[16]

$$\Delta = \Delta_1 \times \cdots \times \Delta_n$$

where Δ_j is a disc in $\mathbb{C}$. The function $g_{\alpha\beta}$ is also a complex-valued smooth function of $z_1, \dots, z_{k-1}$ and a holomorphic function of $z_{k+1}, \dots, z_n$ in the corresponding domains. By Corollary 3.2 there exists a function $h_{\alpha\beta}$ which is smooth in $z_k \in \Delta_k$:

$$h_{\alpha\beta}(z) = h_{\alpha\beta}(z_1, \dots, z_n) = \frac{1}{2\pi i} \iint_{\Delta_k} \frac{g_{\alpha\beta}(z_1, \dots, z_{k-1}, w, z_{k+1}, \dots, z_n)}{w - z_k}\, \mathrm{d}w \wedge \mathrm{d}\overline{w}$$

[15]For its definition see Definition D.1.

[16]In this argument it is important that $\overline{\Delta}$ is a Cartesian product of some compact sets $\overline{\Delta}_1, \dots, \overline{\Delta}_n$ in $\mathbb{C}$, since it enables us to apply Corollary 3.4 in each variable separately, while treating the other variables as parameters [31, p. 241].

such that

$$\frac{\partial h_{\alpha\beta}}{\partial \overline{z}_k} = g_{\alpha\beta}$$

Note that $h_{\alpha\beta}$ is also[17] smooth in $z_1, \dots, z_{k-1}$ and holomorphic in $z_{k+1}, \dots, z_n$ in the same regions as $g_{\alpha\beta}$ is. Replacing each coefficient $g_{\alpha\beta}$ in the differential form θ by such a function $h_{\alpha\beta}$ yields a new $(p, q-1)$-form

$$\sigma = \sum_{\substack{|\alpha|=p \\ \beta_j \in \{1,\dots,k-1\}}} h_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

which by this construction satisfies the equation

$$\overline{\partial}\sigma = \mathrm{d}\overline{z}_k \wedge \theta + \rho$$

for some differential form ρ involving only the conjugate differentials $\mathrm{d}\overline{z}_1, \dots, \mathrm{d}\overline{z}_{k-1}$. Now consider the differential form

$$\nu = \omega - \overline{\partial}\sigma = \xi - \rho$$

Note that ν is a $\overline{\partial}$-closed form since

$$\overline{\partial}\nu = \overline{\partial}\omega - \overline{\partial}^2\sigma = 0$$

and it involves only the conjugate differentials $\mathrm{d}\overline{z}_1, \dots, \mathrm{d}\overline{z}_{k-1}$ since ξ and ρ do. The induction hypothesis implies that ν is $\overline{\partial}$-exact on Δ, i.e. $\nu = \overline{\partial}\lambda$ for some $\lambda \in \Omega^{p,q-1}(\Delta)$. Hence, for $\boxed{\eta = \sigma + \lambda}$ we have $\omega = \overline{\partial}\eta$, completing the proof. ☐

Corollary 3.5. *Let ω be a (p,q)-form such that $\overline{\partial}\omega = 0$ and $q > 0$, then it is locally*[18] *expressible as $\overline{\partial}\eta$ for some $(p, q-1)$-form η.*

Proof. The open polydiscs form a basis for the product topology on $\mathbb{C}^n$. Therefore, this result follows from the previous one. ☐

Theorem 3.3 ($\overline{\partial}$-Poincaré lemma)**.** *Let Δ be an open polydisc in the space $\mathbb{C}^n$, not necessarily having a compact closure, and $\omega \in \Omega^{p,q}(\Delta)$. If $q > 0$ and $\overline{\partial}\omega = 0$, then there is $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega = \overline{\partial}\eta$.*

Proof. Let $\{\Delta_j\}$ be a sequence of open polydiscs in $\mathbb{C}^n$ which have same center as Δ and satisfy the following conditions:

1. $\overline{\Delta}_j \subset \Delta_{j+1}$; and
2. $\Delta = \bigcup_j \Delta_j$

We will divide the proof into two cases:

Case 1. If $q > 1$.

We will inductively construct a sequence of $(p, q-1)$-forms $\{\eta_j\}$ such that

(a) $\eta_j \in \Omega^{p,q-1}(V_j)$ for some open neighborhood V_j of $\overline{\Delta}_j$;
(b) $\overline{\partial}\eta_j = \omega$ on Δ_j; and
(c) $\eta_j\big|_{\Delta_{j-1}} = \eta_{j-1}$ if $j > 1$.

[17] From the proof of Proposition 3.5 it is clear that the function g constructed in holomorphic or smooth in any additional parameters in which f is holomorphic or smooth [10, Lemma I.D.2].

[18] Suppose V is an open set $\mathbb{C}^n$ and $\omega \in \Omega^{p,q}(U)$ such that $q > 0$ and $\overline{\partial}\omega = 0$, and for any point $z \in U$, then in some open neighborhood U of z such that $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{p,q-1}(U)$.

For the base case we get $\eta_1 \in \Omega^{p,q-1}(V_1)$ by Lemma 3.2. Next, as the induction hypothesis, assume that there exist $(p, q-1)$-forms $\{\eta_1, \dots, \eta_k\}$ satisfying the desired conditions. We again apply Lemma 3.2 to get a $(p, q-1)$-form θ on an open neighborhood V of $\overline{\Delta}_{k+1}$ such that $\overline{\partial}\theta = \omega$ on this neighborhood. Since $\overline{\Delta}_k \subset \Delta_{k+1}$, on an open neighborhood of $\overline{\Delta}_k$ we have

$$\overline{\partial}(\theta - \eta_k) = 0$$

So, $\theta - \eta_k$ is a $\overline{\partial}$-closed $(p, q-1)$-form with $q - 1 > 0$. By yet another application of Lemma 3.2 there exits a $(p, q-2)$-form ξ on an open neighborhood U of $\overline{\Delta}_k$ such that $\overline{\partial}\xi = \theta - \eta_k$ on this neighborhood. From Lemma A.3 we know that there exits a real-valued smooth function F in $\mathbb{C}^n$ such that

(a) $0 \le F(z) \le 1$ for all $z \in \mathbb{C}^n$;
(b) $F(z) = 1$ for $z \in \overline{\Delta}_k$; and
(c) $F(z) = 0$ for $z \in \mathbb{C}^n \setminus U$.

Hence we have $F\xi \in \Omega^{p,q-2}(\mathbb{C}^n)$. Then we get the $(p, q-1)$-form $\boxed{\eta_{k+1} = \theta - \overline{\partial}(F\xi)}$ defined on the open neighborhood V of $\overline{\Delta}_{k+1}$, which satisfies the desired conditions:

$$\overline{\partial}\eta_{k+1} = \omega \text{ on } \Delta_{k+1} \quad \text{and} \quad \eta_{k+1}\big|_{\Delta_k} = \theta - \overline{\partial}\xi = \eta_k$$

As a result of the above construction there is $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\eta\big|_{\Delta_j} = \eta_j$ and $\overline{\partial}\eta = \omega$, which concludes the proof of this case.

Case 2. If $q = 1$.

First we will inductively construct a sequence of $(p, 0)$-forms $\{\eta_j\}$ such that

(a) $\eta_j \in \Omega^{p,q-1}(V_j)$ for some open neighborhood V_j of $\overline{\Delta}_j$;
(b) $\overline{\partial}\eta_j = \omega$ on Δ_j; and
(c) If $\eta_j = \sum_\alpha f_\alpha^{(j)}\, \mathrm{d}z_\alpha$ for $\alpha = (\alpha_1, \dots, \alpha_p)$ and $\mathrm{d}z_\alpha = \mathrm{d}z_{\alpha_1} \wedge \dots \wedge \mathrm{d}z_{\alpha_p}$, then

$$\left| f_\alpha^{(j)}(z) - f_\alpha^{(j-1)}(z) \right| < \frac{1}{2^{j-1}} \quad \text{for all } \alpha \text{ and } z \in \overline{\Delta}_{j-1} \text{ if } j > 1$$

For the base case we get $\eta_1 \in \Omega^{p,q-1}(V_1)$ by Lemma 3.2. Next, as the induction hypothesis, assume that there exist $(p, 0)$-forms $\{\eta_1, \dots, \eta_k\}$ satisfying the desired conditions. We again apply Lemma 3.2 to get a $(p, 0)$-form θ on an open neighborhood V of $\overline{\Delta}_{k+1}$ such that $\overline{\partial}\theta = \omega$ on this neighborhood. Let the following be the explicit representation of θ

$$\theta = \sum_\alpha g_\alpha\, \mathrm{d}z_\alpha$$

Then on an open neighborhood of $\overline{\Delta}_k$ all the coefficients of the form $\theta - \eta_k$ are holomorphic by Remark 3.16 since $\overline{\partial}(\theta - \eta_k) = 0$. Observe that each coefficient has a power series expansion centered at the common center of all the polydiscs and converging uniformly in $\overline{\Delta}_k$. Hence choosing suitable partial sums, we find polynomial terms $r_\alpha(z)$ such that

$$\left| g_\alpha(z) - f_\alpha^{(k)}(z) - r_\alpha(z) \right| < \frac{1}{2^k} \quad \text{for all } \alpha \text{ and } z \in \overline{\Delta}_k$$

Let ξ be the $(p, 0)$-form with the polynomials r_α as coefficients

$$\xi = \sum_\alpha r_\alpha\, \mathrm{d}z_\alpha$$

Note that $\overline{\partial}\xi = 0$ since the coefficients are holomorphic. Then we get the $(p, 0)$-form $\boxed{\eta_{k+1} = \theta - \xi}$ defined on the open neighborhood V of $\overline{\Delta}_{k+1}$, which satisfies the desired conditions:

$$\overline{\partial}\eta_{k+1} = \omega \text{ on } \Delta_{k+1} \quad \text{and} \quad \left|f_\alpha^{(k+1)}(z) - f_\alpha^{(k)}(z)\right| < \frac{1}{2^k} \quad \text{for all } \alpha \text{ and } z \in \overline{\Delta}_k$$

Next, fix one α. Then we note that[19] the sequence $\left\{f_\alpha^{(j)}\right\}$ of smooth functions converges uniformly on each Δ_j to a function f_α defined on Δ. Moreover, $f_\alpha^{(j)} - f_\alpha^{(k)}$ is holomorphic on an open neighborhood of $\overline{\Delta}_k$ for each $j > k$ since $\overline{\partial}(\eta_j - \eta_k) = 0$. Thus for each fixed k, $\left\{f_\alpha^{(j)} - f_\alpha^{(k)}\right\}$ is a sequence of holomorphic functions on an open neighborhood of $\overline{\Delta}_k$ which is uniformly convergent on $\overline{\Delta}_k$. Therefore, by Morera's theorem [3, Exercise IV.5.8], the limit function $f_\alpha - f_\alpha^{(k)}$ is holomorphic on Δ_k. Hence, f_α is smooth on Δ_k. Since this is true for each k and $\bigcup_k \Delta_k = \Delta$, we conclude that f_α is a complex-valued smooth function on the whole of Δ.

Finally we define the $(p, 0)$-form

$$\eta = \sum_\alpha f_\alpha \, \mathrm{d}z_\alpha = \lim_{j\to\infty} \eta_j$$

Note that for a fixed k we have

$$\eta - \eta_k = \lim_{j\to\infty} (\eta_j - \eta_k)$$

Since $\eta_j - \eta_k$ have coefficients holomorphic in Δ_k, it follows that in Δ_k, $\boxed{\eta = \eta_k + \sigma_k}$ for some holomorphic form σ_k given by

$$\sigma_k = \sum_\alpha \left(f_\alpha - f_\alpha^{(k)}\right) \mathrm{d}z_\alpha$$

Hence $\overline{\partial}\eta = \overline{\partial}\eta_k = \omega$ in each Δ_k, which completes the proof.

□

Remark 3.22. If we consider $\omega = f \, \mathrm{d}\overline{z} \in \Omega^{0,1}(U)$ for some open set $U \subset \mathbb{C}$, then Theorem 3.2 gives us the "$\overline{\partial}$-Poincaré lemma in one variable." However, due to the lack of purely topological or intrinsic analytical description of the domains in $\mathbb{C}^n$ for $n \geq 2$ on which approximation theorems (like Runge's theorem) hold, we confine ourselves to the simple case of polydiscs [10, §I.F].

Remark 3.23. Unlike the Poincaré lemma we proved earlier (Theorem 1.2), we cannot give a simple topological condition on the domain which will ensure that the $\overline{\partial}$-closed forms are also $\overline{\partial}$-exact. This is because the failure of *Riemann mapping theorem* in $\mathbb{C}^n$ for $n \geq 2$ implies that there is no canonical topologically trivial domain in $\mathbb{C}^n$ for $n \geq 2$, as there is in $\mathbb{C}$ (namely, the disc) [15, §0.3.2].

[19] Recall the following three facts from real analysis: (1). If a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}^n$ satisfies $\sum_{n\geq 1} |x_{n+1} - x_n| < \infty$, then it is Cauchy.; (2). A sequence $\{f_n\}$ converges uniformly if and only if $\{f_n\}$ is uniformly Cauchy; (3). A sequence of functions $\{f_n\}$ from a set A to a metric space X is said to be uniformly Cauchy if for all $\varepsilon > 0$, there exists $N > 0$ such that for all $a \in A$ we have $|f_n(a) - f_m(a)| < \varepsilon$ whenever $m, n > N$.

3.3 Differential forms on complex manifolds

In this section some basic definitions and facts from [12, §2.1, 2.2 and 2.6], [37, §I.2, I.3], [34, §2.1, 2.2, 2.3] and [6, §IV.1] will be stated.

Definition 3.12 (Complex manifold)**.** A *complex manifold* M of dimension n is a second countable Hausdorff space together with a holomorphic structure on it. A *holomorphic structure* $\mathscr{U}$ is the collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ where U_α is an open set of M and ϕ_α is a homeomorphism of U_α onto an open set of $\mathbb{C}^n$ such that

1. the open sets $\{U_\alpha\}_{\alpha\in A}$ cover M.
2. for every pair of indices $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the homeomorphisms
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$
$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$
are holomorphic maps[20] between open subsets of $\mathbb{C}^n$.
3. the family $\mathscr{U}$ is maximal in the sense that it contains all possible pairs (U_α, ϕ_α) satisfying the properties 1. and 2.

Example 3.1. Following two complex manifolds will be used throughout this thesis:

1. The complex space $\mathbb{C}^n$ is a complex manifold with single chart $(\mathbb{C}^n, \mathbb{1}_{\mathbb{C}^n})$, where $\mathbb{1}_{\mathbb{C}^n}$ is the identity map. In other words, $(\mathbb{C}^n, \mathbb{1}_{\mathbb{C}^n}) = (\mathbb{C}^n, z_1, \dots, z_n)$ where $z_1, \dots, z_n$ are the standard coordinates on $\mathbb{C}^n$.
2. Any open subset V of a complex manifold M is also a smooth manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha\cap V})\}$ is an atlas for V, where $\phi_\alpha|_{U_\alpha\cap V} : U_\alpha \cap V \to \mathbb{C}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$.

Remark 3.24. Every complex manifold M is paracompact [6, §IV.1].

Definition 3.13 (Holomorphic function on a manifold)**.** Let M be a complex manifold of dimension n. A function $f : M \to \mathbb{C}$ is said to be a *holomorphic function at a point* w in M if there is a chart (U, ϕ) about w in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{C}^n$, is holomorphic[21] at $\phi(w)$. The function f is said to be holomorphic in M if it is holomorphic at every point of M.

$$\begin{array}{ccc} (U, w) & \xrightarrow{\ \phi\ } & (\mathbb{C}^n, \phi(w)) \\ & \searrow^{f} & \downarrow{\scriptstyle f\circ\phi^{-1}} \\ & & (\mathbb{C}, f(w)) \end{array}$$

Definition 3.14 (Holomorphic map between complex manifolds)**.** Let M and N be complex manifolds of dimension m and n, respectively. A continuous map $F : M \to N$ is said to be *holomorphic at a point* w of M if there are charts (V, ψ) about $F(w)$ in N and (U, ϕ) about w in M such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{C}^m$ to $\mathbb{C}^n$, is holomorphic at $\phi(w)$.

$$\begin{array}{ccc} (U, w) & \xrightarrow{\quad F \quad} & (V, F(w)) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ (\mathbb{C}^m, \phi(w)) & \xrightarrow{\ \psi\circ F\circ\phi^{-1}\ } & (\mathbb{C}^n, \psi(F(w))) \end{array}$$

[20] For the definition of several complex variables holomorphic mapping, see Definition D.8.
[21] For the definition of complex-valued holomorphic function, see Definition D.3.

The continuous map $F : M \to N$ is said to be *holomorphic* if it is holomorphic at every point in M.

Definition 3.15 (Biholomorphic manifolds)**.** Two complex manifolds M and N are called *biholomorphic* if there exists a holomorphic homeomorphism[22] $f : X \to Y$.

Theorem 3.4. *If (U, ϕ) is a chart on a complex manifold M of dimension n, then U is biholomorphic to $\phi(U) \subset \mathbb{C}^n$.*

Remark 3.25. If (U, ϕ) is a chart of a manifold, i.e. $\phi : U \to \mathbb{C}^n$, then let $r_j = z_i \circ \phi$ be the j^{th} component of ϕ and write $\phi = (r_1, \ldots, r_n)$ and $(U, \phi) = (U, r_1, \ldots, r_n)$. Thus, for $w \in U$, $(r_1(w), \ldots, r_n(w))$ is a point in $\mathbb{C}^n$. The functions $r_1, \ldots, r_n$ are called *coordinates* or *local coordinates* on U.

3.3.1 Complex differential forms

Definition 3.16 (Complex vector bundle)**.** A *complex vector bundle* of rank k over a smooth manifold M is a smooth manifold E equipped with a smooth surjective map $\pi : E \to M$ such that for an open cover $\{U_\alpha\}$ of M, there is a *local trivialization* diffeomorphism

$$\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$$

satisfying the following conditions:

1. the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\tau_\alpha} & U_\alpha \times \mathbb{C}^k \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & U_\alpha & \end{array}$$

   where p_1 is the projection onto the first factor,

2. the composite maps

$$\tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta \left(\pi^{-1}(U_\alpha \cap U_\beta)\right) \to \tau_\alpha \left(\pi^{-1}(U_\alpha \cap U_\beta)\right)$$

   are $\mathbb{C}$-linear for each $w \in U_\alpha \cap U_\beta$.

Remark 3.26. For a fixed $w \in U_\alpha \cap U_\beta$, the linear transformation

$$\left(\tau_\alpha \circ \tau_\beta^{-1}\right)_w : \{w\} \times \mathbb{C}^k \to \{w\} \times \mathbb{C}^k$$

must respect the projection onto the first factor, by the first condition above, and is thus described by a complex $k \times k$-matrix, whose coefficients are smooth functions of w. These matrices are called *transition matrices.* In particular, the map $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$ is given by

$$\tau_{\alpha\beta}(w, v) = (w, \sigma_{\alpha\beta}(w)v) \quad \forall w \in U_\alpha \cap U_\beta, v \in \mathbb{C}^k$$

and is completely determined by the map $\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$, called the *transition map.* Since $\tau_{\alpha\beta}$ is smooth, so is $\sigma_{\alpha\beta}$. From now on, we will assume that the transition maps can in fact be used to define a vector bundle. For proof, see [38, §9] and [37, §I.2].

Definition 3.17 (Fiber of a complex vector bundle)**.** If $\pi : E \to M$ is a complex vector bundle and $w \in M$, then $E_w = \pi^{-1}(w)$ is called the *fiber* of E at the point w. It is canonically a vector space, with structure given by any of the trivializations of E in the neighborhood of w.

[22] Note that the inverse of a holomorphic homeomorphism is holomorphic by Proposition D.5.

Remark 3.27. A complex vector bundle is a smooth vector bundle whose fibers are complex vector spaces and the transition maps are complex linear.

Definition 3.18 (Almost complex structure)**.** An *almost complex structure* on a smooth manifold M is a vector bundle endomorphism J of (real) tangent bundle $T_{\mathbb{R}}M$, such that $J^2 = -\mathbb{1}_{T_{\mathbb{R}}M}$, i.e. for all $w \in M$, the linear map $J_w : T_{w,\mathbb{R}}M \to T_{w,\mathbb{R}}M$ is a linear complex structure for $T_{w,\mathbb{R}}M$.

Remark 3.28. Equivalently, the almost complex structure is the structure of a complex vector bundle on $T_{\mathbb{R}}M$ [34, Definition 2.11]. Also, if an almost complex structure exists, then the real dimension of M is even [12, Definition 2.6.1]. However, not every smooth manifold of even dimension admits an almost complex structure [12, Remark 2.6.3].

Definition 3.19 (Almost complex manifold)**.** An *almost complex manifold* is a smooth manifold together with an almost complex structure.

Proposition 3.6. *Let M be an almost complex manifold. Then there exists a direct sum decomposition of the complexified tangent bundle $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ into complex vector bundles*

$$T_{\mathbb{C}}M = (T_{\mathbb{R}}M)^{1,0} \oplus (T_{\mathbb{R}}M)^{0,1}$$

such that the $\mathbb{C}$-linear extension $\tilde{J} = J \otimes \mathbb{1}_{\mathbb{C}}$ satisfies

$$\tilde{J}|_{(T_{\mathbb{R}}M)^{1,0}} = i \cdot \mathbb{1}_{T_{\mathbb{C}}M} \quad \text{and} \quad \tilde{J}|_{(T_{\mathbb{R}}M)^{0,1}} = -i \cdot \mathbb{1}_{T_{\mathbb{C}}M}$$

Proposition 3.7. *Let M be an almost complex manifold. Then the dual of complexified tangent bundle $T^*_{\mathbb{C}}M = T^*_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles*

$$T^*_{\mathbb{C}}M = (T^*_{\mathbb{R}}M)^{1,0} \oplus (T^*_{\mathbb{R}}M)^{0,1}$$

such that the $\mathbb{C}$-linear extension $\tilde{\mathcal{J}} = \mathcal{J} \otimes \mathbb{1}_{\mathbb{C}}$ satisfies

$$\tilde{\mathcal{J}}|_{(T^*_{\mathbb{R}}M)^{1,0}} = i \cdot \mathbb{1}_{T^*_{\mathbb{C}}M} \quad \text{and} \quad \tilde{\mathcal{J}}|_{(T^*_{\mathbb{R}}M)^{0,1}} = -i \cdot \mathbb{1}_{T^*_{\mathbb{C}}M}$$

Remark 3.29. As in Proposition C.8, we get the almost complex structure $\mathcal{J}$ on $T^*_{w,\mathbb{R}}M$ from the almost complex structure J on $T_{w,\mathbb{R}}M$. We will regard this $\mathcal{J}$ as a vector bundle endomorphism of the smooth vector bundle $T^*_{\mathbb{R}}M$ over M.

Definition 3.20 (Differential (p,q)-form)**.** Let M be an almost complex manifold. Over M we define the complex vector bundle of rank $\binom{n}{p}\binom{n}{q}$

$$\bigwedge^{p,q} T^*_{\mathbb{R}}M := \bigwedge^{p} \left((T^*_{\mathbb{R}}M)^{1,0}\right) \otimes_{\mathbb{C}} \bigwedge^{q} \left((T^*_{\mathbb{R}}M)^{0,1}\right)$$

whose fiber is $\bigwedge^{p,q} T^*_{w,\mathbb{R}}M$. The smooth sections of this vector bundle are called the *differential forms of type* (p,q) on M. The space of all smooth differential forms of type (p,q) on M is denoted by $\Omega^{p,q}(M)$.

Remark 3.30. Let $\Omega^k_{\mathbb{C}}(M)$ be the space of sections of vector bundle $\bigwedge^k T^*_{\mathbb{C}}M$. By Remark C.12 we have

$$\bigwedge^{k} T^*_{\mathbb{C}}M \cong \bigoplus_{p+q=k} \bigwedge^{p,q} T^*_{\mathbb{R}}M \quad \Longrightarrow \Omega^k_{\mathbb{C}}(M) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

Thus we have natural projection operators $\bigwedge^k T^*_{\mathbb{C}}M \to \bigwedge^{p,q} T^*_{\mathbb{R}}M$ and $\Omega^k_{\mathbb{C}}(M) \to \Omega^{p,q}(M)$, denoted by $\Pi^{p,q}$ for $p+q=k$.

Definition 3.21 (Differential of a (p, q)-form)**.** Let M be an almost complex manifold, and $\mathrm{d} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$ be the complex linear extension of the usual exterior differential (Definition 1.44). Then

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

are defined as $\partial := \Pi^{p+1,q} \circ \mathrm{d}$ and $\overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}$.

Lemma 3.3. *For an almost complex manifold M, the differential operators ∂ and $\overline{\partial}$ satisfy the Leibniz's rule, i.e.*

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$
$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$$

for $\omega \in \Omega^{p,q}(M)$ and $\eta \in \Omega^{r,s}(M)$.

Proof. As in Lemma 3.1, we will use the properties of d studied earlier. Recall that for $\omega \in \Omega^{p+q}_{\mathbb{C}}(M)$ and $\eta \in \Omega^{r+s}_{\mathbb{C}}(M)$ we have

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \mathrm{d}\eta \in \Omega^{p+q+r+s+1}_{\mathbb{C}}(U)$$

Since $\partial := \Pi^{p+r+1,q+s} \circ \mathrm{d}$, taking the $(p + r + 1, q + s)$-parts on both sides one obtains

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$

Similarly, taking the $(p + r, q + s + 1)$-parts one obtains

$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$$

Hence completing the proof. □

Definition 3.22 (Integrable almost complex structure)**.** An almost complex structure J on M is called *integrable* if $\mathrm{d}\omega = \partial\omega + \overline{\partial}\omega$ for all $\omega \in \Omega^k_{\mathbb{C}}(M)$.

Remark 3.31. By Lemma 3.1 we know that the almost complex structures on the open sets in $\mathbb{C}^n$ are integrable. For more details about this definition, see [12, Proposition 2.6.15], [37, p. 34] and [34, Theorem 2.24].

Definition 3.23 (Complex manifold)**.** A *complex manifold* M of dimension n is a smooth manifold of dimension $2n$ equipped with a holomorphic structure, i.e. if M is covered by open sets U_α which are diffeomorphic via maps called ϕ_α to open sets in $\mathbb{C}^n$, in such a way that the transition diffeomorphisms

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic.

Proposition 3.8. *A complex manifold M induces an almost complex structure on its underlying smooth manifold.*

Proof. This follows from Definition C.4 and Remark C.5. For details, see [37, Proposition I.3.4]. □

Theorem 3.5. *The induced almost complex structure on a complex manifold is integrable.*

Proof. This follows by looking at the local coordinates as in Lemma 3.1. For details, see [37, Theorem I.3.7]. □

Corollary 3.6. *If M is a complex manifold, then $\overline{\partial}^2 = 0$.*

Definition 3.24 (Pullback of a k-form)**.** Let $F : M \to N$ be a holomorphic map between complex manifolds. Then the $\mathbb{C}$-linear extension of the pullback map defined on the underlying smooth manifolds (Definition 1.45)

$$F^* : \Omega^k_{\mathbb{C}}(N) \to \Omega^k_{\mathbb{C}}(M)$$

is called the pullback of a complex k-form

Remark 3.32. Pullback of the identity map is an identity map, i.e. $(\mathbb{1}_M)^* = \mathbb{1}_{\Omega^k_{\mathbb{C}}(M)}$.

Proposition 3.9. *If $F : M \to N$ and $G : N \to N'$ are holomorphic maps between complex manifolds, then $(G \circ F)^* = F^* \circ G^*$.*

$$\begin{array}{ccc} \Omega^k_{\mathbb{C}}(N') & \xrightarrow{G^*} & \Omega^k_{\mathbb{C}}(N) \\ & \searrow^{(G\circ F)^*} & \downarrow F^* \\ & & \Omega^k_{\mathbb{C}}(M) \end{array}$$

Proposition 3.10. *Let $F : M \to N$ be a holomorphic map between complex manifolds. If ω is a differential form on N, then $F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega)$, i.e. the following diagram commutes*

$$\begin{array}{ccc} \Omega^k_{\mathbb{C}}(N) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}_{\mathbb{C}}(N) \\ \downarrow F^* & & \downarrow F^* \\ \Omega^k_{\mathbb{C}}(M) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}_{\mathbb{C}}(M) \end{array}$$

Theorem 3.6. *Let $F : M \to N$ be a holomorphic map between complex manifolds. Then the pullback of differential forms $F^* : \Omega^k_{\mathbb{C}}(N) \to \Omega^k_{\mathbb{C}}(M)$ induces natural $\mathbb{C}$-linear maps $F^* : \Omega^{p,q}(N) \to \Omega^{p,q}(M)$. These maps are compatible with ∂ and $\overline{\partial}$.*

Proof. If F is holomorphic then F^* is compatible with the decomposition [12, Proposition 2.6.10]

$$\Omega^k_{\mathbb{C}}(M) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

In particular, $F^*\left(\Omega^{p,q}(N)\right) \subset \Omega^{p,q}(M)$ and $\Pi^{p+1,q} \circ F^* = F^* \circ \Pi^{p+1,q}$. Thus, for $\omega \in \Omega^{p,q}(M)$ we have

$$\overline{\partial}\left(F^*\left(\omega\right)\right) = \Pi^{p+1,q}\left(\mathrm{d}\left(F^*\left(\omega\right)\right)\right) = \Pi^{p+1,q}\left(F^*\left(\mathrm{d}\left(\omega\right)\right)\right) = F^*\left(\Pi^{p+1,q}\left(\mathrm{d}\left(\omega\right)\right)\right) = F^*\left(\overline{\partial}\left(\omega\right)\right)$$

where, as usual, we are abusing the notations $\overline{\partial}$ and d. Analogously, we can show that $\partial \circ F^* = F^* \circ \partial$. ☐

3.3.2 Holomorphic differential forms

Definition 3.25 (Holomorphic vector bundle)**.** A *holomorphic vector bundle* of rank k is a triple (E, M, π) consisting of a pair of complex manifolds E and M, and a holomorphic surjective map $\pi : E \to M$ satisfying the following conditions

1. for each $w \in M$, the inverse image $E_w = \pi^{-1}(w)$ is an k-dimensional vector space over $\mathbb{C}$,

2. for each $w \in M$, there is an open neighborhood U of w and a biholomorphic map $\tau : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that
   (a) the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\tau\quad} & U \times \mathbb{C}^k \\ & \searrow^{\pi} \quad \swarrow_{p_1} & \\ & U & \end{array}$$

   where p_1 is the projection onto the first factor,

   (b) for each $v \in U$, the induced map $\tau_v : \pi^{-1}(v) \to \mathbb{C}^k$, defined by $\tau(z) = (v, \tau_v(z))$, is a $\mathbb{C}$-linear isomorphism.

Remark 3.33. We can also define it the way we defined the complex vector bundle in Remark 3.26. That is, we have biholomorphic local trivializations

$$\tau_\alpha : \pi_\alpha^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$$

such that the transition maps $\sigma_{\alpha\beta} = U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$ are holomorphic.

Definition 3.26 (Pullback of holomorphic vector bundle)**.** Let $f : M \to N$ be a holomorphic map between complex manifolds and let E be a holomorphic vector bundle on N given by transition maps $\sigma_{\alpha\beta}$ corresponding to an open cover $\{U_\alpha\}$. Then the *pullback* f^*E of E is the holomorphic vector bundle over M that is given by the transition maps $\sigma_{\alpha\beta} \circ f$ corresponding to an open cover $\{f^{-1}(U_\alpha)\}$.

Definition 3.27 (Holomorphic tangent bundle)**.** Let M be a complex manifold of dimension n which is covered by open sets U_α biholomorphic, via maps called ϕ_α, to open sets V_α of $\mathbb{C}^n$. Then the *holomorphic tangent bundle* $\mathcal{T}M$ of M is a holomorphic vector bundle of rank n with the transition maps $\sigma_{\alpha\beta}$ given by

$$\sigma_{\alpha\beta}(w) := \mathrm{Jac}(\phi_{\alpha\beta})(w) = \left[\left. \frac{\partial \phi^\ell_{\alpha\beta}}{\partial z_j} \right|_w \right]_{\substack{1 \le \ell \le n \\ 1 \le j \le n}}$$

is the Jacobian matrix at the point w (see Definition D.10).

Remark 3.34. In this definition, if we replace the complex manifold with the smooth manifold, and the holomorphic Jacobian matrix with the real Jacobian matrix, we will get the definition of smooth tangent bundle [34, §2.1.2]. This definition is equivalent to the one given earlier in Definition 1.31 using *derivations.*

Theorem 3.7. *If M is a complex manifold, then $(T_{\mathbb{R}}M)^{1,0}$ is naturally isomorphic (as a complex vector bundle) to the holomorphic tangent bundle $\mathcal{T}M$.*

Proof. Let $U, V \subset \mathbb{C}^n$ be open subsets and $f : U \to V$ be a biholomorphic map. Then by Proposition 3.2 we get the linear isomorphism

$$\tilde{f}_* : (T_{w,\mathbb{R}}U)^{1,0} \oplus (T_{w,\mathbb{R}}U)^{0,1} \to \left(T_{f(w),\mathbb{R}}V\right)^{1,0} \oplus \left(T_{f(w),\mathbb{R}}V\right)^{0,1}$$

Also, from Remark D.10 we know that

$$\tilde{f}_*(w) = \begin{bmatrix} \mathrm{Jac}(f)(w) & 0 \\ 0 & \overline{\mathrm{Jac}(f)(w)} \end{bmatrix}$$

Let $\{(U_\alpha, \phi_\alpha)\}$ be a holomorphic atlas of M, i.e. U_α is biholomorphic to $\phi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}^n$. Then $\left(\phi_\alpha^{-1}\right)^* \left((T_\mathbb{R} U_\alpha)^{1,0}\right) \cong (T_\mathbb{R} V_\alpha)^{1,0}$. With respect to the canonical trivialization the induced isomorphisms $\left(T_{\phi_\beta(w),\mathbb{R}} V_\beta\right)^{1,0} \cong \left(T_{\phi_\alpha(w),\mathbb{R}} V_\alpha\right)^{1,0}$ are given by the transition maps of $\mathcal{T}M$ [12, Definition 2.2.14, Proposition 2.6.4(ii)]. Therefore, both $(T_\mathbb{R} M)^{1,0}$ and $\mathcal{T}M$ are naturally isomorphic. □

Remark 3.35. We call the bundles $(T_\mathbb{R} M)^{1,0}$ and $(T_\mathbb{R} M)^{1,0}$ the *holomorphic* and *antiholomorphic* tangent bundle of the complex manifold M.

Definition 3.28 (Holomorphic cotangent bundle)**.** The *holomorphic cotangent bundle* $\mathcal{T}^*M$ is the dual of $\mathcal{T}M$. That is, for all $w \in M$ we have $\mathcal{T}_w^*M = \mathrm{Hom}_\mathbb{C}(\mathcal{T}_w M, \mathbb{C})$.

Definition 3.29 (Holomorphic p-forms)**.** Over M we consider the holomorphic vector bundle $\bigwedge^p \mathcal{T}^*M$ whose fiber is $\bigwedge^p \mathcal{T}_w^*M$. The holomorphic sections[23] of this vector bundle are called the *holomorphic p-forms* on M. The space of all holomorphic p-forms on M is denoted by $\mathcal{O}^p(M)$.

Remark 3.36. We note that holomorphic 0-forms on M are the holomorphic complex-valued functions on M, i.e. $\mathcal{O}^0(M) = \mathcal{O}(M)$. As in Remark 3.25, let $(U, r_1, \ldots, r_n)$ be a coordinate chart of M. Then the differentials $\{\mathrm{d}r_1, \ldots, \mathrm{d}r_n\}$ are 1-forms on U. At each point $w \in U$, the 1-forms $\{\mathrm{d}r_1\big|_w, \ldots, \mathrm{d}r_n\big|_w\}$ form a basis of $\bigwedge^1(\mathcal{T}_w^*M) = \mathcal{T}_w^*M$, dual to the basis $\{\partial/\partial r_1|_w, \ldots, \partial/\partial r_n|_w\}$ for the tangent space $\mathcal{T}_w M$. Hence, a 1-form on U is a linear combination $\omega = \sum_{\alpha=1}^n f_\alpha dr_\alpha$ where f_α are complex-valued holomorphic functions on U. In general[24], any holomorphic p-form $\omega \in \mathcal{O}^p(M)$ can be written uniquely as

$$\omega = \sum_{|\alpha|=p} f_\alpha \,\mathrm{d}r_\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ are multi-indices with $1 \le \alpha_j \le n$, $\mathrm{d}r_\alpha = \mathrm{d}r_{\alpha_1} \wedge \ldots \wedge \mathrm{d}r_{\alpha_p}$ and f_α is a complex-valued holomorphic function on U, i.e. $f_\alpha \in \mathcal{O}(U)$.

3.4 $\overline{\partial}$-closed and exact forms on complex manifolds

In this section some basic definitions and facts from [9, p. 25], [34, §2.3.3], [12, §2.6] and [15, §6.3] will be stated.

Definition 3.30 ($\overline{\partial}$-closed forms)**.** Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$ is called $\overline{\partial}$-closed if $\overline{\partial}\omega = 0$.

Remark 3.37. Given a complex manifold M, denote the set of all $\overline{\partial}$-closed (p,q)-forms on M by $\mathcal{Z}^{p,q}(M)$. The sum of two such (p,q)-forms is another $\overline{\partial}$-closed (p,q)-form, and so is the product of a $\overline{\partial}$-closed (p,q)-form by a scalar. Hence $\mathcal{Z}^{p,q}(M)$ is the vector sub-space of $\Omega^{p,q}(M)$. Also, if we write the elements of $\mathcal{Z}^{p,0}(M)$ is terms of local coordinates, then from Theorem D.2 it follows that it is the space of $(p,0)$-forms whose coefficients are complex-valued holomorphic functions in M, i.e. $\mathcal{O}^p(M) = \mathcal{Z}^{p,0}(M)$ by Remark 3.36. In particular, note that $\mathcal{Z}^{0,0}(M) = \mathcal{O}(M)$, the space of complex-valued functions holomorphic in M.

Definition 3.31 ($\overline{\partial}$-exact forms)**.** Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$, for $q > 0$, is called $\overline{\partial}$-exact if $\omega = \overline{\partial}\eta$ for some differential form $\eta \in \Omega^{p,q-1}(M)$.

[23] Replace "smooth" by "holomorphic" in Definition 1.38.

[24] In the case of $M = \mathbb{C}^n$ the expression was much more straightforward because $\mathcal{T}_w M \cong \mathbb{C}^n$ (vector space isomorphism) and we could replace r_j by z_j.

Remark 3.38. Given a complex manifold M, denote the set of all $\overline{\partial}$-exact (p,q)-forms on M by $\mathcal{B}^{p,q}(M)$. The sum of two such (p,q)-forms is another $\overline{\partial}$-exact (p,q)-form, and so is the product of a $\overline{\partial}$-exact (p,q)-form by a scalar. Hence $\mathcal{B}^{p,q}(M)$ is the vector sub-space of $\Omega^{p,q}(M)$. Moreover, the trivial form $\omega \equiv 0$ is the only $(p,0)$-form which is $\overline{\partial}$-exact for any value of $p = 0, 1, \ldots, n$. That is, $\mathcal{B}^{p,0}(M)$ consists only of zero.

Theorem 3.8. *On a complex manifold M, every $\overline{\partial}$-exact form is $\overline{\partial}$-closed.*

Proof. Let M be a complex manifold and $\omega \in \mathcal{B}^{p,q}(M)$ such that $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{p,q-1}(M)$. From Corollary 3.6 we know that $\overline{\partial}\omega = \overline{\partial}(\overline{\partial}\eta) = 0$ hence $\omega \in \mathcal{Z}^{p,q}(M)$ for all $q \geq 1$. For $q = 0$, the statement is trivially true. □

Lemma 3.4. *Let $F : M \to N$ be a holomorphic map of complex manifolds, then the pullback map F^* sends $\overline{\partial}$-closed forms to $\overline{\partial}$-closed forms, and $\overline{\partial}$-exact forms to $\overline{\partial}$-exact forms.*

Proof. Suppose ω is $\overline{\partial}$-closed. From Theorem 3.6 we know that F^* commutes with $\overline{\partial}$

$$\overline{\partial}F^*\omega = F^*\overline{\partial}\omega = 0$$

Hence, $F^*\omega$ is also $\overline{\partial}$-closed. Next suppose $\theta = \overline{\partial}\eta$ is $\overline{\partial}$-exact. Then

$$F^*\theta = F^*\overline{\partial}\eta = \overline{\partial}F^*\eta$$

Hence, $F^*\theta$ is $\overline{\partial}$-exact. □

3.4.1 Dolbeault cohomology

Definition 3.32 (Dolbeault cohomology of a complex manifold)**.** The $(p,q)^{th}$ *Dolbeault cohomology group*[25] of a complex manifold M is the quotient group

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$

Remark 3.39. Hence, the Dolbeault cohomology of a complex manifold measures the extent to which $\overline{\partial}$-closed forms are not $\overline{\partial}$-exact on that manifold.

Proposition 3.11. *If M is a complex manifold then its Dolbeault cohomology group in degree $(p,0)$ is the group of holomorphic p-forms on M.*

Proof. Since there are no non-zero $\overline{\partial}$-exact $(0,p)$-forms

$$H^{p,0}_{\overline{\partial}}(M) = \mathcal{Z}^{p,0}(M) = \mathcal{O}^p(M)$$

□

Remark 3.40. Though the definitions of de Rham and Dolbeault cohomology are similar, they measure different things. The de Rham cohomology is a topological invariant, whereas the Dolbeault cohomology measures the holomorphic complexity[26].

Proposition 3.12. *On a complex manifold M of dimension n, the Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(M)$ vanishes for $q > n$.*

Proof. It follows from the fact that if $q > n$ then $\bigwedge^{p,q}(T^*_{\mathbb{R}}M) = 0$. Hence for $q > n$, the only (p,q)-form on M is the zero form. □

[25] It is also a vector space over $\mathbb{C}$.

[26] Donu Arapura, "de Rham vs Dolbeault Cohomology", `https://mathoverflow.net/q/95432`, 28 April 2012.

3.4.2 $\overline{\partial}$-Poincaré lemma for complex manifolds

Definition 3.33 (Pullback map in cohomology)**.** Let $F : M \to N$ be a holomorphic map of complex manifolds, then its pullback F^* induces[27] a linear map of quotient spaces, denoted by $F^\#$

$$F^\# : \frac{\mathcal{Z}^{p,q}(N)}{\mathcal{B}^{p,q}(N)} \to \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$
$$[\![\omega]\!] \mapsto [\![F^*(\omega)]\!]$$

This is a map in cohomology $F^\# : H^{p,q}_{\overline{\partial}}(N) \to H^{p,q}_{\overline{\partial}}(M)$ called the *pullback map in cohomology.*

Remark 3.41. From Remark 3.32 and Proposition 3.9 it follows that:

1. If $\mathbb{1}_M : M \to M$ is the identity map, then $\mathbb{1}_M^\# : H^{p,q}_{\overline{\partial}}(M) \to H^{p,q}_{\overline{\partial}}(M)$ is also the identity map.
2. If $F : M \to N$ and $G : N \to N'$ are holomorphic maps, then $(G \circ F)^\# = F^\# \circ G^\#$.

Proposition 3.13 (Invariance of Dolbeault cohomology for biholomorphic manifolds)**.** *Let $F : M \to N$ be a biholomorphic map of manifolds, then the pullback map in cohomology $F^\# : H^{p,q}_{\overline{\partial}}(N) \to H^{p,q}_{\overline{\partial}}(M)$ is an isomorphism.*

Proof. Since F is a biholomorphic map, $F^{-1} : N \to M$ is also a holomorphic map of manifolds. Therefore, using Remark 3.41 we have

$$\mathbb{1}_{H^{p,q}_{\overline{\partial}}(M)} = \mathbb{1}_M^\# = (F^{-1} \circ F)^\# = F^\# \circ (F^{-1})^\#$$

This implies that $(F^{-1})^\#$ is the inverse of $F^\#$, i.e. $F^\#$ is an isomorphism. □

Theorem 3.9 ($\overline{\partial}$-Poincaré lemma for complex manifolds)**.** *Let M be a complex manifold, then for all $w \in M$ there exists an open neighborhood U such that every $\overline{\partial}$-closed (p, q)-form on U is $\overline{\partial}$-exact for $q \geq 1$.*

Proof. Let (U, ϕ) be a chart on the complex manifold M of dimension n such that $w \in U$. By Theorem 3.4 we know that the coordinate map $\phi : U \to \phi(U) \subset \mathbb{C}^n$ is biholomorphic. We choose U such that $\phi(U)$ is an open polydisc in $\mathbb{C}^n$. Then by Theorem 3.3 every $\overline{\partial}$-closed (p, q)-form on $\phi(U)$ is exact for $q \geq 1$, i.e. $H^{p,q}_{\overline{\partial}}(\phi(U)) = 0$ for $q \geq 1$. Now we can use Proposition 3.13 to conclude that $H^{p,q}_{\overline{\partial}}(U) = 0$ for $q \geq 1$, i.e. every $\overline{\partial}$-closed (p, q)-form on U is $\overline{\partial}$-exact for $q \geq 1$. □

[27] Follows from Lemma 3.4.

Chapter 4

Cousin problems

4.1 Cousin problems for $\mathbb{C}$

In this section some basic definitions and facts from [31, §1.6], [16, §13.1], [15, §0.3.4] and [3, §VII.5, VIII.3] will be stated.

4.1.1 Mittag-Leffler theorem

Consider the following problem:

> Let U be an open subset of $\mathbb{C}$ and $\{a_k\}$ be a sequence of distinct points in U such that $\{a_k\}$ has no limit points in U. For each integer $k \geq 1$ consider the rational function
> $$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_k)^j}$$
> where m_k is some positive integer and $A_{1k}, \ldots, A_{m_k k}$ are arbitrary complex coefficients. Is there a meromorphic function f on U whose poles are exactly the points $\{a_k\}$ and such that the singular part[1] of f at $z = a_k$ is $S_k(z)$?

The answer to this problem is yes and was solved by Gösta Mittag-Leffler during 1876-1884, building on the work of his mentor Karl Weierstrass [33]. Here we will discuss a proof which will illustrate the general method for solving the Cousin problems.

Theorem 4.1 (Single variable Cousin I). *Let $U \subset \mathbb{C}$ be an open set with an open covering $\{U_\alpha\}$. Suppose that for each U_α, U_β with nonempty intersection there is a holomorphic function $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ satisfying*

1. *$g_{\alpha\beta} + g_{\beta\alpha} = 0$ for each pair (α, β);*
2. *$g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).*

Then there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ for each α such that $g_{\alpha\beta} = f_\beta - f_\alpha$ on $U_\alpha \cap U_\beta$ whenever the intersection is nonempty.

Proof. As in Theorem A.3, let $\{V_k\}$ be a locally finite refinement of $\{U_\alpha\}$ and $\{\psi_k\}$ be a smooth partition of unity of U subordinate to the open cover $\{V_k\}$. Then for a fixed k, ψ_k has a compact

[1] Let f has a pole of order m at $z = a$ such that f has the Laurent series expansion in an open neighborhood V of a give by
$$f(z) = \frac{A_m}{(z-a)^m} + \ldots + \frac{A_{m-1}}{(z-a)^{m-1}} + \ldots + \frac{A_1}{(z-a)} + g(z)$$
where g is analytic in V and $A_m \neq 0$. Then $\sum_{j=1}^{m} \frac{A_j}{(z-a)^j}$ is called *singular part* or *principal part* of f at $z = a$.

support contained in $V_k \subset U_{r(k)}$. We can then define the smooth functions $\{h_{k\alpha}\}$ in the open sets $\{U_\alpha\}$ by

$$h_{k\alpha}(z) = \begin{cases} \psi_k(z) g_{r(k)\alpha}(z) & \text{if } z \in V_k \cap U_\alpha \\ 0 & \text{if } z \in U_\alpha \setminus (V_k \cap U_\alpha) \end{cases}$$

Since ψ_k vanishes in an open neighborhood of $U \setminus V_k$, ψ_k will also vanish in an open neighborhood of $U_\alpha \setminus (U_\alpha \cap V_k)$. Therefore, the function $\boxed{h_{k\alpha} = \psi_k g_{r(k)\alpha}}$ is a smooth function U_α, and for each α we have the smooth function

$$h_\alpha := \sum_k h_{k\alpha} \quad \text{on } U_\alpha$$

Then, on $U_\alpha \cap U_\beta$, using the properties of $\{g_{\alpha\beta}\}$ we get

$$\begin{aligned} h_\beta - h_\alpha &= \sum_k (h_{k\beta} - h_{k\alpha}) = \sum_k \psi_k \left(g_{r(k)\beta} - g_{r(k)\alpha}\right) \\ &= \sum_k \psi_k \left(-g_{\beta r(k)} - g_{r(k)\alpha}\right) = \sum_k \psi_k \left(g_{\alpha\beta}\right) = g_{\alpha\beta} \end{aligned}$$

since $\sum_k \psi_k = 1$. This gives us a smooth solution $\{h_\alpha\}$ to the first Cousin problem.

Next, since $g_{\alpha\beta}$ is holomorphic, by Theorem D.2 we have

$$\frac{\partial h_\alpha}{\partial \bar{z}} = \frac{\partial h_\beta}{\partial \bar{z}} \quad \text{on } U_\alpha \cap U_\beta$$

Hence there exists a function $h \in C^\infty(U)$ such that

$$h = \frac{\partial h_\alpha}{\partial \bar{z}} \quad \text{on } U_\alpha \text{ for each } \alpha \tag{4.1}$$

Also, from Theorem 3.2 we get $f \in C^\infty(U)$ such that

$$\frac{\partial f}{\partial \bar{z}} = h \tag{4.2}$$

Comparing (4.1) and (4.2) we get that

$$f_\alpha = h_\alpha - f \in \mathcal{O}(U_\alpha) \text{ for each } \alpha$$

Since $f_\beta - f_\alpha = g_{\alpha\beta}$, the set $\{f_\alpha\}$ is the required holomorphic solution to the Cousin problem. ☐

Theorem 4.2 (Mittag-Leffler theorem). *Let U be an open subset of $\mathbb{C}$ and $\{a_k\}$ be a sequence of distinct points in U such that $\{a_k\}$ has no limit points in U. For each integer $k \geq 1$ consider the rational function*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}$$

where m_k is some positive integer and $A_{1k}, \ldots, A_{m_k k}$ are arbitrary complex coefficients. Then there is a meromorphic function f on U whose poles are exactly the points $\{a_k\}$ and such that the singular part of f at $z = a_k$ is $S_k(z)$.

Proof. Choose an open cover $\{U_\alpha\}$ of U with the property that each U_α contains at most one point of $\{a_k\}$. Assign a meromorphic function h_α on U_α for each α such that $h_\alpha = S_k$ if U_α contains a_k, otherwise $f_\alpha \equiv 0$. We can then define the Cousin data for the cover $\{U_\alpha\}$ by setting

$$g_{\alpha\beta} = h_\beta - h_\alpha \quad \text{on } U_\alpha \cap U_\beta \tag{4.3}$$

Note that for each U_α, U_β with nonempty intersection $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ since there doesn't exist any pole $a_k \in U_\alpha \cap U_\beta$. Moreover, $\{g_{\alpha\beta}\}$ satisfies the conditions

1. $g_{\alpha\beta} + g_{\beta\alpha} = 0$ for each pair (α, β);

2. $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).

Therefore, by Theorem 4.1, there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ for each α such that

$$g_{\alpha\beta} = f_\beta - f_\alpha \quad \text{on } U_\alpha \cap U_\beta \tag{4.4}$$

Comparing (4.3) and (4.4) we get that

$$h_\beta - h_\alpha = f_\beta - f_\alpha \quad \text{on } U_\alpha \cap U_\beta$$

for each pair (α, β). Hence, we can define a meromorphic function f on U such that

$$f(z) = h_\alpha(z) - f_\alpha(z) \quad \text{for } z \in U_\alpha$$

for each α. Since subtracting a holomorphic function f_α from h_α doesn't affect the poles and singular parts, f is the desired meromorphic function on U whose poles are exactly the points $\{a_k\}$ and the singular part at $z = a_k$ is S_k. □

4.1.2 Weierstrass theorem

Consider the following problem:

> Let U be an open subset of $\mathbb{C}$ and $\{a_k\}$ be a sequence of distinct points in U such that $\{a_k\}$ has no limit points in U. Given a sequence of integers $\{m_k\}$, is there a function f which is holomorphic on U such that the only zeros of f are the points a_k with multiplicity m_k?

The answer to this problem is yes and was solved by Karl Weierstrass in 1876. Though this problem was solved before Mittag-Leffler theorem, we will deduce it from Cousin I following [16, Theorem 13.1.6].

Lemma 4.1. *Let $U \subset \mathbb{C}$ be simply connected open set and $f : U \to \mathbb{C}$ be a holomorphic and non-vanishing function.Then there is a holomorphic function g on U such that* $\exp(g) = f$.

Proof. This is a standard result in single variable complex analysis, see [3, Theorem VIII.2.2(g)] or [16, Lemma 13.1.5]. □

Theorem 4.3 (Single variable Cousin II)**.** *Let $U \subset \mathbb{C}$ be an open set with an open covering $\{U_\alpha\}$. Suppose that for each U_α, U_β with nonempty intersection there is a non-vanishing holomorphic function $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying*

1. *$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ for each pair (α, β);*

2. *$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).*

Then there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ for each α such that $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$ on $U_\alpha \cap U_\beta$ whenever the intersection is nonempty.

Proof. Let $\{V_j\}$ be a refinement of $\{U_\alpha\}$ such that for each j, V_j is an open ball and $V_j \subset U_{r(j)}$. Next, we define $h_{jk} : V_j \cap V_k \to \mathbb{C}$ by $h_{jk}(z) = g_{r(j)r(k)}(z)$. Then $\{h_{jk}\}$ is a set of holomorphic functions satisfying

1. $h_{jk} \cdot h_{kj} = 1$ for each pair (j, k);

2. $h_{jk} \cdot h_{k\ell} \cdot h_{\ell j} = 1$ on $V_j \cap V_k \cap V_\ell$ for each triple (j, k, ℓ).

`Step 1:` *There exist* $u_j \in \mathcal{O}^*(V_j)$ *for each* j *such that* $h_{jk} = u_k/u_j$ *on* $V_j \cap V_k$ *whenever the intersection is nonempty.*

Since each open ball V_j is simply connected, by Lemma 4.1, there exists $\tilde{h}_{jk} \in \mathcal{O}(V_j \cap V_k)$ such that $h_{jk} = \exp(\tilde{h}_{jk})$. Then $\{\tilde{h}_{jk}\}$ satisfies the condition of Cousin I data for the covering $\{V_j\}$, and by Theorem 4.1 there exist $\tilde{u}_j \in \mathcal{O}(V_j)$ for each j such that $\tilde{h}_{jk} = \tilde{u}_k - \tilde{u}_j$ on $V_j \cap V_k$ whenever the intersection is nonempty. Then the set $\{u_j\}$ for $u_j = \exp(\tilde{u}_j)$ is the required holomorphic solution to the Cousin problem.

`Step 2:` *There exist* $f_\alpha \in \mathcal{O}^*(U_\alpha)$ *for each* α *such that* $g_{\alpha\beta} = f_\beta/f_\alpha$ *on* $U_\alpha \cap U_\beta$ *whenever the intersection is nonempty.*

Note that, for $z \in U_\alpha \cap V_j \cap V_k$ we have

$$\begin{aligned}\left(\frac{u_k}{u_j} g_{r(k)\alpha} g_{\alpha r(j)}\right)(z) &= \left(\frac{u_k}{u_j} \frac{1}{g_{r(j)r(k)}}\right)(z) \\ &= \left(\frac{u_k}{u_j} g_{r(k)r(j)}\right)(z) \\ &= \left(\frac{u_k}{u_j} h_{kj}\right)(z) \\ &= 1\end{aligned}$$

Therefore, we have $u_k g_{r(k)\alpha}(z) = u_j g_{r(j)\alpha}(z)$ on $U_\alpha \cap V_j \cap V_k$. Since this is true for any pair (j,k), for any α we define non-vanishing holomorphic function $f_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$f_\alpha(z) = u_j g_{r(j)\alpha}(z) \quad \text{for } z \in U_\alpha \cap V_j$$

Finally, $\{f_\alpha\}$ is the required holomorphic solution to the Cousin problem since

$$\frac{f_\beta}{f_\alpha}(z) = \frac{u_j g_{r(j)\beta}}{u_j g_{r(j)\alpha}}(z) = \frac{1}{g_{\beta r(j)} g_{r(j)\alpha}}(z) = g_{\alpha\beta}(z) \quad \text{for } z \in U_\alpha \cap U_\beta \cap V_j$$

where j is arbitrary. □

Theorem 4.4 (Weierstrass theorem)**.** *Let* U *be an open subset of* $\mathbb{C}$ *and* $\{a_k\}$ *be a sequence of distinct points in* U *such that* $\{a_k\}$ *has no limit points in* U*. Given a sequence of integers* $\{m_k\}$*, there is a function* f *which is holomorphic on* U *such that the only zeros of* f *are the points* a_k *with multiplicity* m_k*.*

Proof. Choose an open cover $\{U_\alpha\}$ of U with the property that each U_α contains at most one point of $\{a_k\}$. Assign a holomorphic function h_α on U_α for each α such that $h_\alpha = (z - a_k)^{m_k}$ if U_α contains a_k, otherwise $h_\alpha \equiv 1$. We can then define the Cousin data for the cover $\{U_\alpha\}$ by setting

$$g_{\alpha\beta} = \frac{h_\beta}{h_\alpha} \quad \text{on } U_\alpha \cap U_\beta \tag{4.5}$$

Note that for each U_α, U_β with nonempty intersection $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ since there doesn't exist any zero $a_k \in U_\alpha \cap U_\beta$, and $\{g_{\alpha\beta}\}$ satisfies the conditions

1. $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ for each pair (α, β);

2. $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).

Therefore, by Theorem 4.3, there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ for each α such that

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta \tag{4.6}$$

Comparing (4.5) and (4.6) we get that

$$\frac{h_\beta}{h_\alpha} = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta$$

for each pair (α, β). Hence, we can define a holomorphic function f on U such that

$$f(z) = \frac{h_\alpha(z)}{f_\alpha(z)} \quad \text{for } z \in U_\alpha$$

for each α. Since dividing h_α by a non-vanishing holomorphic function f_α doesn't affect the zeros of h_α and their multiplicities, f is the desired holomorphic function on U whose only zeros are the points a_k with multiplicity m_k. □

Corollary 4.1. *Let $U \subset \mathbb{C}$ be any open set. Let $Y \subset U$ be a discrete set. Then there is a holomorphic function f on all of U such that $Y = \{z \in U : f(z) = 0\}$.*

4.2 Cousin problems for $\mathbb{C}^n$

In this section some basic definitions and facts from [15, §6.1] and [10, §I.E] will be stated.

4.2.1 Cousin I

Consider the following problem:

> Let $U \subset \mathbb{C}^n$ be an open set with an open covering $\{U_\alpha\}$. Suppose that for each U_α, U_β with nonempty intersection there is a holomorphic function $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ satisfying
>
> 1. $g_{\alpha\beta} + g_{\beta\alpha} = 0$ for each pair (α, β);
> 2. $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).
>
> Then does there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ for each α such that $g_{\alpha\beta} = f_\beta - f_\alpha$ on $U_\alpha \cap U_\beta$ whenever the intersection is nonempty?

The answer to this problem is yes when U is a polydisc. Moreover, in general, this is true when U is a domain of holomorphy[2], for details see [15, Proposition 6.1.8]. In fact, the solution to Cousin I is exactly same as the single variable case since in the theory of single variable holomorphic functions, every open set is a domain of holomorphy.

Theorem 4.5 (Cousin I for a polydisc). *Let $\Delta \subset \mathbb{C}^n$ be an open polydisc with an open covering $\{U_\alpha\}$. Suppose that for each U_α, U_β with nonempty intersection there is a holomorphic function $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ satisfying*

1. *$g_{\alpha\beta} + g_{\beta\alpha} = 0$ for each pair (α, β);*
2. *$g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).*

Then there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ for each α such that $g_{\alpha\beta} = f_\beta - f_\alpha$ on $U_\alpha \cap U_\beta$ whenever the intersection is nonempty.

[2] An open set $U \subset \mathbb{C}^n$ is called a *domain of holomorphy* is there doesn't exist non-empty open sets U_1, U_2 with U_2 connected, $U_2 \not\subset U_1$, $U_1 \subset U_2 \cap U$, such that for every holomorphic function f on U there is a holomorphic function f_2 on U_2 such that $h = h_2$ on U_1, see [15, §0.3.1] and [31, §2.5].

Proof. As in Theorem A.3, let $\{V_k\}$ be a locally finite refinement of $\{U_\alpha\}$ and $\{\psi_k\}$ be a smooth partition of unity of Δ subordinate to the open cover $\{V_k\}$. Then for a fixed k, ψ_k has a compact support contained in $V_k \subset U_{r(k)}$. We can then define the smooth functions $\{h_{k\alpha}\}$ in the open sets $\{U_\alpha\}$ by

$$h_{k\alpha}(z) = \begin{cases} \psi_k(z) g_{r(k)\alpha}(z) & \text{if } z \in V_k \cap U_\alpha \\ 0 & \text{if } z \in U_\alpha \setminus (V_k \cap U_\alpha) \end{cases}$$

Since ψ_k vanishes in an open neighborhood of $\Delta \setminus V_k$, ψ_k will also vanish in an open neighborhood of $U_\alpha \setminus (U_\alpha \cap V_k)$. Therefore, the function $\boxed{h_{k\alpha} = \psi_k g_{r(k)\alpha}}$ is a smooth function U_α, and for each α we have the smooth function

$$h_\alpha = \sum_k h_{k\alpha} \quad \text{on } U_\alpha$$

Then, on $U_\alpha \cap U_\beta$, using the properties of $\{g_{\alpha\beta}\}$ we get

$$\begin{aligned} h_\beta - h_\alpha &= \sum_k (h_{k\beta} - h_{k\alpha}) = \sum_k \psi_k \left(g_{r(k)\beta} - g_{r(k)\alpha}\right) \\ &= \sum_k \psi_k \left(-g_{\beta r(k)} - g_{r(k)\alpha}\right) = \sum_k \psi_k \left(g_{\alpha\beta}\right) = g_{\alpha\beta} \end{aligned}$$

since $\sum_k \psi_k = 1$. This gives us a smooth solution $\{h_\alpha\}$ to the first Cousin problem.

Next, for each set U_α consider the differential form $\omega_\alpha \in \overline{\partial} h_\alpha \in \Omega^{0,1}(U_\alpha)$. In each intersection $U_\alpha \cap U_\beta$ we note that $\omega_\alpha = \overline{\partial}(h_\beta + g_{\alpha\beta}) = \omega_\beta$, since $g_{\alpha\beta}$ are holomorphic functions. Hence there exists a global differential form $\omega \in \Omega^{p,q}(\Delta)$ such that

$$\omega = \overline{\partial} h_\alpha \quad \text{on } U_\alpha \text{ for each } \alpha \tag{4.7}$$

Also, since $\overline{\partial}\omega = 0$, from Theorem 3.3 we get $f \in \Omega^{0,0}(\Delta) = C^\infty(\Delta)$ such that

$$\overline{\partial} f = \omega \tag{4.8}$$

Comparing (4.7) and (4.8) we get that

$$f_\alpha = h_\alpha - f \in \mathcal{O}(U_\alpha) \text{ for each } \alpha$$

Since $f_\beta - f_\alpha = g_{\alpha\beta}$, the set $\{f_\alpha\}$ is the required holomorphic solution to the Cousin problem. ☐

4.2.2 Cousin II

Consider the following problem:

> Let $U \subset \mathbb{C}^n$ be an open set with an open covering $\{U_\alpha\}$. Suppose that for each U_α, U_β with nonempty intersection there is a non-vanishing holomorphic function $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying
>
> 1. $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ for each pair (α, β);
> 2. $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).
>
> Then does there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ for each α such that $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$ on $U_\alpha \cap U_\beta$ whenever the intersection is nonempty?

The answer to this problem is yes when U is a polydisc. However, in general, this is not true when U is any domain of holomorphy. Unlike the single variable case, Cousin I doesn't imply Cousin II for $n \geq 2$. For the counterexample given by Kiyoshi Oka, see [15, pp. 250-253].

Lemma 4.2. *Let $U \subset \mathbb{C}^n$ be simply connected open set and $f : U \to \mathbb{C}$ be a holomorphic and non-vanishing function. Then there is a holomorphic function g on U such that* $\exp(g) = f$.

Proof. Since this is a topological fact, we are able to generalize the proof of Lemma 4.1. For details, see [16, Lemma 13.1.5] and [15, Lemma 6.1.10]. □

Theorem 4.6 (Cousin II for a polydisc)**.** *Let $\Delta \subset \mathbb{C}^n$ be an open polydisc with an open covering $\{U_\alpha\}$. Suppose that for each U_α, U_β with nonempty intersection there is a non-vanishing holomorphic function $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying*

1. $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ *for each pair* (α, β)*;*

2. $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ *on* $U_\alpha \cap U_\beta \cap U_\gamma$ *for each triple* (α, β, γ)*.*

Then there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ for each α such that $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$ on $U_\alpha \cap U_\beta$ whenever the intersection is nonempty.

Instead of proving this theorem[3], we will directly prove the generalization of Corollary 4.1 in the next section.

4.3 Cousin problem for analytic hypersurface in $\mathbb{C}^n$

Consider the following problem:

> Is any analytic subvariety Y of a complex manifold M the zero-locus of some global holomorphic functions defined on M?

The answer to this problem is yes when Y is a hypersurface and M is $\mathbb{C}^n$.

4.3.1 Analytic subvariety of a complex manifold

In this subsection some definitions and properties from [12, §2.1, 2.3] and [6, §I.8, IV.1] will be discussed.

Definition 4.1 (Analytic subvariety)**.** Let M be a n-dimensional complex manifold. An *analytic subvariety* of M is a closed subset $Y \subset M$ such that for every point $w \in Y$ there exists an open neighborhood $w \in U \subset M$ and $f_1, \ldots, f_m \in \mathcal{O}(U)$ with

$$U \cap Y = \{z \in U : f_j(z) = 0 \text{ for } j = 1, \ldots, m\}$$

Remark 4.1. A more natural definition of an analytic subvariety of M is that it is a subset $Y \subset M$ such that for every point $w \in M$ there exists an open neighborhood $w \in U \subset M$ and $f_1, \ldots, f_m \in \mathcal{O}(U)$ with

$$U \cap Y = \{z \in U : f_j(z) = 0 \text{ for } j = 1, \ldots, m\}$$

This definition is equivalent to the earlier one because we can prove that $w \in M \setminus Y$ if and only if Y is a closed subset of M [6, p. 36].

Definition 4.2 (Analytic hypersurface)**.** An analytic subvariety Y of M is called *analytic hypersurface* if we can always take $m = 1$, i.e. for every point $w \in Y$ there exists an open neighborhood $w \in U \subset M$ and $f \in \mathcal{O}(U)$ with

$$U \cap Y = \{z \in U : f(z) = 0\}$$

[3]For an outline of the proof, see [10, pp. 33-36]. Here, unlike the single variable case, we also need to show the existence of non-vanishing continuous solution before proving the existence of non-vanishing smooth solution. For details, see [15, Proposition 6.1.11(Part I)].

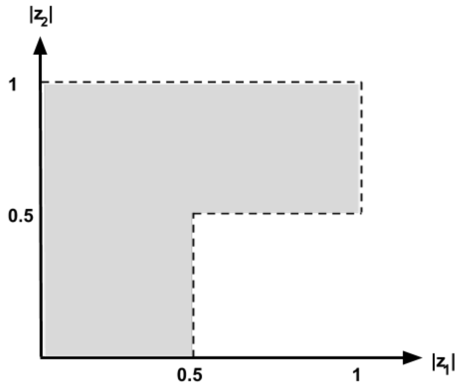
Remark 4.2. In general, analytic subvariety cannot be given by global equations. For example, if M is compact and connected, there are no non-constant holomorphic functions on M. For an example in which the ambient manifold M is not compact, consider the complex manifold $M := U_1 \cup U_2$ with

$$U_1 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < \frac{1}{2} \text{ and } |z_2| < 1 \right\}$$
$$U_2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \text{ and } \frac{1}{2} < |z_2| < 1 \right\}$$

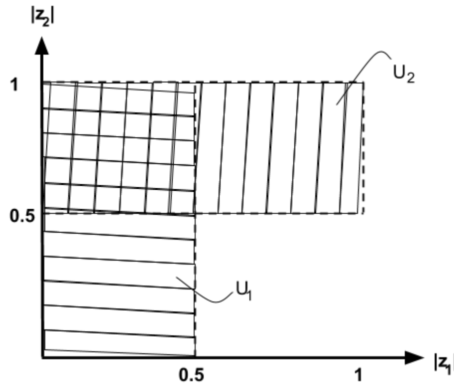
Next, consider the closed subset[4] $Y = \{(z_1, z_2) \in U_2 : z_1 = z_2\} \subset M$. Note that U_1 and U_2 give an open covering of M with $Y \subset U_2$, i.e. for all $p \in Y$ we can use U_2 since $Y \cap U_2 = \{(z_1, z_2) \in U_2 : f(z_1, z_2) = z_1 - z_2 = 0\}$ where $f \in \mathcal{O}(U_2)$. Therefore, Y is an analytic hypersurface of M.

`Claim:` There does not exist $g \in \mathcal{O}(M)$ such that $Y = \{(z_1, z_2) \in M : g(z_1, z_2) = 0\}$.

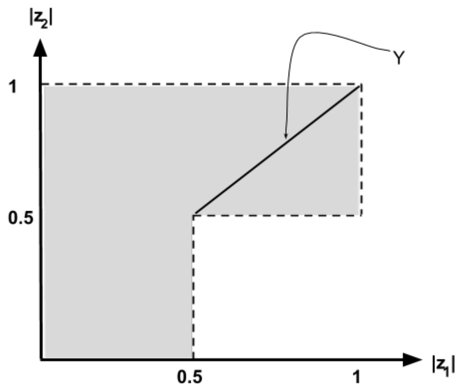
On the contrary, let there exist $g \in \mathcal{O}(M)$ such that g vanishes exactly on Y. Note that $M \subset \Delta(0; 1)$. Hence, by Theorem D.4, there exists $G \in \mathcal{O}(\Delta(0; 1))$ such that $G|_M = g$. In particular, for $z_1 = z_2 = z$, $G(z, z) = h(z)$ is a single variable holomorphic functions which vanishes for $\frac{1}{2} < |z| < 1$ in M. Since zero function is the only single variable holomorphic function with uncountably many zeros, $h(z)$ vanishes for $0 \leq |z| < 1$ in $\Delta(0; 1)$, i.e. $G|_M = g$ vanishes on $Z = \{(z_1, z_1) \in M : z_1 = z_2\} \supset Y$. Contradiction.



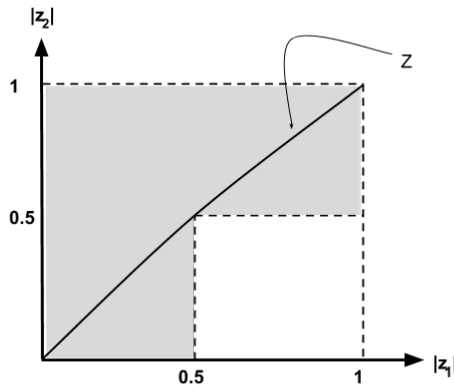
The region corresponding to M



The regions corresponding to U_1 and U_2.



The analytic subvariety Y of X



The vanishing set Z of G in M

[4]Recall that for a continuous function the inverse image of a closed set is closed. In particular, the set of zeros of a continuous function is closed.

4.3.2 Sheaf theory and Čech cohomology

In this subsection we will revisit the results from sheaf theory and Čech cohomology that were discussed in section 2.1 and section 2.2.

Example 4.1 (Sheaves on complex manifold)**.** In Example 2.4 we saw that if one has a presheaf of functions (or forms) on a topological space M which is defined by some local property, then the presheaf is also a sheaf. In particular, if M is a complex manifold then:

- $\mathcal{O}$ is the sheaf of holomorphic functions on M such that for every open subset U of M we have the additive abelian group $\mathcal{O}(U)$ of holomorphic functions on U along with the natural restriction maps as the group homomorphisms for the nested open subsets.
- $\mathcal{O}$ is the sheaf of non-vanishing holomorphic functions on M such that for every open subset U of M we have the multiplicative abelian group $\mathcal{O}^*(U)$ of non-vanishing holomorphic functions on U along with the natural restriction maps as the group homomorphisms for the nested open subsets.
- $\Omega^{p,q}$ is the sheaf of complex (p,q)-forms on M such that for every open subset U of M we have the additive abelian group $\Omega^{p,q}(U)$ of smooth (p,q)-forms on U (smooth sections of a exterior power of a vector bundle, i.e. smooth maps of manifolds) along with the natural restriction maps as the group homomorphisms for the nested open subsets.
- $\mathcal{O}^p$ is the sheaf of holomorphic p-forms on M such that for every open subset U of M we have the additive abelian group $\mathcal{O}^p(U)$ of holomorphic p-forms on U (holomorphic sections of an exterior power of holomorphic cotangent bundle, i.e. holomorphic maps of manifolds) along with the natural restriction maps as the group homomorphisms for the nested open subsets.

Example 4.2 (Sheaf maps)**.** Recall that a sheaf map is collection of group homomorphisms which commute with the restriction maps. Then for a complex manifold M we have:

- The exponential map $\exp : \mathcal{O} \to \mathcal{O}^*$ defined by the collection of group homorphisms $\{\exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U)\}_{U \subset M}$ where $\exp_U(f) = \exp(f)$ is defined via charts. This is a sheaf map since for $U \subset V \subset M$, $\exp_U$ and $\exp_V$ commute with the restriction maps.
- In Remark 1.29 we saw that the exterior derivative is a local operator, it commutes with restriction. Therefore, as in Example 2.6, $\mathrm{d} : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}}$ is a map of sheaves. In particular, $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ is a sheaf map between the sheaf of complex differential forms on a complex manifold M.

Example 4.3 (Kernel sheaf)**.** For a complex manifold M we have the sheaf of closed (p,q)-forms on M given by $\ker(\overline{\partial}) = \mathcal{Z}^{p,q}$ corresponding to the sheaf map $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$. In particular, $\mathcal{Z}^{p,0} = \mathcal{O}^p$ is the sheaf of holomorphic p-forms on M.

Example 4.4 (Exact sequence of sheaves)**.** For a complex manifold M we have:

- The short exact sequence, called *exponential sheaf sequence*

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

  Note that the sheaf map exp is surjective by Lemma 4.2, since locally M is biholomorphic to an open set in $\mathbb{C}^n$ and for every point $w \in \mathbb{C}^n$ we can find a simply connected open neighborhood U such that every $f \in \mathcal{O}^*(U)$ can be written as $\exp(g) = f$ for some $g \in \mathcal{O}(U)$.

- The exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \Omega^{p,2} \xrightarrow{\bar\partial} \cdots$$

where the exactness follows from Theorem 3.8, Theorem 3.9, and Remark 3.37.

Remark 4.3 (Long exact sequence of Čech cohomology)**.** By Remark 3.24 we know that complex manifolds are paracompact. Hence we can use Serre's theorem (Theorem 2.1) to get the long exact sequence of Čech chohomology corresponding to a short exact sequence of sheaves of a complex manifold.

- The exponential sheaf sequence on M will induce the following long exact sequence of cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(M,\mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(M,\mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M,\underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(M,\mathcal{O}) \longrightarrow \cdots$$

- Using Example 4.3 and Example 4.4 we get the following short exact sequence of sheaves on a complex manifold M

$$0 \longrightarrow \mathcal{Z}^{p,\ell} \hookrightarrow \Omega^{p,\ell} \xrightarrow{\bar\partial} \mathcal{Z}^{p,\ell+1} \longrightarrow 0$$

for every $\ell \geq 0$. This induces the following long exact sequence of cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(M,\Omega^{p,\ell}) \longrightarrow \check{\mathrm{H}}^q(M,\mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M,\mathcal{Z}^{p,\ell}) \longrightarrow \check{\mathrm{H}}^{q+1}(M,\Omega^{p,\ell}) \longrightarrow \cdots$$

for each ℓ.

Remark 4.4 (Fine sheaves)**.** Note that, for $p,\ell \geq 0$, $\Omega^{p,\ell}$ are smooth sections of vector bundles and hence are fine sheaves. Therefore we can use Theorem 2.2 to get $\check{\mathrm{H}}^q(M,\Omega^{p,\ell}) = 0$ for all $q \geq 1$.

Theorem 4.7 (Homotopy invariance of Čech cohomology)**.** *Let M and N be two smooth manifolds, and assume that $f : M \to N$ is a homotopy equivalence. If $\underline{G}$ is a constant sheaf on N, then $\check{\mathrm{H}}^q(M, f^{-1}\underline{G}) \cong \check{\mathrm{H}}^q(N,\underline{G})$ for all $q \geq 0$. In other words, the Čech cohomology of locally constant sheaves on smooth manifolds is a homotopy invariant.*

Proof. From Theorem 2.1 and Theorem 2.2 it follows that Čech cohomology of a manifold is isomorphic to its sheaf cohomology [35, §5.18, 5.33]. Moreover, it is a well known fact that sheaf cohomology of locally constant sheaves is a homotopy invariant [36, §10.2, 11.3]. Therefore, sheaf cohomology of locally constant sheaves is a homotopy invariant [29, §6.3]. □

Corollary 4.2. *If a smooth manifold M is contractible and $\underline{G}$ is a constant sheaf on M, then $\check{\mathrm{H}}^0(M,\underline{G}) \cong \underline{G}(M)$ and $\check{\mathrm{H}}^q(M,\underline{G}) \cong 0$ for $q > 0$.*

Proof. Since $M \simeq \{*\}$, for some point $* \in M$, we know that $\check{\mathrm{H}}^q(M,\underline{G}) \cong \check{\mathrm{H}}^q\left(\{*\}, f^{-1}\underline{G}\right)$ for all $q \geq 0$. From Proposition 2.6, we know that $\check{\mathrm{H}}^0(M,\underline{G}) \cong \underline{G}(M)$. Therefore, we just need to show that $\check{\mathrm{H}}^q(\{*\},\underline{G}) = 0$ for $q > 0$. Fortunately, when we calculate Čech cohomology of a point we don't need to take direct limit because the system is trivial, i.e. there is only one covering with only one open subset:

$$\check{\mathrm{H}}^q(\{*\},\underline{G}) = \frac{\check{Z}^q(\{\{*\}\},\underline{G})}{\check{B}^q(\{\{*\}\},\underline{G})} = \frac{\ker\{\delta : \check{\mathrm{C}}^q(\{\{*\}\},\underline{G}) \to \check{\mathrm{C}}^{q+1}(\{\{*\}\},\underline{G})\}}{\mathrm{im}\{\delta : \check{\mathrm{C}}^{q-1}(\{\{*\}\},\underline{G}) \to \check{\mathrm{C}}^q(\{\{*\}\},\underline{G})\}}$$

Note that $\check{\mathrm{C}}^q(\{\{*\}\}, \underline{G}) = \{f | f : \{*\} \to \underline{G}$ is a constant map$\}$. Hence for $q > 0$ we have

$$\check{Z}^q(\{\{*\}\}, \underline{G}) = \begin{cases} \check{\mathrm{C}}^q(\{\{*\}\}, \underline{G}) & \text{if } q \text{ is odd} \\ \{f | f \equiv 0 \text{ where 0 is the identity element of } G\} & \text{if } q \text{ is even} \end{cases}$$

and

$$\check{B}^q(\{\{\{*\}\}, \underline{G}) = \begin{cases} \{f | f \equiv 0 \text{ where 0 is the identity element of } G\} & \text{if } q-1 \text{ is odd} \\ \check{\mathrm{C}}^q(\{\{*\}\}, \underline{G}) & \text{if } q-1 \text{ is even} \end{cases}$$

Therefore, $\check{Z}^q(\{\{*\}\}, \underline{G}) = \check{B}^q(\{\{*\}\}, \underline{G})$ for all $q > 0$. Hence completing the proof. ☐

Remark 4.5. In section 2.3 we proved de Rham-Čech isomorphism, which says that if M is a smooth manifold then for each $k \geq 0$ there exists a group isomorphism $H^k_{dR}(M) \cong \check{\mathrm{H}}^k(M, \underline{\mathbb{R}})$. By the above theorem we can conclude that de Rham cohomology is in fact a homotopy invariant.

4.3.3 Dolbeault isomorphism

In this subsection we will prove Dolbeault's theorem, following [15, §6.3] and [9, p. 45]. This is a complex analogue of de Rham's theorem (Theorem 2.3), and asserts that the Dolbeault cohomology is isomorphic to the Čech cohomology of the sheaf of holomorphic differential forms.

Theorem 4.8. *Let M be a complex manifold. Then for each $p, q \geq 0$ there exists a group isomorphism*

$$H^{p,q}_{\overline{\partial}}(M) \cong \check{\mathrm{H}}^q(M, \mathcal{O}^p)$$

Proof. For $q = 0$, from Proposition 3.11 and Proposition 2.6, we know that both $H^{p,0}_{\overline{\partial}}(M)$ and $\check{\mathrm{H}}^0(M, \mathcal{O}^p)$ are isomorphic to the group of holomorphic p-forms on M. That is

$$\boxed{H^{p,0}_{\overline{\partial}}(M) \cong \check{\mathrm{H}}^0(M, \mathcal{O}^p)}$$

Now let's restrict our attention to $q \geq 1$. From Example 4.4 we know that the $\overline{\partial}$-Poincaré lemma implies the existence of the following long exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

Then, as noted in Remark 4.3, we have a family of short exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^p & \hookrightarrow & \Omega^{p,0} & \xrightarrow{\overline{\partial}} & \mathcal{Z}^{p,1} & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathcal{Z}^{p,1} & \hookrightarrow & \Omega^{p,1} & \xrightarrow{\overline{\partial}} & \mathcal{Z}^{p,2} & \longrightarrow & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & \mathcal{Z}^{p,\ell} & \hookrightarrow & \Omega^{p,\ell} & \xrightarrow{\overline{\partial}} & \mathcal{Z}^{p,\ell+1} & \longrightarrow & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

which will induce the respective long exact sequences of Čech cohomology

$$\begin{array}{l} \cdots \longrightarrow \check{\mathrm{H}}^q(M, \Omega^{p,0}) \longrightarrow \check{\mathrm{H}}^q(M, \mathcal{Z}^{p,1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M, \mathcal{O}^p) \longrightarrow \check{\mathrm{H}}^{q+1}(M, \Omega^{p,0}) \longrightarrow \cdots \\ \cdots \longrightarrow \check{\mathrm{H}}^q(M, \Omega^{p,1}) \longrightarrow \check{\mathrm{H}}^q(M, \mathcal{Z}^{p,2}) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M, \mathcal{Z}^{p,1}) \longrightarrow \check{\mathrm{H}}^{q+1}(M, \Omega^{p,1}) \longrightarrow \cdots \\ \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \cdots \longrightarrow \check{\mathrm{H}}^q(M, \Omega^{p,\ell}) \longrightarrow \check{\mathrm{H}}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow \check{\mathrm{H}}^{q+1}(M, \Omega^{p,\ell}) \longrightarrow \cdots \\ \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}$$

Now let's study one of these long exact sequence of Čech cohomology. By Proposition 2.6 we have $\check{\mathrm{H}}^0(M, \Omega^{p,\ell}) \cong \Omega^{p,\ell}(M)$ and $\check{\mathrm{H}}^0(M, \mathcal{Z}^{p,\ell}) \cong \mathcal{Z}^{p,\ell}(M)$. Also by Remark 4.4 we have $\check{\mathrm{H}}^q(M, \Omega^{p,\ell}) = 0$ for all $q \geq 1$ and $\ell \geq 0$. Hence for any $\ell \geq 0$ we get the exact sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0 \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell+1})$$

$$\big\downarrow \Delta$$

$$\cdots \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^3(M, \mathcal{Z}^{p,\ell}) \xleftarrow{\Delta} \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell+1}) \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell})$$

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

Since this sequence is exact, the map $\Delta : \mathcal{Z}^{p,\ell+1}(M) \to \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell})$ is a surjective group homomorphism and $\mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \mathcal{Z}^{p,\ell+1}(M)\} = \ker(\Delta)$. Hence by the *first isomorphism theorem* we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong \frac{\mathcal{Z}^{p,\ell+1}(M)}{\ker(\Delta)} \quad \text{for all } \ell \geq 0$$

Since $\mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \mathcal{Z}^{p,\ell+1}(M)\} = \mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \Omega^{p,\ell+1}(M)\} = \mathcal{B}^{p,\ell+1}(M)$, we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong H^{p,\ell+1}_{\overline{\partial}}(M) \quad \text{for all } \ell \geq 0 \tag{4.9}$$

Note that $\mathcal{Z}^{p,0} = \mathcal{O}^p$, hence from (4.9) we get

$$\boxed{\check{\mathrm{H}}^1(M, \mathcal{O}^p) \cong H^{p,1}_{\overline{\partial}}(M)}$$

Next we consider the remaining parts of the long exact sequence, i.e. for $q \geq 1$ and $\ell \geq 0$ we have

$$0 \longrightarrow \check{\mathrm{H}}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{\mathrm{H}}^{q+1}(M, \mathcal{Z}^{p,\ell}) \cong \check{\mathrm{H}}^q(M, \mathcal{Z}^{p,\ell+1}) \quad \text{for all } q \geq 1, \ell \geq 0 \tag{4.10}$$

Again substituting $\mathcal{Z}^{p,0} = \mathcal{O}^p$ and restricting our attention to $q \geq 2$, we apply (4.10) recursively to get

$$\begin{aligned}
\check{\mathrm{H}}^q(M, \mathcal{O}^p) &\cong \check{\mathrm{H}}^{q-1}\left(M, \mathcal{Z}^{p,1}\right) \\
&\cong \check{\mathrm{H}}^{q-2}\left(M, \mathcal{Z}^{p,2}\right) \\
&\;\vdots \\
&\cong \check{\mathrm{H}}^{1}\left(M, \mathcal{Z}^{p,q-1}\right)
\end{aligned}$$

Then using (4.9) we get

$$\boxed{\check{\mathrm{H}}^q(M, \mathcal{O}^p) \cong H^{p,q}_{\overline{\partial}}(M)} \quad \text{for all } q \geq 2$$

Hence completing the proof. □

4.3.4 Solution of the problem

We can now solve the Cousin problem, following the solution outlined in [9, p. 47].

Lemma 4.3. $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ *for* $q > 0$.

Proof. Consider the long exact sequence associated to the exponential sheaf sequence on $\mathbb{C}^n$

$$\cdots \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

By the $\overline{\partial}$-Poincaré lemma (Theorem 3.3), we get $H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for all $p \geq 0$ and $q > 0$. Then using Dolbeault isomorphism (Theorem 4.8) for $p = 0$, we get $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) = 0$ for $q > 0$. Moreover, since $\mathbb{C}^n$ is contractible, we can use Corollary 4.2 to get $\check{\mathrm{H}}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0$ for $q > 0$. Substituting these in the sequence and using exactness, we conclude that $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$. □

Theorem 4.9. *Any analytic hypersurface in* $\mathbb{C}^n$ *is the zero locus of an entire function* $f : \mathbb{C}^n \to \mathbb{C}$.

Proof. Let H be the analytic hypersurface in $\mathbb{C}^n$, then $H \subset \mathbb{C}^n$ such that for every point $w \in \mathbb{C}^n$ there exists an open neighborhood $w \in U \subset \mathbb{C}^n$ and $f \in \mathcal{O}(U)$ with

$$U \cap H = \{z \in U : h(z) = 0\}$$

By Theorem D.7 we know that $\mathcal{O}_w$ is a unique factorization domain. Therefore, if h is a representative element of the equivalence classes in $\mathcal{O}_w$, then $h = h_1 \cdots h_k$ for some irreducible representative functions in $\mathcal{O}_w$. Hence we can choose h such that it is not divisible by the square of any non-unit[5] in $\mathcal{O}_w$.

Next, choose an open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{C}^n$ and functions $h_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$U_\alpha \cap H = \{z \in U : h_\alpha(z) = 0\}$$

where h_α is not divisible by the square of any non-unit. We can then define the Cousin data for the cover $\mathcal{U} = \{U_\alpha\}$ by setting

$$g_{\alpha\beta} = \frac{h_\beta}{h_\alpha} \quad \text{on } U_\alpha \cap U_\beta \tag{4.11}$$

Note that for each U_α, U_β with nonempty intersection $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ since[6] h_α and h_β vanish at the same points in $U_\alpha \cap U_\beta$, and $\{g_{\alpha\beta}\}$ satisfies the conditions

1. $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ for each pair (α, β);
2. $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for each triple (α, β, γ).

Therefore, $(g_{\alpha\beta}) \in \check{Z}^1(\mathcal{U}, \mathcal{O}^*)$. But since $\check{\mathrm{H}}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$ by Lemma 4.3, after some refinement of $\mathcal{U}$ if necessary[7], there exists a cochain $(f_\alpha) \in \check{\mathrm{C}}^0(\mathcal{U}, \mathcal{O}^*)$ such that

$$g_{\alpha\beta} = \delta(f_\alpha) = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta \tag{4.12}$$

[5]Recall that the non-vanishing functions at $w \in \mathbb{C}^n$ are the unit elements in $\mathcal{O}_w$.

[6]We can prove this by contradiction. On the contrary assume that there exists $z \in U_\alpha \cap U_\beta$ such that $h_\alpha(z) = 0$ but $h_\beta(z) \neq 0$. Then $z \in U_\alpha \cap H$ but $z \notin U_\beta \cap H$. Which contradicts our assumption that $z \in U_\alpha \cap U_\beta$.

[7]If we need a refinement $\mathcal{V}$ of $\mathcal{U}$, then just start whole argument with the open cover $\mathcal{V}$ instead of $\mathcal{U}$.

Comparing (4.11) and (4.12) we get that

$$\frac{h_\beta}{h_\alpha} = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta$$

for each pair (α, β). Hence, we can define a global holomorphic function f on $\mathbb{C}^n$ such that

$$f(z) = \frac{h_\alpha(z)}{f_\alpha(z)} \quad \text{for } z \in U_\alpha$$

for each α. Since dividing h_α by a non-vanishing holomorphic function f_α doesn't affect the vanishing set of h_α, f is the desired holomorphic function on $\mathbb{C}^n$ whose vanishing set is H. $\square$

Future work

In the Remark 3.26 and Remark 3.33 we noted that transition maps can be used to define vector bundles. Following is the more precise statement:

Theorem 29. *If M be a smooth manifold and $\pi : E \to M$ is a complex*[8] *vector bundle of rank k. Then there exists an open cover $\{U_\alpha\}$ of M and a collection of smooth transition maps $\{\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})\}$ such that*

1. $\sigma_{\alpha\alpha} = I_k$
2. $\sigma_{\alpha\beta} \cdot \sigma_{\beta\gamma} \cdot \sigma_{\gamma\alpha} = I_k$

where I_k is a $k \times k$ identity matrix. This collection $\{g_{\alpha\beta}\}$ is called transition data. *Conversely, given an open cover $\{U_\alpha\}$ of M and a collection of smooth maps $\{\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})\}$ satisfying the above two conditions, there exists a complex rank k vector bundle $\pi' : E' \to M$ whose transition data is given by $\{\sigma_{\alpha\beta}\}$. Moreover, these two processes are well-defined and are inverses of each other when applied to the set of* equivalence classes of vector bundles[9] *and the set of* equivalence classes of transition data[10].

Now, if we use this result to define vector bundles using transition data, then we get the following [37, Lemma III.4.4]:

Theorem 30. *There is one-to-one correspondence between the equivalence classes of holomorphic line bundles on a complex manifold M and the elements of the cohomology group $\check{\mathrm{H}}^1(M, \mathcal{O}^*)$ where $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic functions.*

Also, by considering the underlying complex vector bundle of rank 1, we get:

Theorem 31. *There is one-to-one correspondence between the equivalence classes of complex line bundles on a smooth manifold M and the elements of the cohomology group $\check{\mathrm{H}}^1(M, \mathcal{E}^*)$ where $\mathcal{E}^*$ is the sheaf of non-vanishing smooth functions.*

We can generalize this result by generalizing the definition of Čech cohomology. In section 2.2 we defined Čech cohomology for a sheaf of abelian groups. Note that we can't define Čech cohomology in a similar way if $\mathcal{F}$ is a sheaf of non-abelian groups, since $\delta \circ \delta \neq 0$ if the sheaf is not abelian. However, we have the following general definition of the zeroth and first Čech cohomolgy [27, Remark 5.5(2)]:

(a). $\check{\mathrm{H}}^0(X, \mathcal{F}) := \mathcal{F}(X)$

[8]Same argument is valid for smooth and holomorphic vector bundles. For the case of smooth vector bundles, replace $\mathbb{C}$ by $\mathbb{R}$, and for the case of holomorphic vector bundles consider holomorphic transition maps and holomorphic isomorphism of vector bundles.

[9]Two vector bundles over M are said to be equivalent if they are isomorphic as vector bundles over M.

[10]Two sets of transition data $\{\sigma_{\alpha\beta}\}$ and $\{\sigma'_{\alpha\beta}\}$ are said to be equivalent if there exists a collection of smooth functions $\{\rho_\alpha : U_\alpha \to GL(k, \mathbb{C})\}$ such that $\sigma'_{\alpha\beta} = \rho_\alpha \cdot \sigma_{\alpha\beta} \cdot \rho_\beta^{-1}$ for all α, β.

(b). $\check{\mathrm{H}}^1(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^1(\mathcal{U},\mathcal{F})$ where the direct limit is indexed over all the open covers of X with order relation induced by refinement, i.e. $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$, and $\check{\mathrm{H}}^1(\mathcal{U},\mathcal{F})$ is a pointed set[11] defined as

$$\check{\mathrm{H}}^1(\mathcal{U},\mathcal{F}) := \ker\{\delta : \check{\mathrm{C}}^1(\mathcal{U},\mathcal{F}) \to \check{\mathrm{C}}^2(\mathcal{U},\mathcal{F})\}\Big/\sim$$

$$(g_{\alpha\beta}) \sim (h_{\alpha\beta}) \Leftrightarrow \exists (f_\alpha) \in \check{\mathrm{C}}^0(\mathcal{U},\mathcal{F}) \text{ such that } f_\alpha * g_{\alpha\beta} = h_{\alpha\beta} * f_\beta \text{ on } U_\alpha \cap U_\beta$$

with $*$ being the group operation. Therefore, $\check{\mathrm{H}}^1(X,\mathcal{F})$ is a group if and only if $\mathcal{F}$ is an abelian sheaf.

Using this new definition we get the following more general correspondence between vector bundles and Čech cohomology [38, §24]:

Theorem 32. *Let M be a smooth manifold, then*

(a). there is one-to-one correspondence between the equivalence classes of rank k smooth vector bundles over M and the elements of the first cohomology set $\check{\mathrm{H}}^1(M,O(k))$ where $O(k)$ is the sheaf of smooth functions to the Lie group $O(k)$ of orthogonal matrices.

(b). there is one-to-one correspondence between the equivalence classes of rank k complex vector bundles over M and the elements of the first cohomology set $\check{\mathrm{H}}^1(M,U(k))$ where $U(k)$ is the sheaf of smooth functions to the Lie group $U(k)$ of unitary matrices.

Clearly this is a generalization of the previous result, since for $k = 1$ we get $\check{\mathrm{H}}^1(M,U(1)) = \check{\mathrm{H}}^1(M,S^1) = \check{\mathrm{H}}^1(M,\mathcal{E}^*)$.

Definition (Picard group)**.** The set of isomorphic classes of line bundles on a manifold M form a group under the tensor product[12] operation, where the inverse of a line bundle is its dual bundle[13]. This group of isomorphism classes of holomorphic line bundles on M is called the *Picard group* of M, denoted by $\mathrm{Pic}(M)$.

In fact, the one-to-one correspondence that we get in Theorem 30 is a group isomorphism, i.e. $\mathrm{Pic}(M) \cong \check{\mathrm{H}}^1(M,\mathcal{O}^*)$ [9, p. 133]. This enables us to define the first Chern class of holomorphic line bundles as follows [12, Definition 2.2.13]:

Definition (First Chern class of holomorphic line bundle)**.** The exponential sheaf sequence on a complex manifold M

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

gives a long exact sequence in cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(M,\mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(M,\mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M,\underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(M,\mathcal{O}) \longrightarrow \cdots$$

Therefore, we have the connecting homomorphism

$$\begin{aligned} \Delta : \check{\mathrm{H}}^1(M,\mathcal{O}^*) &\to \check{\mathrm{H}}^2(M,\underline{\mathbb{Z}}) \\ [\![L]\!] &\mapsto c_1(L) \end{aligned}$$

where $c_1(L)$ is called the *first Chern class* of the holomorphic line bundle L on M.

[11] For details regarding its construction, refer to the lecture notes by Zinger [38, §24].

[12] If $\pi : L \to M$ and $\pi' : L' \to M$ are smooth line bundles, then their tensor product, $L \otimes L'$ is defined such that $(L \otimes L')_w = L_w \otimes L'_w$ for all $w \in M$ [38, §13].

[13] If $\pi : L \to M$ is a smooth line bundles of rank k, the dual bundle of L^* is a line bundle $L^* \to M$ such that $(L^*)_w = L^*_w = \mathrm{Hom}_{\mathbb{R}}(L_w, \mathbb{R})$ for all $w \in M$. For complex and holomorphic line bundles, replace $\mathbb{R}$ by $\mathbb{C}$ [38, §12].

The immediate consequences of this definition are [9, p. 139]:

$$c_1(L \otimes L') = c_1(L) + c_1(L') \quad \text{and} \quad c_1(L^*) = -c_1(L)$$

Note that, in Theorem 2.1 we only proved the existence of connecting homomorphism Δ. However, to be able to calculate the first Chern class of a holomorphic line bundle we must know the exact definition of Δ, which turns out to be a challenging task [37, p. 104].

Similarly, the one-to-one correspondence that we get in Theorem 31 is a group isomorphism. This enables us to define the first Chern class of complex line bundles as follows [37, p. 105]:

Definition (First Chern class of complex line bundle)**.** The exponential sheaf sequence on a smooth manifold M

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{E} \xrightarrow{\exp} \mathcal{E}^* \longrightarrow 0$$

gives a long exact sequence in cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(M, \mathcal{E}) \longrightarrow \check{\mathrm{H}}^q(M, \mathcal{E}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(M, \underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(M, \mathcal{E}) \longrightarrow \cdots$$

Therefore, we have the connecting homomorphism

$$\begin{aligned} \Delta : \check{\mathrm{H}}^1(M, \mathcal{E}^*) &\to \check{\mathrm{H}}^2(M, \underline{\mathbb{Z}}) \\ [\![L]\!] &\mapsto c_1(L) \end{aligned}$$

where $c_1(L)$ is called the *first Chern class* of the complex line bundle L on M.

Since $\mathcal{E}$ is a fine sheaf, by Theorem 2.2, $\check{\mathrm{H}}^k(M, \mathcal{E}) = 0$ for $k > 0$. Therefore, the connecting homomorphism $\Delta : \check{\mathrm{H}}^1(M, \mathcal{E}^*) \to \check{\mathrm{H}}^2(M, \underline{\mathbb{Z}})$ is a group isomorphism, and the equivalence classes of complex line bundles are determined by their first Chern class in $\check{\mathrm{H}}^2(M, \underline{\mathbb{Z}})$ [9, p. 140].

Theorem 33. *There is a natural group isomorphism between the equivalence classes of complex line bundles on a smooth manifold M and the elements of the cohomology group $\check{\mathrm{H}}^2(M, \underline{\mathbb{Z}})$. That is, a complex line bundle is determined upto smooth vector bundle isomorphism by its first Chern class.*

In Theorem 2.3 we proved that $H^k_{dR}(M) \cong \check{\mathrm{H}}^k(M, \underline{\mathbb{R}})$ for $k \geq 0$. Also note that there is a natural homomorphism $j : \check{\mathrm{H}}^2(M, \underline{\mathbb{Z}}) \to \check{\mathrm{H}}^2(M, \underline{\mathbb{R}})$ induced by the inclusion of constant sheaves $\underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{R}}$. Combining these with the fact that $\check{\mathrm{H}}^1(M, \mathcal{E}^*) \cong \check{\mathrm{H}}^2(M, \underline{\mathbb{Z}})$, we can compute the Chern classes of complex line bundles using differential forms [37, Theorem III.4.5].

$$\begin{aligned} c_1 : \{\text{isomorphism classes of complex line bundles over } M\} &\to H^2_{dR}(M) \\ [\![L]\!] &\mapsto c_1(L) \end{aligned}$$

Since a complex vector bundle L of rank 1 over a smooth manifold M can be thought of as a smooth vector bundle L of rank 2 over M, we can use the following result for computing the first Chern class of a complex line bundle [1, pp. 71-73]:

Theorem 34. *Let $\pi : L \to M$ be an oriented smooth oriented rank 2 vector bundle over M, and $\{U_\alpha\}$ be a coordinate open cover of M that trivializes E. If $\{\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2)\}$ are the transition functions*[14] *of L and $\{\eta_\gamma\}$ is a parition of unity of M subordinate to $\{U_\gamma\}$, then*

$$c_1(L) = -\frac{1}{2\pi i} \sum_\gamma \mathrm{d}\left(\eta_\gamma \, \mathrm{d} \log(\sigma_{\gamma\alpha})\right) \text{ on } U_\alpha \text{ for each } \alpha$$

where $\sigma_{\alpha\beta}$ are thought of as complex valued functions by identifying $SO(2)$ with S^1 via $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = e^{i\theta}$ and $c_1(L)$ is a closed form representing a cohomology class in $H^2_{dR}(M)$.

[14] The structure group of every smooth rank k vector bundle $\pi : E \to M$ can be reduced to the orthogonal group $O(k)$ using Gram-Schmidt process. This is also a key step of the proof of Theorem 32(a). Moreover, if the vector bundle is orientable then the structure group can be further reduced to $SO(k)$ [1, Proposition 6.4].

Appendix A

Topology

A.1 Paracompact spaces

In this section some definitions and facts from [23, §39 and 41] will be stated. Here X denotes a topological space.

Definition A.1 (Locally finite collection)**.** Let X be a topological space. A collection $\mathcal{U}$ of subsets of X is said to be *locally finite* in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{U}$.

Lemma A.1. *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite collection of subsets of X. Then*

1. *any subcollection of $\mathcal{U}$ is locally finite.*
2. *the collection $\mathcal{V} = \left\{\overline{U_\alpha}\right\}_{\alpha \in A}$ of the closures of the elements of $\mathcal{U}$ is locally finite.*
3. $\overline{\bigcup_{\alpha \in A} U_\alpha} = \bigcup_{\alpha \in A} \overline{U}$

Definition A.2 (Refinement of a collection)**.** Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. A collection $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ of subsets of X is said to be a *refinement* of $\mathcal{U}$ if for each element V_β of $\mathcal{V}$, there is an element U_α of $\mathcal{U}$ containing V_β.

Remark A.1. If elements of $\mathcal{V}$ are open sets, the $\mathcal{V}$ is called an *open refinement* of $\mathcal{U}$; if they are closed, $\mathcal{V}$ is called a *closed refinement*.

Definition A.3 (Paracompact space)**.** The space X is *paracompact* is every open covering $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ that covers X.

Remark A.2. In most algebraic geometry textbooks, following the lead of Bourbaki, the requirement that the space be Hausdorff is included as part of the definition of the term *compact* and *paracompact*. We shall not follow this convention.

Theorem A.1 (Shrinking lemma)**.** *Let X be a paracompact Hausdorff space; let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed family of pen sets covering X. Then there exists a locally finite indexed family $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ of open sets covering X such that $\overline{V_\alpha} \subseteq U_\alpha$ for each α.*

Definition A.4 (Continuous partition of unity)**.** Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed open covering of X. An indexed family of continuous functions $\{\phi_\alpha : X \to [0,1]\}$ is said to be a *continuous partition of unity* on X, dominated by $\{U_\alpha\}$, if

1. $\operatorname{supp}(\phi_\alpha) \subseteq U_\alpha$ for each α
2. the indexed family $\{\operatorname{supp}(\phi_\alpha)\}_{\alpha \in A}$ is locally finite

3. $\sum_{\alpha \in A} \phi_\alpha(x) = 1$ for each $x \in X$.

where $\text{supp}(\phi_\alpha)$ is the closure of the set of those $x \in X$ for which $\phi_\alpha(x) \neq 0$.

Theorem A.2. *Let X be a paracompact Hausdorff space; let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed open covering of X. Then there exists a continuous partition of unity on X dominated by $\{U_\alpha\}$*

A.2 Topological results for $\mathbb{C}^n$

In this section, for the sake of completeness, the proofs of a few simple standard results[1] for $\mathbb{C}^n$ have been discussed.

Lemma A.2. *If U is an open set in $\mathbb{C}$, then there exists a sequence $\{K_n\}$ of compact subsets of U such that*

1. *$K_n \subset \text{int}(K_{n+1})$ for each n;*
2. *$\bigcup\limits_{n \in \mathbb{N}} \text{int}(K_n) = U$; and*
3. *each bounded component of the complement of K_n contains a point of the complement of U.*

Proof. For each $n \in \mathbb{N}$, define the open set,

$$V_n := \Delta(\infty; n) \cup \bigcup_{z \in \mathbb{C} \setminus U} \Delta\left(z; \frac{1}{n}\right)$$

where $\Delta(z; \frac{1}{n}) = \{w \in \mathbb{C} : |z - w| < \frac{1}{n}\}$, and $\Delta(\infty, n) = \{w \in \mathbb{C} : |w| > n\}$ is the "disk at ∞". Then we define

$$K_n := \mathbb{C} \setminus V_n$$

which is a closed and bounded (hence compact[2]) subset of U for all n. Now we will verify the three desired properties:

1. If $z \in K_n$ and $r = \frac{1}{n} - \frac{1}{n+1}$, then $\Delta(z; r) \subset K_{n+1}$. The interior of K_{n+1} is, by definition, the largest open subset of K_{n+1}. Therefore, $K_n \subset \bigcup_{z \in K_n} \Delta(z; r) \subset \text{int}(K_{n+1})$.
2. As $n \to \infty$ we get $V_n \to \mathbb{C} \setminus U$. Therefore, $\bigcup\limits_{n \in \mathbb{N}} K_n = U$. Now since $K_n \subset \text{int}(K_{n+1})$, we have $\bigcup\limits_{n \in \mathbb{N}} \text{int}(K_n) = U$.
3. We need to show that every bounded connected component $\mathcal{C}$ of V_n meets $\mathbb{C} \setminus U$. To prove this, pick a $w \in \mathcal{C}$. Note that w, being an element of V_n, must be contained in $\Delta(z; \frac{1}{n})$ for some $z \in \mathbb{C} \setminus U$ or in $\Delta(\infty; n)$. Since $\mathcal{C}$ is bounded, we have[3] $w \in \Delta(z; \frac{1}{n})$ for some $z \in \mathbb{C} \setminus U$. Observe that $\mathcal{C} \cup \Delta(z; \frac{1}{n})$ is a connected subset of V_n, since it is the union of two connected open subsets of V_n with non-empty intersection. Since $\mathcal{C}$ is a connected component of V_n, we know that $\mathcal{C}$ is a maximal connected set of V_n. Therefore, $\Delta(z, \frac{1}{n})$ must be contained in $\mathcal{C}$. Hence $\mathcal{C}$ contains z, which is in $\mathbb{C} \setminus U$.

□

[1]These results are also true for $\mathbb{R}^n$.
[2]Note that K_n can be empty also.
[3]If $\mathbb{C} \setminus U \neq \emptyset$ then there is no bounded component of V_n to begin with.

Remark A.3. We can't guarantee that the third property will hold for an unbounded component, unless we replace $\mathbb{C}$ by the Riemann sphere[4] $\mathbb{C} \cup \{\infty\}$. For example, if $U = \{z \in \mathbb{C} : |z| > 1/2\}$ then for $n = 1$ the unbounded connected component $\mathcal{C} = \Delta(\infty; 1)$ doesn't intersect with $\mathbb{C} \setminus U$.

Lemma A.3. *Let K be a compact subset of an open set $U \subset \mathbb{C}^n$. Then there exists a real-valued smooth function $F(z)$ in $\mathbb{C}^n$ such that*

1. $0 \le F(z) \le 1$ *for all* $z \in \mathbb{C}^n$*;*
2. $F(z) = 1$ *for* $z \in K$*; and*
3. $F(z) = 0$ *for* $z \in \mathbb{C}^n \setminus U$*.*

Proof. Consider the following smooth[5] function defined on $\mathbb{R}$:

$$h(x) = \begin{cases} e^{\frac{-1}{(x-r)}} e^{\frac{-1}{(x-R)}} & \text{for } r < x < R \\ 0 & \text{otherwise} \end{cases}$$

Consequently the function defined as[6]

$$g(x) = \frac{\int_x^R h(t)\,\mathrm{d}t}{\int_r^R h(t)\,\mathrm{d}t}$$

is a smooth function such that

1. $0 \le g(x) \le 1$ for all $x \in \mathbb{R}$;
2. $g(x) = 1$ for $x \le r$; and
3. $g(x) = 0$ for $x \ge R$.

Next, consider the special case in which K is a closed ball of radius r centered at origin, and U is an open ball of radius $R > r$, i.e.

$$K = \{z \in \mathbb{C}^n : |z| \le r\} \quad \text{and} \quad U = \{z \in \mathbb{C}^n : |z| < R\}$$

Then for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, the function

$$f(z) = g(\|z\|) = g\left(\sqrt{|z_1|^2 + \cdots + |z_n|^2}\right)$$

satisfies the required conditions

1. $0 \le f(z) \le 1$ for all $z \in \mathbb{C}^n$;
2. $f(z) = 1$ for $\|z\| \le r$; and
3. $f(z) = 0$ for $\|z\| \ge R$.

[4]The same construction works for the case of Riemann sphere. In fact we can prove a stronger statement: for each $n \in \mathbb{N}$, every connected component of $\mathbb{C} \cup \{\infty\} \setminus K_n$ contains a connected component of $\mathbb{C} \cup \{\infty\} \setminus U$. For details, see [3, Proposition VII.1.2], there this theorem is used to prove Runge's theorem.

[5]This is a standard exercise in real analysis, for example, see [32, Problem 1.2].

[6]The same construction can be used for bump functions on smooth manifolds, see [22, Lemma 2.1.8].

Now for the general case, select a finite number of pairs of concentric balls $K_j \subset U_j$ such that $K \subset \bigcup K_j$ and $U_j \subset U$. Let $f_j(z)$ be the functions satisfying the desired conditions on these pairs of balls, as constructed for the special case. Then the function

$$F(z) = 1 - \prod_j (1 - f_j(z))$$

is the desired function, hence completing the proof. □

Theorem A.3. *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of an open subset $U \subset \mathbb{C}^n$, then there is a smooth partition of unity $\{\psi_k\}_{k=1}^{\infty}$ with every ψ_k having compact support such that for each k, $\operatorname{supp}(\psi_k) \subset U_\alpha$ for some $\alpha \in A$.*

Proof. Since any open subset $U \subset \mathbb{C}^n$ is paracompact, every open covering $\{U_\alpha\}$ has a locally finite refinement $\{V_k\}$. Then the smooth partition of unity $\{\psi_k\}$ of U subordinate to $\{V_k\}$ will have compact support. For details, see [10, Appendix A]. □

Remark A.4. This result has also been used in Theorem 13. However, there we don't require the support to be compact.

Appendix B

Direct limit

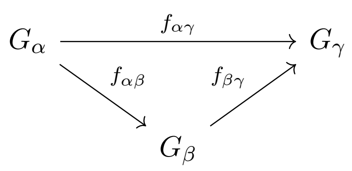
In this appendix some definitions and facts from [25, §73] and [26, §IV.2] will be stated.

Definition B.1 (Directed set)**.** A directed set A is a set with relation $<$ such that

1. $\alpha < \alpha$ for all $\alpha \in A$
2. $\alpha < \beta$ and $\beta < \gamma$ implies $\alpha < \gamma$
3. Given α and β, there exists δ such that $\alpha < \delta$ and $\beta < \delta$. The element δ is called an *upper limit* for α and β.

Definition B.2 (Direct system)**.** A *direct system* of abelian groups and group homomorphisms, corresponding to the directed set A, is an indexed family $\{G_\alpha\}_{\alpha\in A}$ of abelian groups, along with the family of homomorphisms $\{f_{\alpha\beta} : G_\alpha \to G_\beta\}_{\alpha,\beta\in A,\ \alpha<\beta}$ such that

1. $f_{\alpha\alpha} : G_\alpha \to G_\alpha$ is identity
2. If $\alpha < \beta < \gamma$ then $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$; i.e. the following diagram commutes:



Definition B.3 (Direct limit)**.** Given a directed set A and the associated direct system of abelian groups and homomorphisms $\{(G_\alpha, f_{\alpha\beta})\}$, the *direct limit* is defined to be the quotient

$$\varinjlim_{\alpha\in A} G_\alpha = \coprod_{\alpha\in A} G_\alpha \Big/ \sim$$

where, given $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$, $g_\alpha \sim g_\beta$ if there exists an upper bound δ of α and β such that $f_{\alpha\delta}(g_\alpha) = f_{\beta\delta}(g_\beta)$. Also, $g_\alpha \sim g_\beta$ implies that they belong to same equivalence class, i.e. $[\![g_\alpha]\!] = [\![g_\beta]\!]$. The direct limit is again an abelian group under addition defined as

$$[\![g_\alpha]\!] + [\![g_\beta]\!] := [\![f_{\alpha\delta}(g_\alpha) + f_{\beta\delta}(g_\beta)]\!]$$

for some upper bound δ of α and β.

Remark B.1. Just as in case of definition of sheaf, the definition of direct limit can be generalized to any category like groups, rings, modules, and algebras instead of abelian groups.

Proposition B.1. *Given a directed set A and the associated direct system $\{(G_\alpha, f_{\alpha\beta})\}$ of abelian groups and homomorphisms such that all the maps $f_{\alpha\beta}$ are isomorphisms, then $\varinjlim G_\alpha$ is isomorphic to any one of the groups G_α.*

Proposition B.2. *Given a directed set A and the associated direct system $\{(G_\alpha, f_{\alpha\beta})\}$ of abelian groups and homomorphisms such that all the maps $f_{\alpha\beta}$ are zero-homomorphisms, then $\varinjlim G_\alpha$ is the trivial group. More generally, if for each α there is a β such that $\alpha < \beta$ and $f_{\alpha\beta}$ is the zero homomorphism, then $\varinjlim G_\alpha$ is the trivial group.*

Definition B.4 (Map of direct systems)**.** Let A and B be two directed sets. Let $\{(G_\alpha, f_{\alpha\beta})\}$ and $\{(G'_\gamma, f'_{\gamma\delta})\}$ be the associated direct systems of abelian groups and homomorphisms, respectively. A *map of direct systems* $\Phi = (\phi, \{\phi_\alpha\}) : \{(G_\alpha, f_{\alpha\beta})\} \to \{(G'_\gamma, f'_{\gamma\delta})\}$ is a collection of maps such that

1. the set map $\phi : A \to B$ that preserves order relation
2. for each $\alpha \in A$, $\phi_\alpha : G_\alpha \to G'_{\phi(\alpha)}$ is a group homomorphism such that the following diagram commutes

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\phi_\alpha} & G'_\gamma \\ \downarrow{\scriptstyle f_{\alpha\beta}} & & \downarrow{\scriptstyle f'_{\gamma\delta}} \\ G_\beta & \xrightarrow{\phi_\beta} & G'_\delta \end{array}$$

   for $\alpha < \beta$, $\gamma = \phi(\alpha)$ and $\delta = \phi(\beta)$

Definition B.5 (Direct limit of direct system homomorphisms)**.** The map of direct systems $\Phi : \{(G_\alpha, f_{\alpha\beta})\} \to \{(G'_\gamma, f'_{\gamma\delta})\}$ induces a homomorphism, called the *direct limit of the homomorphisms ϕ_α*

$$\underrightarrow{\Phi} : \varinjlim_{\alpha \in A} G_\alpha \to \varinjlim_{\gamma \in B} G'_\gamma$$

It maps the equivalence class of $g_\alpha \in G_\alpha$ to the equivalence class of $\phi_\alpha(g_\alpha)$.

Theorem B.1 (Universal property of direct limits)**.** *Let A be a directed set and $\{(G_\alpha, f_{\alpha\beta})\}$ be the associated direct system of abelian groups and homomorphisms. If $G = \varinjlim_{\alpha \in A} G_\alpha$, then the inclusion $i_\alpha : G_\alpha \hookrightarrow \coprod_{\alpha \in A} G_\alpha$ induces a family of group homomorphisms $\{\chi_\alpha : G_\alpha \to G\}_{\alpha \in A}$. If H is an abelian group such that for each $\alpha \in A$ there is a group homomorphism $\psi_\alpha : G_\alpha \to H$ satisfying $\psi_\alpha = \psi_\beta \circ f_{\alpha\beta}$, whenever $\alpha < \beta$. Then there exists a unique group homomorphism*

$$\Psi : G \to H$$

satisfying $\psi_\alpha = \Psi \circ \chi_\alpha$ for all $\alpha \in A$.

Remark B.2. We observe that this universal property is a special case of the preceding construction, in which second direct system consists of the single group H. Hence, we have $\Psi = \underrightarrow{\Psi}$. One can also observe that the family of group homomorphisms $\{\chi_\alpha : G_\alpha \to G\}_{\alpha \in A}$ satisfies the condition $\chi_\alpha = \chi_\beta \circ f_{\alpha\beta}$ for all $\alpha < \beta$ since the following diagram commutes

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\chi_\alpha} & G \\ \downarrow{\scriptstyle f_{\alpha\beta}} & & \downarrow{\scriptstyle \mathbb{1}_G} \\ G_\beta & \xrightarrow{\chi_\beta} & G \end{array}$$

Theorem B.2 (Direct limit is as an exact functor)**.** *Let A be a directed set*[1]*. Let $\{(G'_\alpha, f'_{\alpha\beta})\}$, $\{(G_\alpha, f_{\alpha\beta})\}$ and $\{(G''_\alpha, f''_{\alpha\beta})\}$ be three direct systems of abelian groups and homomorphisms associated with A, with the maps of direct systems*

$$\Phi : \{(G'_\alpha, f'_{\alpha\beta})\} \to \{(G_\alpha, f_{\alpha\beta})\} \quad \text{and} \quad \Psi : \{(G_\alpha, f_{\alpha\beta})\} \to \{(G''_\alpha, f''_{\alpha\beta})\}$$

such that the sequence of abelian groups

$$G'_\alpha \xrightarrow{\phi_\alpha} G_\alpha \xrightarrow{\psi_\alpha} G''_\alpha$$

is exact for every $\alpha \in A$. Then the induced sequence

$$\varinjlim_{\alpha\in A} G'_\alpha \xrightarrow{\underrightarrow{\Phi}} \varinjlim_{\alpha\in A} G_\alpha \xrightarrow{\underrightarrow{\Psi}} \varinjlim_{\alpha\in A} G''_\alpha$$

is also exact.

Proof. Let $G' = \varinjlim_{\alpha\in A} G'_\alpha$, $G = \varinjlim_{\alpha\in A} G_\alpha$ and $G'' = \varinjlim_{\alpha\in A} G''_\alpha$. We consider the commutative diagram, for all $\alpha \in A$

$$\begin{CD}
G'_\alpha @>{\phi_\alpha}>> G_\alpha @>{\psi_\alpha}>> G''_\alpha \\
@V{\chi'_\alpha}VV @V{\chi_\alpha}VV @V{\chi''_\alpha}VV \\
G' @>{\underrightarrow{\Phi}}>> G @>{\underrightarrow{\Psi}}>> G''
\end{CD}$$

where $\chi'_\alpha, \chi_\alpha$ and χ''_α are the homomorphisms induced by the inclusion maps into the respective disjoint union (as in Theorem B.1). Given to us is that $\operatorname{im}\phi_\alpha = \ker\psi_\alpha$ for all $\alpha \in A$.

`Claim:` $\operatorname{im}\underrightarrow{\Phi} = \ker\underrightarrow{\Psi}$

($\ker\underrightarrow{\Psi} \subseteq \operatorname{im}\underrightarrow{\Phi}$) Let $g \in G$, then by the definition of direct limit there exists $\alpha \in A$ such that for some $g_\alpha \in G_\alpha$ we have $\chi_\alpha(g_\alpha) = g$. Also, let $\underrightarrow{\Psi}(g) = 0_{G''}$. By the commutative diagram above, we have

$$\chi''_\alpha(\psi_\alpha(g_\alpha)) = \underrightarrow{\Psi}(\chi_\alpha(g_\alpha)) = \underrightarrow{\Psi}(g) = 0_{G''}$$

The direct limit is a collection of equivalence classes, hence we have

$$\chi''_\alpha(\psi_\alpha(g_\alpha)) = [\![\psi_\alpha(g_\alpha)]\!] = [\![0_{G''_\alpha}]\!]$$

Since $\psi_\alpha(g_\alpha), 0_{G''_\alpha} \in G''_\alpha$, we have $f''_{\alpha\delta}(\psi_\alpha(g_\alpha)) = f''_{\alpha\delta}(0_{G''_\alpha}) = 0_{G''_\delta}$ for some δ such that $\alpha < \delta$. But $\psi_\delta \circ f_{\alpha\delta} = f''_{\alpha\delta} \circ \psi_\alpha$, hence we have $\psi_\delta(f_{\alpha\delta}(g_\alpha)) = 0_{G''_\delta}$. Hence $f_{\alpha\delta}(g_\alpha) \in \ker\psi_\delta = \operatorname{im}\phi_\delta$. So there exist $h_\delta \in G'_\delta$ such that $\phi_\delta(h_\delta) = f_{\alpha\delta}(g_\alpha)$. Using $\chi_\alpha = \chi_\delta \circ f_{\alpha\delta}$ and commutativity of diagram we get we get

$$g = \chi_\alpha(g_\alpha) = \chi_\delta(f_{\alpha\delta}(g_\alpha)) = \chi_\delta(\phi_\delta(h_\delta)) = \underrightarrow{\Phi}(\chi'_\delta(h_\delta))$$

($\operatorname{im}\underrightarrow{\Phi} \subseteq \ker\underrightarrow{\Psi}$) Suppose $g \in \operatorname{im}\underrightarrow{\Phi}$. Then $g = \underrightarrow{\Phi}(h)$, and by definition of direct limit we have $h = \chi'_\alpha(h_\alpha)$ for some $h_\alpha \in G'_\alpha$. Now by the commutativity of diagram we have

$$g = \underrightarrow{\Phi}(\chi'_\alpha(h_\alpha)) = \chi_\alpha(\phi_\alpha(h_\alpha))$$

Since $\psi_\alpha \circ \phi_\alpha = 0_{G''_\alpha}$ by exactness, we have

$$\underrightarrow{\Psi}(g) = \underrightarrow{\Psi}(\chi_\alpha(\phi_\alpha(h_\alpha))) = \chi''_\alpha(\psi_\alpha(\phi_\alpha(h_\alpha))) = \chi''_\alpha(0_{G''_\alpha}) = 0_{G''}$$

Hence completing the proof. □

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[1]To avoid too many new symbols, let all the direct systems be associated with the same directed set, i.e. $A = B = C$ and $\phi = \mathbb{1}_A$.

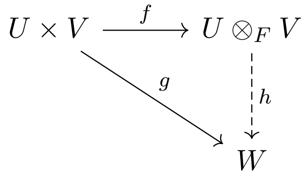
Appendix C

Algebra

C.1 Complexification of vector space

In this section some definitions and facts from [28, Chapter 14] will be stated.

Definition C.1 (Tensor product of vector spaces)**.** Let U and V be vector spaces over a field F. The tensor product $U \otimes_F V$ is a vector space over F equipped with a bilinear map $f : U \times V \to U \otimes_F V$ such that for each bilinear map from $U \times V$ to any vector space W over F there is a unique linear map $h : U \otimes V \to W$ making the following diagram commute.



Remark C.1. We use the symbol $\otimes$ to denote the image of any ordered pair (u, v) under the tensor map, i.e. $u \otimes v = f(u, v)$ for any $u \in U$ and $v \in V$. Not all members of $U \otimes_F V$ are of this form. In general, if $\{u_i : i \in I\}$ is a basis for U and $\{v_j : i \in J\}$ is a basis for V, then any vector $w \in U \otimes_F V$ has a unique expression as a sum

$$w = \sum_{i \in I} \sum_{j \in J} r_{i,j}(u_i \otimes v_j)$$

where only a finite number of the coefficients $r_{i,j}$ are non-zero.

Proposition C.1. *For finite dimensional vector spaces U and V over a field F*

$$\dim_F(U \otimes_F V) = \dim_F(U) \dim_F(V)$$

Proposition C.2 (Bilinearity on $U \times V$ equals linearity on $U \otimes_F V$)**.** *Let U, V and W be vector spaces over a field F. Let $\mathrm{Hom}_F(U, V; W)$ be the set of all bilinear maps from $U \times V$ to W, and $\mathrm{Hom}_F(U \otimes V; W)$ be the set of all linear maps from $U \otimes V$ to W. Then the mediating map*

$$\begin{aligned} \phi : \mathrm{Hom}_F(U, V; W) &\to \mathrm{Hom}_F(U \otimes_F V; W) \\ g &\mapsto h \end{aligned}$$

where h is the unique linear map satisfying $g = h \circ f$ for the tensor map $f : U \times V \to U \otimes_F V$, is an isomorphism.

Proposition C.3 (Linear functionals on tensor product)**.** *Let U and V be finite dimensional vector spaces over a field F. Then the linear transformation*

$$\psi : U^* \otimes_F V^* \to (U \otimes_F V)^*$$

defined by $\psi(f \otimes g)(u \otimes v) = f(u)g(v)$, is an isomorphism. Thus, the tensor product of linear functionals is a linear functional on tensor products.

Corollary C.1. *For a finite dimensional vector spaces U and V over a field F, we have*

$$U^* \otimes_F V^* \cong \mathrm{Hom}_F(U, V; F)$$

Proof. From Proposition C.3 we know that $U^* \otimes_F V^* \cong (U \otimes_F V)^*$. Note that $(U \otimes_F V)^* = \mathrm{Hom}_F(U \otimes_F V; F)$, hence we can use Proposition C.2 to conclude that $U^* \otimes_F V^* \cong (U \otimes_F V)^* \cong \mathrm{Hom}_F(U, V; F)$ □

Theorem C.1 (Extending the base field)**.** *Let V be vector space over a field F and K be a finite extension of F. Then $W = V \otimes_F K$ is a vector space over K such $\dim_K(W) = \dim_F(V)$. Moreover, if W_F is the vector space obtained by restricting the the scalar multiplication for W to scalars from F, then W_F contains an isomorphic copy of V.*

Proof. Since K is a vector space over F, we can form the tensor product

$$W_F = V \otimes_F K$$

where all relevant maps are F-bilinear and F-linear. By definition of tensor product W_F is a vector space over F. However, since V is not a K-space, we can't have a K-tensor product. We just need to show that W_F can be made into a vector space over K.

`Claim:` *For $\alpha \in K$, the scalar multiplication operation $\alpha(v \otimes \beta) = v \otimes (\alpha\beta)$ is well defined.*

To prove the claim, we need to check that

$$v \otimes \beta = w \otimes \gamma \quad \Rightarrow \quad v \otimes (\alpha\beta) = w \otimes (\alpha\gamma)$$

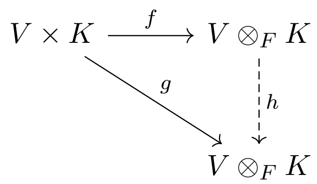
Note that for a fixed α, the map

$$\begin{aligned} g : V \times K &\to V \otimes_F K \\ (v, \beta) &\mapsto v \otimes (\alpha\beta) \end{aligned}$$

is F-bilinear. Now the definition of tensor product implies that there exists a unique F-linear map

$$\begin{aligned} h : V \otimes_F K &\to V \otimes_F K \\ v \otimes \beta &\mapsto v \otimes (\alpha\beta) \end{aligned}$$

since the following diagram commutes



We define this map h to be scalar multiplication by α, under which $W = V \otimes_F K$ is a vector space over the field K. Note that W_F and W are identical as sets and as abelian groups, only the scalar multiplication operation is different. Moreover, we recover W_F from W simply by restricting scalar multiplication to scalars from F.

If K is a degree d field extension of F, then using Proposition C.1 we get

$$\dim_F(W_F) = \dim_F(V \otimes_F K) = \dim_F(V) \cdot d$$

Hence, if $\{v_i : i \in I\}$ is a basis for V, then $\{v_i \otimes 1\}$ is a basis for W, that is,

$$\dim_K(W) = \dim_F(V)$$

The map $\mu : V \to W_F$ defined by $\mu(v) = v \otimes 1$ is an injective F-linear map, so W_F contains an isomorphic copy of V. □

Remark C.2. We can also think of μ as mapping of V into W, in which case μ is called the *K-extension map* of V.

Theorem C.2 (Extending the linear map)**.** *Let U and V be two vector spaces over the field F, with K-extension maps μ_U and μ_V, respectively. Then for any F-linear map $\tau : U \to V$, the map $\tau \otimes \mathbb{1}_K : U \otimes_F K \to V \otimes_F K$ is the unique K-linear map that makes the following diagram commute*

$$\begin{array}{ccc} U & \xrightarrow{\quad\tau\quad} & V \\ \Big\downarrow{\scriptstyle \mu_U} & & \Big\downarrow{\scriptstyle \mu_V} \\ U \otimes_F K & \xrightarrow{\tau\otimes\mathbb{1}_K} & V \otimes_F K \end{array}$$

Thus, $\tau \otimes \mathbb{1}_K$ is the extension of the F-linear map τ to a K-linear map.

Definition C.2 (Complexification of a real vector space)**.** To each real vector space V, we can associate a complex vector space $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ called the *complexification of V*.

Proposition C.4. *Let V be a real vector space, and $\widetilde{V} = V \oplus V$ be a complex vector space with multiplication law $(a + ib)(v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$. Then there is a unique isomorphism $\phi : \widetilde{V} \to V_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces which makes the diagram*

$$\begin{array}{ccccc} & & V & & \\ & \swarrow & & \searrow{\scriptstyle \mu} & \\ \widetilde{V} & & \xrightarrow{\quad\phi\quad} & & V_{\mathbb{C}} \end{array}$$

commute. Explicitly,

$$\phi(v_1, v_2) = v_1 \otimes 1 + v_2 \otimes i$$

Proof. Firstly we will verify that ϕ is $\mathbb{C}$-linear

$$\begin{aligned} \phi\left((a + ib)(v_1, v_2)\right) &= \phi\left(av_1 - bv_2, bv_1 + av_2\right) \\ &= (av_1 - bv_2) \otimes 1 + (bv_1 + av_2) \otimes i \\ &= a(v_1 \otimes 1) - b(v_2 \otimes 1) + b(v_1 \otimes i) + a(v_2 \otimes i) \\ &= a(v_1 \otimes 1) + ib(v_2 \otimes i) + ib(v_1 \otimes 1) + a(v_2 \otimes i) \\ &= a\left(v_1 \otimes 1 + v_2 \otimes i\right) + ib\left(v_2 \otimes i + v_1 \otimes 1\right) \\ &= (a + ib)\ \phi(v_1, v_2) \end{aligned}$$

To show that ϕ is an isomorphism, we will write down the inverse map:

$$\begin{aligned} \psi : V_{\mathbb{C}} &\to \widetilde{V} \\ v \otimes \alpha &\mapsto \alpha(v, 0) \end{aligned}$$

which is extended by linearity. Using the definition of scalar multiplication for $V_{\mathbb{C}}$ we verify that ψ is $\mathbb{C}$-linear. Let $\beta \in \mathbb{C}$ then

$$\begin{aligned} \psi(\beta(v \otimes \alpha)) &= \psi(v \otimes \beta\alpha) \\ &= \beta\alpha(v, 0) \\ &= \beta\ \psi(v \otimes \alpha) \end{aligned}$$

Finally, we show that ϕ and ψ are inverse of each other:

$$\psi\left(\phi\left(v_1, v_2\right)\right) = \psi\left(v_1 \otimes 1 + v_2 \otimes i\right) = (v_1, 0) + i\left(v_2, 0\right) = (v_1, 0) + (0, v_2) = (v_1, v_2)$$

$$\phi(\psi(v \otimes \alpha)) = \phi(\alpha(v, 0)) = \alpha\phi(v, 0) = \alpha(v \otimes 1) = v \otimes \alpha$$

Note that it suffices to verify $\phi \circ \psi = \mathbb{1}_{V_{\mathbb{C}}}$ for elementary tensors. □

Proposition C.5. *The complexification of the dual space* V^* *of a real vector space* V *is naturally isomorphic to the space of all* $\mathbb{R}$*-linear maps from* V *to* $\mathbb{C}$*. That is,* $(V^*)_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$*.*

Proof. The isomorphism is given by

$$\begin{aligned} \Phi : (V^*)_{\mathbb{C}} &\to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \\ \varphi_1 \otimes 1 + \varphi_2 \otimes i &\mapsto \varphi_1 + i\varphi_2 \end{aligned}$$

where φ_1 and φ_2 are elements of $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. □

Corollary C.2. *The complexification of the dual space* V^* *of a real vector space* V *is naturally isomorphic to the dual of the dual space of* $V_{\mathbb{C}}$*. That is,* $(V^*)_{\mathbb{C}} = (V_{\mathbb{C}})^*$*.*

Proof. Given a $\mathbb{R}$-linear map $\varphi : V \to \mathbb{C}$, we can extend by linearity to obtain a $\mathbb{C}$-linear map

$$\begin{aligned} \tilde{\varphi} : V_{\mathbb{C}} &\to \mathbb{C} \\ v \otimes \alpha &\mapsto \alpha\varphi(v) \end{aligned}$$

This extension gives an isomorphism from $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ to $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$. The latter is just the complex dual space to $V_{\mathbb{C}}$, hence giving the isomorphism $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$. □

Remark C.3. More generally, given real vector spaces V and W there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{R}}(V, W)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$$

Proposition C.6. *Complexification commutes with the operations of taking tensor products. That is, if* V *and* W *are real vector spaces then there is a natural isomorphism* $(V \otimes_{\mathbb{R}} W)_{\mathbb{C}} \cong V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$*, where the left-hand tensor product is taken over* $\mathbb{R}$ *while the right-hand one is taken over* $\mathbb{C}$*.*

C.2 Linear complex structure

In this section some definitions and facts from [37, §I.3] and [12, §1.2] will be stated.

Definition C.3 (Complex structure)**.** Let V be a real vector space and suppose that J is an $\mathbb{R}$-linear endomorphism $J : V \to V$ such that $J^2 = -\mathbb{1}_V$. Then J is called a *complex structure* on V.

Lemma C.1. *If* J *is a complex structure on a real vector space* V*, then* V *admits in a natural way the structure of a complex vector space.*

Proof. We can equip V with the structure of a complex vector space in the following manner:

$$(\alpha + i\beta)v := \alpha v + \beta J(v), \quad \alpha, \beta \in \mathbb{R}, i = \sqrt{-1}$$

Thus scalar multiplication on V by complex numbers is well defined, and V becomes a complex vector space. □

Lemma C.2. *If* V *is a complex vector space, then we can define a complex structure* J *on* V *when it is considered as a real vector space.*

Proof. Since V is a complex vector space and $\mathbb{R} \subset \mathbb{C}$, it can also be considered as a vector space over $\mathbb{R}$, and the operation of multiplication by $i = \sqrt{-1}$ is an $\mathbb{R}$-linear endomorphism of V onto itself, which we can call J,

$$\begin{aligned} J : V &\to V \\ v &\mapsto iv \end{aligned}$$

This is a complex structure. □

Remark C.4. Moreover, if $\{v_1, \dots, v_n\}$ is a basis for V over $\mathbb{C}$, then

$$\{v_1, \dots, v_n, J(v_1), \dots, J(v_n)\}$$

will be a basis for V over $\mathbb{R}$, i.e. $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Hence a complex structure can only exist on an even dimensional real vector space.

Definition C.4 (Standard complex structure on $\mathbb{R}^{2n}$)**.** Let $\mathbb{C}^n$ be the usual Euclidean space of n-tuples of complex numbers, $\{(z_1, \dots, z_n)\}$, and let $z_j = x_j + iy_j$, $j = 1, \dots, n$, where x_j, y_j are the real and imaginary parts. Then $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n} = \{(x_1, y_1, \dots, x_n, y_n)\}$. Scalar multiplication by i in $\mathbb{C}^n$ induces a mapping $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$J(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, \dots, -y_n, x_n)$$

and, with $J^2 = -\mathbb{1}$. This is the *standard complex structure* on $\mathbb{R}^{2n}$.

Remark C.5. Given a basis $\{e_1, e_2, \dots, e_n\}$ for the complex space $\mathbb{C}^n$, this set, together with these vectors multiplied by i namely $\{ie_1, ie_2, \dots, ie_n\}$, form a basis for the real space $\mathbb{R}^{2n}$. There are two natural ways to order this basis:

1. If one orders the basis as $\{e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n\}$, then the matrix for the standard complex structure J on $\mathbb{R}^{2n}$ takes the block diagonal form:

$$J = \begin{bmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix}_{2n \times 2n}$$

2. On the other hand, if one orders the basis as $\{e_1, e_2, \dots, e_n, ie_1, ie_2, \dots, ie_n\}$, then the matrix for the standard complex structure J on $\mathbb{R}^{2n}$ takes the block-antidiagonal form:

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}_{2n \times 2n}$$

Remark C.6. If J is a complex structure on V, then $J \in \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the general linear group[1] of V. Moreover, the coset space[2] $\mathrm{GL}(2n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{C})$ determines all complex structures on $\mathbb{R}^{2n}$ by the mapping $[A] \mapsto A^{-1}JA$, where $[A]$ is the equivalence class of $A \in \mathrm{GL}(2n, \mathbb{R})$.

Proposition C.7. *Let V be a real vector space with a complex structure J. Then we have*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where

$$V^{1,0} = \{w \in V_{\mathbb{C}} : (J \otimes \mathbb{1}_{\mathbb{C}})(w) = i \cdot w\} \quad \text{and} \quad V^{0,1} = \{w \in V_{\mathbb{C}} : (J \otimes \mathbb{1}_{\mathbb{C}})(w) = -i \cdot w\}$$

Moreover, the complex conjugation on $V_{\mathbb{C}}$, defined as $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$ for $v \in V$ and $\alpha \in \mathbb{C}$, induces $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$.

[1] If V is a vector space over the field F, $\mathrm{GL}(V)$ or $\mathrm{Aut}(V)$ is the group of all automorphisms of V, i.e. the set of all bijective linear transformations from V onto V, together with functional composition as group operation. If V has finite dimension n, then $\mathrm{GL}(V)$ and $\mathrm{GL}(n, F)$ are isomorphic.

[2] $\mathrm{GL}(n, \mathbb{C})$, is a complex Lie group of complex dimension n^2. As a real Lie group (through realification) it has dimension $2n^2$. In fact, we have $\mathrm{GL}(n, \mathbb{R}) < \mathrm{GL}(n, \mathbb{C}) < \mathrm{GL}(2n, \mathbb{R})$, which have real dimensions $n^2, 2n^2$ and $(2n)^2 = 4n^2$. See, John Lee's *Introduction to Smooth Manifolds* (2nd Edition), Example 7.18(d), p. 158.

Proof. Note that $\tilde{J} = J \otimes \mathbb{1}_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of the $\mathbb{R}$-linear map J, which still has the property that $\tilde{J}^2 = -\mathbb{1}_{V_{\mathbb{C}}}$. It follows that $\tilde{J}$ has two eigenvalues $\{i, -i\}$. Also, $V^{1,0}$ is the eigenspace corresponding to the eigenvalue i and $V^{0,1}$ is the eigenspace corresponding to $-i$. Since the minimal polynomial $p(t) = t^2 + 1$ of $\tilde{J}$ is product of distinct linear factors, $\tilde{J}$ is diagonalizable [28, Theorem 8.11]. Hence $V_{\mathbb{C}}$ is the direct sum of eigenspaces corresponding to the distinct eigenvalues [28, Theorem 8.10].

In particular, every vector w of $V_{\mathbb{C}}$ can be written as :

$$w = \frac{w - i\tilde{J}(w)}{2} + \frac{w + i\tilde{J}(w)}{2}$$

where $(w - i\tilde{J}(w))/2$ is an eigenvector with eigenvalue i while $(w + i\tilde{J}(w))/2$ is an eigenvector with eigenvalue $-i$. Note that

$$\overline{\left(\frac{w - i\tilde{J}(w)}{2}\right)} = \frac{\overline{w} + i\tilde{J}(\overline{w})}{2}$$

Hence, complex conjugation interchanges the two factors, and induces $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$. □

Remark C.7. Note that the complex vector space obtained from V by means of the complex structure J, denoted by V_J, is $\mathbb{C}$-linear isomorphic to $V^{1,0}$. Hence we can identify V_J with $V^{1,0}$.

Proposition C.8. *Let V be a real vector space endowed with a complex structure J. Then the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ has a natural complex structure given by $\mathcal{J}(f)(v) = f(J(v))$ for all $f \in V^*$ and $v \in V$. The induced decomposition on $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$ is given by*

$$(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

where

$$(V^*)^{1,0} \cong \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(J(v)) = if(v)\} \cong (V^{1,0})^*$$
$$(V^*)^{0,1} \cong \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(J(v)) = -if(v)\} \cong (V^{0,1})^*$$

C.3 Exterior algebra

By replacing bilinearity with multilinearity in Definition C.1, we can extend the definition of tensor product to more than two vector spaces. In this section some facts about tensor spaces will be stated from [28, Chapter 14] and [12, §1.2]. Unlike the rest of the thesis, here the letter T denote "tensor" space instead of "tangent" space.

Definition C.5 (**(p, q)-tensor**)**.** Let V be a finite dimensional vector space over a field F. For non-negative integers p and q, the tensor product

$$T^p_q(V) = \underbrace{V \otimes_F \cdots \otimes_F V}_{p \text{ factors}} \otimes_F \underbrace{V^* \otimes_F \cdots \otimes_F V^*}_{q \text{ factors}} = V^{\otimes p} \otimes (V^*)^{\otimes q}$$

where $V^{\otimes k}$ is k-fold tensor product of V with itself, is called the space of tensors of type (p, q), where p is the *contravariant type* and q is the *covariant type*. If $p = q = 0$, then $T^p_q(V) = F$.

Remark C.8. For a finite dimensional vector space V over a field F, we have $V \cong V^{**}$, hence we can generalize Corollary C.1 to get:

$$T^p_q(V) = V^{\otimes p} \otimes_F (V^*)^{\otimes q} \cong ((V^*)^{\otimes p} \otimes_F V^{\otimes q})^* \cong \mathrm{Hom}_F\left((V^*)^{\times p} \times V^{\times q}, F\right)$$

where $V^{\times k}$ is k-fold cartesian product of V with itself. Therefore, the k-tensor defined in Definition 1.6 is in fact a $(0, k)$-tensor, i.e. a vector belonging to $(V^*)^{\otimes k}$. In other words, as seen in Remark 1.4, $T^0_k(V) = T^k_0(V^*) = \mathcal{L}^k(V)$.

Proposition C.9. *Let V be a finite dimensional vector space over a field F. Then*

1. $\dim_F(T^p_q(V)) = (\dim_F(V))^{p+q}$
2. $T^p_q(V) \otimes T^r_s(V) \cong T^{p+r}_{q+s}(V)$

Definition C.6 (Tensor algebra)**.** The external direct sum

$$T(V) = \bigoplus_{p=0}^{\infty} T^p_0(V)$$

is a graded algebra, where $T^p_0(V)$ are the elements of grade p. This graded algebra $T(V)$ is called the *tensor algebra* over V.

Remark C.9. Since

$$T^0_q(V) = (V^*)^{\otimes q} = T^q_0(V^*)$$

there is no need to look separately at $T^0_q(V)$.

Definition C.7 (Antisymmetric tensor)**.** Let V be a finite dimensional vector space and $\tau \in T^p_0(V)$. For each $\sigma \in S_p$, we have the isomorphism on $T^p_0(V)$ defined as

$$\begin{aligned} \lambda_\sigma : T^p_0(V) &\to T^p_0(V) \\ x_1 \otimes \cdots \otimes x_p &\mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} \end{aligned}$$

which we extend by linearity. A tensor $\tau \in T^p_0(V)$ is said to be *antisymmetric* $(p, 0)$-tensor if $\lambda_\sigma(\tau) = (\operatorname{sgn}\sigma)\tau$ for all permutations $\sigma \in S_p$.

Remark C.10. The set of all antisymmetric $(p, 0)$-tensors

$$\bigwedge^p(V) := \{\tau \in T^p_0(V) \mid \lambda_\sigma(\tau) = (\operatorname{sgn}\sigma)\tau \text{ for all } \sigma \in S_p\}$$

is a subspace of $T^p_0(V)$, called the *antisymmetric tensor space* or *exterior product space* of degree $(p, 0)$ over V.

Remark C.11. Note that if $\operatorname{char}(F) \neq 2$ then alternating and skew symmetric tensors are the same [28, pp. 391, 398]. Since we have $F = \mathbb{R}$ or $\mathbb{C}$, the alternating k-tensor defined in Definition 1.9 is in fact an antisymmetric $(0, k)$-tensor, i.e. a vector belonging to $\bigwedge^k(V^*)$. In other words, as seen in Definition 1.37, $\bigwedge^k(V^*) = \mathcal{A}^k(V)$. Hence the definition and properties of wedge product (or exterior product) stated in subsection 1.1.2, like $\dim_F(\bigwedge^p(V)) = \binom{n}{p}$ and $\bigwedge^p(V) = 0$ for $p > n$ where $n = \dim_F(V)$, hold here also.

Definition C.8 (Antisymmetric tensor algebra)**.** The graded algebra

$$\bigwedge(V) = \bigoplus_{p=0}^{n} \bigwedge^p(V)$$

where $\dim_F(V) = n$, is called *antisymmetric tensor algebra* or *exterior algebra* of V.

Proposition C.10. *The exterior algebra of a direct sum is isomorphic to the tensor product of the exterior algebras. That is, if V and W are vector spaces over a field F, then*

$$\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes_F \bigwedge(W)$$

This is a graded isomorphism; i.e.,

$$\bigwedge^k(V \oplus W) \cong \bigoplus_{p+q=k} \bigwedge^p(V) \otimes_F \bigwedge^q(W)$$

Proposition C.11. *Complexification commutes with the operations of taking exterior powers. That is, if V is a real vector space there is a natural isomorphism $\left(\bigwedge_{\mathbb{R}}^{p} V\right)_{\mathbb{C}} \cong \bigwedge_{\mathbb{C}}^{p}(V_{\mathbb{C}})$, where the left-hand exterior power is taken over $\mathbb{R}$ while the right-hand one is taken over $\mathbb{C}$.*

Remark C.12. If V is endowed with a complex structure J, then we introduce the notation

$$\bigwedge^{p,q} V := \bigwedge^{p}(V^{1,0}) \otimes_{\mathbb{C}} \bigwedge^{q}(V^{0,1})$$

where $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ as shown in Proposition C.7. Hence we have

$$\bigwedge^{k} V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \bigwedge^{p,q} V$$

Definition C.9 (Natural projection)**.** With respect to the direct sum decomposition of $\bigwedge V_{\mathbb{C}} = \bigoplus_{k=0}^{n} \bigwedge^{k} V_{\mathbb{C}}$ one defines the *natural projections*

$$\Pi^{k} : \bigwedge V_{\mathbb{C}} \to \bigwedge^{k} V_{\mathbb{C}} \quad \text{and} \quad \Pi^{p,q} : \bigwedge V_{\mathbb{C}} \to \bigwedge^{p,q} V$$

Remark C.13. The operator Π^{k} does not depend on the complex structure J, but the operator $\Pi^{p,q}$ certainly do.

Appendix D

Analysis

D.1 Several variable holomorphic functions

In this section some definitions and facts from [10, §I.A], [12, §1.1] and [15, §1.2] will be stated.

Definition D.1 (Open polydisc)**.** An *open polydisc* or *open polycylinder* in $\mathbb{C}^n$ is a subset $\Delta(z;r) \subset \mathbb{C}^n$ of the form

$$\Delta(z;r) = \Delta(z_1,\dots,z_n;r_1,\dots,r_n) = \{w \in \mathbb{C}^n : |w_j - z_j| < r_j, 1 \le j \le n\}$$

Definition D.2 (Closed polydisc)**.** The closure of $\Delta(z;r)$ is called the *closed polydisc* with center z and polyradius r, and is denoted by $\overline{\Delta}(z;r)$.

Remark D.1. The open polydiscs form a basis for the product topology on $\mathbb{C}^n$. Considered only as a topological space (or as a real vector space), $\mathbb{C}^n$ is the same as $\mathbb{R}^{2n}$, the ordinary Euclidean space of $2n$ dimensions.

Definition D.3 (Several variable holomorphic function)**.** A complex-valued function f defined on an open subset $U \subset \mathbb{C}^n$ is called *holomorphic* in U if each point $w = (w_1,\dots,w_n) \in U$ has an open neighborhood W, $w \in W \subset U$, such that the function f has a power series expansion

$$f(z) = f(z_1,\dots,z_n) = \sum_{j_1,\dots,j_n=0}^{\infty} a_{j_1\dots j_n}(z_1 - w_1)^{j_1}\cdots(z_n - w_n)^{j_n}$$

which converges for all $z \in W$.

Remark D.2. The set of all complex-valued functions holomorphic in U is denoted by $\mathcal{O}(U)$. Clearly, if f is holomorphic in $U \subset \mathbb{C}^n$, then f is smooth in U, i.e. $f \in \mathcal{O}(U)$ implies that $f \in C^\infty(U)$.

Proposition D.1. *If a complex-valued function f is holomorphic in an open subset $U \subset \mathbb{C}^n$, then it is continuous in U and is holomorphic in each variable separately.*

Proof. The function f has a power series expansion of the form

$$f(z) = f(z_1,\dots,z_n) = \sum_{j_1,\dots,j_n=0}^{\infty} a_{j_1\dots j_n}(z_1 - w_1)^{j_1}\cdots(z_n - w_n)^{j_n}$$

which is absolutely uniformly convergent in all suitably small open polydiscs $\Delta(w;r)$ [3, Theorem III.1.3]. Therefore, the function f is continuous in such polydiscs $\Delta(w;r)$, and hence any function holomorphic in U is also continuous in U. Moreover, the power series can be rearranged arbitrarily and will still represent the function f. In particular, if the coordinates

$z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$ are given any fixed values $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$, then this power series can be rearranged as a convergent power series in the variable z_j alone, for z_j sufficiently close to w_j and each a_k sufficiently close to z_k for $k = 1, \dots, j-1, j+1, \dots, n$. Therefore, the function f is holomorphic in each variable separately throughout the domain in which it is analytic. □

Definition D.4 (Complex partial differential operators)**.** As in Definition 3.4, we define the following two first-order linear partial differential operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for $z_j = x_j + iy_j$ and $j = 1, \dots, n$.

Remark D.3. The previous result implies that the operation $\partial/\partial z_j$ is well-defined for each complex-valued holomorphic function. Therefore, when applied to holomorphic functions, the operator $\partial/\partial z_j$ coincides with the ordinary complex derivative with respect to one of the variables z_j. For example,

$$\begin{aligned}
\frac{\partial}{\partial z_j} z_j^n &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(x_j + iy_j)^n \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_j}(x_j + iy_j)^n - i\frac{\partial}{\partial y_j}(x_j + iy_j)^n\right) \\
&= \frac{1}{2}\left(n(x_j + iy_j)^{n-1} - i \cdot n(x_j + iy_j)^{n-1} i\right) \\
&= n(x_j + iy_j)^{n-1} \\
&= n z_j^{n-1}
\end{aligned}$$

Proposition D.2 (Cauchy formula for polydisc)**.** *Let $w \in \mathbb{C}^n$ and f be a complex-valued holomorphic function in an open neighborhood of a closed polydisc $\overline{\Delta}(w; r)$. Then, for any $z \in \Delta(w; r)$, it holds that*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_n - w_n| = r_n} \cdots \int\limits_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

Proof. From the previous result we know that f is holomorphic in each variable in an open neighborhood of $\overline{\Delta}(w; r)$. By repeated application of Cauchy integral formula for functions of one variable leads to the formula

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int\limits_{|\zeta_n - w_n| = r_n} \frac{f(z_1, \dots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} d\zeta_n \\
&= \frac{1}{(2\pi i)^2} \int\limits_{|\zeta_n - w_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} \int\limits_{|\zeta_{n-1} - w_{n-1}| = r_{n-1}} \frac{f(z_1, \dots, \zeta_{n-1}, \zeta_n)}{\zeta_{n-1} - z_{n-1}} d\zeta_{n-1} \\
&\;\vdots \\
&\;\vdots \\
&= \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_n - w_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} \cdots\cdots \int\limits_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_{n-1}, \zeta_n)}{\zeta_1 - z_1} d\zeta_1
\end{aligned}$$

for all $z \in \Delta(w;r)$. For any fixed point $z = (z_1, \ldots, z_n)$, from the the previous result, it follows that this integrand is continuous on the compact domain of integration. Hence the iterated integral can be replaced by a single multiple integral

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_n - w_n| = r_n} \cdots \int\limits_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

completing the proof. □

Theorem D.1 (Osgood's lemma)**.** *If a complex-valued function f is continuous in an open set $U \subset \mathbb{C}^n$ and is holomorphic in each variable separately, then it is holomporphic in U.*

Proof. Select any point $w \in U$ and any closed polydisc $\overline{\Delta}(w;r) \subset U$. Since f is holomorphic in each variable separately in an open neighborhood of $\overline{\Delta}(w;r)$, a repeated application of Cauchy integral formula leads to the formula

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1 - w_1| = r_1} \frac{d\zeta_1}{\zeta_1 - z_1} \cdots\cdots \int\limits_{|\zeta_n - w_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} f(\zeta)$$

for all $z \in \Delta(w;r)$. For any fixed point $z = (z_1, \ldots, z_n)$, this integrand is continuous on the compact domain of integration. Hence the iterated integral can be replaced by a single multiple integral

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1 - w_1| = r_1} \cdots \int\limits_{|\zeta_n - w_n| = r_n} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \tag{D.1}$$

Note that $|z_j - w_j| < |\zeta_j - w_j|$ for all $j = 1, \ldots, n$. Therefore, we have

$$\sum_{k=0}^{\infty} \left(\frac{z_j - w_j}{\zeta_j - w_j} \right)^k = \frac{1}{1 - \frac{z_j - w_j}{\zeta_j - w_j}} = \frac{\zeta_j - w_j}{\zeta_j - z_j} \quad \forall j = 1, \ldots, n$$

Hence for a fixed $z \in \Delta(w;r)$, we have the following absolutely uniformly convergent series expansion for all points ζ on the domain of integration

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}}{(\zeta_1 - w_1)^{k_1 + 1} \cdots (\zeta_n - w_n)^{k_n + 1}} \tag{D.2}$$

Using (D.2) in (D.1), and interchanging the orders of summation and integration, we get the power series expansion of f

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1 - w_1| = r_1} \cdots \int\limits_{|\zeta_n - w_n| = r_n} f(\zeta) d\zeta_1 \cdots d\zeta_n \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}}{(\zeta_1 - w_1)^{k_1 + 1} \cdots (\zeta_n - w_n)^{k_n + 1}} \\ &= \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1 \ldots k_n} (z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n} \\ &\quad \text{where } a_{k_1 \ldots k_n} = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1 - w_1| = r_1} \cdots \int\limits_{|\zeta_n - w_n| = r_n} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - w_1)^{k_1 + 1} \cdots (\zeta_n - w_n)^{k_n + 1}} \end{aligned}$$

Therefore, f is a holomorphic function in U. □

Remark D.4. The hypothesis that the function f be continuous in U is not required, i.e. Goursat's theorem [3, §IV.8] can be generalized to several variables. However, this stronger result, called Hartogs's theorem, is much more difficult to prove [15, Theorem 1.2.5].

Corollary D.1. *The power series expansion of a holomorphic function $f : U \to \mathbb{C}$ at $w \in U \subset \mathbb{C}^n$ is uniquely determined by that function and it converges within the polydisc $\Delta(w; r)$ contained in U.*

Proof. By differentiating (D.1) it follows that

$$\frac{\partial^{k_1+\cdots+k_n} f(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} = \frac{k_1! \cdots k_n!}{(2\pi i)^n} \int\limits_{|\zeta_1 - w_1| = r_1} \cdots \int\limits_{|\zeta_n - w_n| = r_n} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{k_1+1} \cdots (\zeta_n - z_n)^{k_n+1}}$$

Comparing this with the final statement of the above theorem, we get

$$a_{k_1 \ldots k_n} = \frac{1}{k_1! \cdots k_n!} \frac{\partial^{k_1+\cdots+k_n} f(w)}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}}$$

Therefore, all the power series expansion convergent within any fixed compact subset of $\Delta(w; r)$ must coincide. □

Theorem D.2 (Cauchy-Riemann criterion)**.** *A complex-valued smooth*[1] *function f defined in an open subset $U \subset \mathbb{C}^n$ is holomorphic in U if and only if it satisfies the system of partial differential equations*

$$\frac{\partial}{\partial \overline{z}_j} f(z) = 0, \quad \forall\ j = 1, \ldots, n$$

Proof. At any point in U, we consider $f(z)$ to be a function of the single variable z_j, holding the other variables constant. Next, we decompose f into its real and imaginary parts by writing $f(z) = u(z) + iv(z)$, and observe that

$$\frac{\partial}{\partial \overline{z}_j} f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)(u(z) + iv(z)) = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j}\right)$$

Therefore, $\partial f(z)/\partial \overline{z}_j = 0$, for all $j = 1, \ldots, n$ is equivalent to the classical Cauchy-Riemann equations for each variable separately. This is equivalent to the function f being holomorphic in each variable separately. The desired result follows from Proposition D.1 and Theorem D.1. □

Remark D.5. The transition from the real partial differentials to the complex partial differentials can be illustrated for the simplest case. For some open set $U \subset \mathbb{C} = \mathbb{R}^2$, consider the differentiable map $f : U \to \mathbb{R}^2$ such that $f(x, y) = (u(x, y), v(x, y))$. Then the total derivative[2] $Df(w)$ at point $w = (r, s) \in U$ is a $\mathbb{R}$-linear map between tangent spaces $Df(w) : T_w\mathbb{R}^2 \to T_{f(w)}\mathbb{R}^2$. With respect to the standard basis we get the real Jacobian matrix

$$Df(w) = \begin{bmatrix} \left.\dfrac{\partial u}{\partial x}\right|_w & \left.\dfrac{\partial u}{\partial y}\right|_w \\ \left.\dfrac{\partial v}{\partial x}\right|_w & \left.\dfrac{\partial v}{\partial y}\right|_w \end{bmatrix}$$

Next, we extend $Df(w)$ to a $\mathbb{C}$-linear map $\widetilde{Df(w)} : T_w\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} \to T_{f(w)}\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$. If we consider $f = u + iv$ and $z = x + iy$, then with respect to the new basis we get the complexified Jacobian matrix

$$\widetilde{Df(w)} = \begin{bmatrix} \left.\dfrac{\partial f}{\partial z}\right|_w & \left.\dfrac{\partial f}{\partial \overline{z}}\right|_w \\ \left.\dfrac{\partial \overline{f}}{\partial z}\right|_w & \left.\dfrac{\partial \overline{f}}{\partial \overline{z}}\right|_w \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial f}{\partial z}\right|_w & \left.\dfrac{\partial f}{\partial \overline{z}}\right|_w \\ \overline{\left(\left.\dfrac{\partial f}{\partial \overline{z}}\right|_w\right)} & \overline{\left(\left.\dfrac{\partial f}{\partial z}\right|_w\right)} \end{bmatrix}$$

[1]That is, continuously differentiable in the underlying real coordinates of $\mathbb{C}^n$. In other words, $f \in C^\infty(U)$.
[2]This is same as what we defined as *pushforward of a vector* in Definition 1.5.

Therefore, if f is holomorphic, then the differential in the new base system is given by the diagonal matrix

$$\begin{bmatrix} \left.\frac{\partial f}{\partial z}\right|_w & 0 \\ 0 & \left.\frac{\partial \overline{f}}{\partial \overline{z}}\right|_w \end{bmatrix}$$

Proposition D.3. *Let U be an open set in $\mathbb{C}^n$. Then:*

1. *$\mathcal{O}(U)$ is a ring under the operations $(f+g)(z) = f(z) + g(z)$ and $(fg)(z) = f(z)g(z)$.*
2. *If $f \in \mathcal{O}(U)$ and is nowhere vanishing, then $1/f \in \mathcal{O}(U)$*
3. *If $f \in \mathcal{O}(U)$ and is real-valued or has constant modulus, then f is constant.*

Theorem D.3 (Identity theorem)**.** *Let U be a connected open set in $\mathbb{C}^n$ and $f, g \in \mathcal{O}(U)$. If $f(z) = g(z)$ for all points z in an open subset $V \subset U$, then $f(z) = g(z)$ for all points $z \in U$.*

Proof. This is a straight-forward generalization of the single-variable identity theorem, see [10, Theorem I.A.6] for the proof. □

Theorem D.4 (Hartogs's extension theorem)**.** *Let $U \subset \mathbb{C}^n$ for $n > 1$ be a bounded open set and K be a compact subset U with the property that $U \setminus K$ is connected. If f is a complex-valued holomorphic function on $U \setminus K$, then there is a unique complex-valued holomorphic function F on U such that $F|_{U \setminus K} = f$.*

Proof. The proof involves a typical $\overline{\partial}$-argument as seen in the proof of $\overline{\partial}$-Poincaré lemma, see [15, Theorem 1.2.6] and [31, §2.2]. □

Remark D.6. This extension does not hold when $n = 1$. For example, consider the function $f(z) = 1/z$, which is clearly holomorphic in $\mathbb{C} \setminus \{0\}$, but cannot be continued as a holomorphic function on the whole $\mathbb{C}$.

This extension also does not hold when $U \setminus K$ is not connected. For example, consider the open ball $U = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ and the compact set $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1/2\}$. Then $U \setminus K = U_1 \cup U_2$ where

$$U_1 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1/2\}$$
$$U_2 := \{(z_1, z_2) \in \mathbb{C}^2 : 1/2 < |z_1|^2 + |z_2|^2 < 1\}$$

such that $U_1 \cap U_2 = \emptyset$. Now consider the holomorphic function f defined on $U \setminus K$ as

$$f(z) = \begin{cases} 0 & \text{if } z \in U_1 \\ 1 & \text{if } z \in U_2 \end{cases}$$

But this clearly can't be extended to a holomorphic function on U.

D.2 Algebraic properties of $\mathcal{O}_w$

In this section some definitions and facts from [15, §6.4], [10, §II.A, II.B] and [12, §1.1] will be stated.

Definition D.5 (Ring of germs of holomorphic functions)**.** For $w \in \mathbb{C}^n$, consider the set

$$\mathcal{O}_w := \{(U, f) | w \in U \subset \mathbb{C}^n \text{open }, f \in \mathcal{O}(U)\}/\sim$$

where $(U, f) \sim (V, g)$ if $\exists W$ open, $w \in W$ such that $W \subset V \cap U$ and $f|_W = g|_W$. The representative function of an equivalence class is called a *germ of holomorhic functions at* w and $\mathcal{O}_w$ is called the *ring of germs of holomorphic functions* at $w \in \mathbb{C}^n$ with the following operations: $[(U, f)] + [(V, g)] := [(U \cap V, f + g)]$ and $[(U, f)] \cdot [(V, g)] := [(U \cap V, fg)]$.

Remark D.7. The ring $\mathcal{O}_w$ is a commutative ring with an identity element. The zero of this ring is the germ of the function which vanishes identically, and the identity of the ring is the germ of the function which is identically one.

Lemma D.1. *$\mathcal{O}_w$ is an integral domain.*

Proof. Consider two arbitrary germs $[(U, f)]$ and $[(V, g)]$ such that

$$[(U, f)] \cdot [(V, g)] = [(U \cap V, fg)] = [(W, 0)]$$

for some open neighborhood W of w. Hence $f(z)g(z) = 0$ in some connected open neighborhood $W' \subset W \cap V \cap U$ of w. If $f(z_0) \neq 0$ for a single point $z_0 \in W'$, then by continuity $f(z) \neq 0$ in an open neighborhood of z_0 and therefore $g(z) = 0$ in that open neighborhood. By Theorem D.3, therefore, it follows that $g(z) = 0$ in W', hence that $(V, g) \sim (W', 0)$. □

Lemma D.2. *A germ $[(U, f)] \in \mathcal{O}_w$ is a unit if and only if $f(w) \neq 0$.*

Proof. We need to show that the multiplicative inverse of $[(U, f)]$ exists if and only if f does not vanish at w. Suppose that $[(U, f)] \in \mathcal{O}_w$ such $f(w) \neq 0$. By continuity, $f(z) \neq 0$ in an open neighborhood $V \subset U$ of w; and hence $1/f(z)$ is continuous in V and is holomorphic in each variable separately in V. An application of Proposition D.3(2) shows that $1/f(z)$ is holomorphic in V, hence $[(V, 1/f)] \in \mathcal{O}_w$. □

Lemma D.3. *$\mathcal{O}_w$ is a local ring.*

Proof. Since a germ $[(U, f)]$ is a unit if and only if $f(w) \neq 0$, any proper ideal $\mathfrak{a}$ of $\mathcal{O}_w$ consists only of germs which vanish at w. So the unique maximal ideal in $\mathcal{O}_w$ is

$$\mathfrak{m} := \{[(U, f)] \in \mathcal{O}_w | f(w) = 0\}$$

Therefore, $\mathcal{O}_w$ is a local ring. □

Definition D.6 (Order of a holomorphic function)**.** Let f be a holomorphic function in a neighborhood of w in $\mathbb{C}^n$ such that

$$f(z) = f(z_1, \dots, z_n) = \sum_{j_1,\dots,j_n=0}^{\infty} a_{j_1 \dots j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

Then the order of f is defined to be the least value of $j_1 + \ldots + j_n$ for which $a_{j_1 \dots j_n} \neq 0$, i.e.

$$\operatorname{ord}(f) := \min\{j_1 + \ldots + j_n | a_{j_1 \dots j_n} \neq 0\}$$

Remark D.8. If $\operatorname{ord}(f) = k$, then there exists a non-singular linear change of coordinates so that in the new coordinates, the coefficient of z_n^k is 1. When f is of this form it is said to be *normalized* (with respect to the variable z_n) *of order k*.

Definition D.7 (Weierstrass polynomial)**.** A function W, holomorphic in a neighborhood of $w \in \mathbb{C}^n$ is called a *Weierstrass polynomial* of degree m, if we have

$$W(z_1, \dots, z_n) = W(z', z_n) = z_n^m + a_{m-1}(z') z_n^{m-1} + \ldots + a_1(z') z_n + a_0(z')$$

where $z' = (z_1, \dots, z_{n-1})$ and a_j are holomorphic functions in a neighborhood of $w' = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$ and $a_j(0) = 0$ for $j = 0, \dots, m-1$.

Remark D.9. If we denote the ring of germs of holomorphic functions in the variables $z_1, \dots, z_{n-1}$ by $\mathcal{O}_{w'}$, then[3] the Weierstrass polynomial $W \in \mathcal{O}_{w'}[z_n]$ such that the coefficients are non-unit elements of $\mathcal{O}_{w'}$. Note that $\mathcal{O}_{w'} \subset \mathcal{O}_{w'}[z_n] \subset \mathcal{O}_w$.

[3] From now onwards we will abuse the notation for germs, i.e. instead of writing $[(U, f)] \in \mathcal{O}_w$ we will simply write $f \in \mathcal{O}_w$ such that f is an holomorphic function in an open neighborhood of w.

Theorem D.5 (Weierstrass preparation theorem). *Let f be a normalized holomorphic function of order k in a neighborhood of $w \in \mathbb{C}^n$. Then in a small neighborhood of w, f can be written uniquely as*

$$f(z) = u(z) \cdot W(z)$$

where $u \in \mathcal{O}_w$ is a unit and $W \in \mathcal{O}_{w'}[z_n]$ is a Weierstrass polynomial of degree k.

Proof. To prove this we will need Hartogs's extension theorem [15, Theorem 6.4.5] or Riemann extension theorem [10, Theorem II.B.2]. □

Theorem D.6 (Weierstrass division theorem). *Let $W \in \mathcal{O}_{w'}[z_n]$ be a Weierstrass polynomial in z_n of degree k. Then any $f \in \mathcal{O}_w$ can be written in a unique manner in the form $f = g \cdot W + r$, for some $g \in \mathcal{O}_w$ and $r \in \mathcal{O}_{w'}[z_n]$ a polynomial of degree less than k. Moreover, if $f \in \mathcal{O}_{w'}[z_n]$ then necessarily $g \in \mathcal{O}_{w'}[z_n]$.*

Proof. For a proof, see [10, Theorem II.B.3]. □

Lemma D.4. *A Weierstrass polynomial $W \in \mathcal{O}_{w'}[z_n]$ is reducible over $\mathcal{O}_w$ if and only if it is reducible over $\mathcal{O}_{w'}[z_n]$. Moreover, if W is reducible, then all of its non-unit factors are Weierstrass polynomials of $\mathcal{O}_{w'}[z_n]$.*

Proof. ($\Rightarrow$) Suppose that W is reducible over $\mathcal{O}_w$, and write $W = f_1 f_2$ for some non-units $f_1, f_2 \in \mathcal{O}_w$. Since W is a Weierstrass polynomial, it is normalized and hence both f_1 and f_2 are also normalized. Applying Theorem D.5, we get $f_1 = u_1 W_1$ and $f_2 = u_2 W_2$ for some units $u_1, u_1 \in \mathcal{O}_w$ and Weierstrass polynomials $W_1, W_2 \in \mathcal{O}_{w'}[z_n]$. Thus $W = (u_1 u_2)(W_1 W_2)$. But since $W_1 W_2$ is also a Weierstrass polynomial, the uniqueness part of the Theorem D.5 implies that[4] $u_1 u_1 = 1$ and $W_1 W_2 = W$. Therefore W is reducible in the ring of polynomials $\mathcal{O}_{w'}[z_n]$ as well, and its factors are Weierstrass polynomials.

($\Leftarrow$) Suppose that W is reducible over $\mathcal{O}_{w'}[z_n]$, and write $W = g_1 g_2$ for some non-units $g_1, g_2 \in \mathcal{O}_{w'}[z_n]$. If g_1 was a unit in $\mathcal{O}_w$, then $W/g_1 = g_2$ and by the application of Theorem D.6 it would follow that $1/g_1 \in \mathcal{O}_{w'}[z_n]$. This is impossible, since g_1 is a non-unit element of $\mathcal{O}_{w'}[z_n]$. Therefore g_1 is a non-unit element of $\mathcal{O}_w$. Similarly, g_2 is non-unit element of $\mathcal{O}_w$. Therefore, W is reducible in $\mathcal{O}_w$ as well. □

Theorem D.7. *The local ring $\mathcal{O}_w$ is a unique factorization domain.*

Proof. Note that for any fixed point $w \in \mathbb{C}^n$ the linear change of variable $\zeta_j = z_j - w_j$ induces a canonical isomorphism between the rings $\mathcal{O}_w$ and $\mathcal{O}_0$. Hence, for the local theory, it is sufficient to consider only the ring $\mathcal{O}_0$ for $0 \in \mathbb{C}^n$. We will proceed by induction on n.

For $n = 1$, the theorem is trivial: if $f \in \mathcal{O}_0$ has order k then $f(z) = z^k g(z)$ where $g(0) \neq 0$, so that g is a unit in $\mathcal{O}_0$.

Let $\mathcal{O}_0^{n-1}$ denote the ring of germs of holomorphic functions at $0 \in \mathbb{C}^{n-1}$. We will continue the abuse of notations by writing $g \in \mathcal{O}_0^{n-1}$ instead of $[(U, g)] \in \mathcal{O}_0^{n-1}$. Now assume that the result is true for $n - 1$, i.e. $\mathcal{O}_0^{n-1}$ is a unique factorization domain. Let $f \in \mathcal{O}_0^n$. Without loss of generality, we can assume that f is normalized of order k. Then by Theorem D.5 we have $f = u \cdot W$, where $W \in \mathcal{O}_0^{n-1}[z_n]$. From Gauss Lemma[5] it follows that $\mathcal{O}_0^{n-1}[z_n]$ is a unique factorization domain, and $W = W_1 \cdots W_m$ where $W_j \in \mathcal{O}_0^{n-1}[z_n]$ are irreducible elements. By Lemma D.4, it follows that the W_j's are Weierstrass polynomials. Therefore, $f = u \cdot W_1 \cdots W_m$. If f could also be written as $f = V_1 \cdots V_\ell$, then we apply Theorem D.5 to each $V_j \in \mathcal{O}_0^n$ to obtain $V_j = u'_j \cdot W'_j$, that is, $f = u' \cdot W'_1 \cdots W'_\ell$, where u' is a unit and $W'_j \in \mathcal{O}_0^{n-1}[z_n]$ are Weierstrass

[4]Here again we are abusing notations. Actually, the constant function 1 and $u_1 u_2$ will represent the same equivalence class in $\mathcal{O}_w$, and $W_1 W_2$ and W will represent same equivalence class in $\mathcal{O}_{w'}[z_n]$. That is, in some small enough neighborhood of w, all these equalities, like $W = f_1 f_2$, will hold.

[5]It implies that R is a unique factorization domain if and only if $R[x]$ is a unique factorization domain. For proof, see Theorem 9.3.7 on p. 304 of Dummit and Foote's book "Abstract Algebra".

polynomials. Since there is only one way to write f as a unit times a Weierstrass polynomial, we conclude that

$$W_1 \cdots W_m = W_1' \cdots W_\ell'$$

By induction hypothesis $\mathcal{O}_0^{n-1}[z_n]$ is a unique factorization domain, and hence $\{W_1, \ldots, W_m\} = \{W_1', \ldots, W_\ell'\}$. □

D.3 Several variable holomorphic mappings

In this section some definitions and facts from [10, §I.A, I.B], [12, §1.1] and [6, §I.7] will be stated.

Definition D.8 (Several varaible holomorphic mapping)**.** Let $U \subset \mathbb{C}^n$ be an open set, and $g : U \to \mathbb{C}^m$ be any mapping such that

$$g(z) = g(z_1, \ldots, z_n) = (g_1(z), \ldots, g_m(z))$$

where $g_j : U \to \mathbb{C}$ for all $j = 1, \ldots, m$. The mapping g is called a *holomorphic mapping* if the m complex-valued functions $g_1, \ldots, g_m$ are holomorphic in U.

Proposition D.4 (Chain rule)**.** *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subsets. If $g : U \to V$ is a holomorphic mapping and $f : V \to \mathbb{C}$ is a holomorphic function, then*

$$\frac{\partial (f \circ g)}{\partial z_j} = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial \overline{w}_k} \frac{\partial \overline{g}_k}{\partial z_j} \right) \quad \text{and} \quad \frac{\partial (f \circ g)}{\partial \overline{z}_j} = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial \overline{z}_j} + \frac{\partial f}{\partial \overline{w}_k} \frac{\partial \overline{g}_k}{\partial \overline{z}_j} \right)$$

where $w_k = g_k(z_1, \ldots, z_n)$ for $k = 1, \ldots, m$.

Proof. We have the following composite maps

$$\begin{array}{ccccc} U & \xrightarrow{\quad g \quad} & V & \xrightarrow{\quad f \quad} & \mathbb{C} \\ (z_1, \ldots, z_n) & \longmapsto & (w_1, \ldots, w_m) & \longmapsto & f(w) \end{array}$$

where $w_k = g_k(z_1, \ldots, z_n)$ for $k = 1, \ldots, m$. We can separate each g_k into real and imaginary parts by writing $g_k(z) = u_k(z) + iv_k(z)$. Since all the functions involved are differentiable in the underlying real coordinates, the usual chain rule for differentiation can be applied as follows:

$$\begin{aligned} \frac{\partial (f \circ g)}{\partial z_j} &= \sum_{k=1}^{m} \left(\frac{\partial f}{\partial u_k} \frac{\partial u_k}{\partial z_j} + \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial z_j} \right) \\ &= \sum_{k=1}^{m} \frac{1}{2} \left(\frac{\partial f}{\partial u_k} - i \frac{\partial f}{\partial v_k} \right) \frac{\partial g_k}{\partial z_j} + \sum_{k=1}^{m} \frac{1}{2} \left(\frac{\partial f}{\partial u_k} + i \frac{\partial f}{\partial v_k} \right) \frac{\partial \overline{g}_k}{\partial z_j} \\ &= \sum_{k=1}^{m} \left(\frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial \overline{w}_k} \frac{\partial \overline{g}_k}{\partial z_j} \right) \end{aligned}$$

Similarly we can prove for $\partial / \partial \overline{z}$. □

Corollary D.2. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subsets. If $g : U \to V$ is a holomorphic mapping and $f : V \to \mathbb{C}$ is a holomorphic function, then the composition $f \circ g \in \mathcal{O}(U)$.*

Definition D.9 (Several complex variables biholomorphic mapping)**.** Let $U, V \subset \mathbb{C}^n$ be two open sets. A holomorphic mapping $f : U \to V$ is called *biholomorphic* if it is bijective and its inverse $f^{-1} : V \to U$ is also holomorphic.

Definition D.10 (Jacobian matrix of a holomorphic mapping)**.** Let $g : U \to \mathbb{C}^m$ be a holomorphic mapping, where U is an open subset of $\mathbb{C}^n$. The Jacobian matrix of the mapping g at a point $w \in U$ is defined to be the matrix

$$\mathrm{Jac}(g)(w) := \left[\left. \frac{\partial g_j}{\partial z_k} \right|_w \right]_{\substack{1\leq j\leq m \\ 1\leq k\leq n}}$$

Remark D.10. This Jacobian matrix is also related to the complexified Jacobian matrix for total derivative discussed in Remark D.5. For some open set $U \subset \mathbb{C}^n = \mathbb{R}^{2n}$, consider the differentiable map $f : U \to \mathbb{C}^m = \mathbb{R}^{2m}$ such that

$$g(z) = g(z_1, \dots, z_n) = g(\underline{x}, \underline{y}) = \big(u_1(\underline{x}, \underline{y}), \dots, u_m(\underline{x}, \underline{y}), v_1(\underline{x}, \underline{y}), v_m(\underline{x}, \underline{y})\big)$$

where $\underline{x} = (x_1, \dots, x_n)$, $\underline{y} = (y_1, \dots, y_n)$. Then the total derivative $Dg(z)$ at point $w = (\underline{r}, \underline{s}) \in U$ is a $\mathbb{R}$-linear map between tangent spaces $Dg(w) : T_w\mathbb{R}^{2n} \to T_{f(w)}\mathbb{R}^{2m}$. With respect to the standard basis we get the real Jacobian matrix

$$Dg(w) = \begin{bmatrix} \left[\left. \frac{\partial u_j}{\partial x_k} \right|_w \right]_{j,k} & \left[\left. \frac{\partial u_j}{\partial y_k} \right|_w \right]_{j,k} \\ \left[\left. \frac{\partial v_j}{\partial x_k} \right|_w \right]_{j,k} & \left[\left. \frac{\partial v_j}{\partial x_k} \right|_w \right]_{j,k} \end{bmatrix}$$

Next, we extend $Dg(w)$ to a $\mathbb{C}$-linear map $\widetilde{Dg(w)} : T_w\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T_{f(w)}\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$. If we consider $g_j = u_j + iv_j$ for all $j = 1, \dots, m$ and $z_k = x_k + iy_k$ for $k = 1, \dots, n$, then with respect to the new basis we get the complexified Jacobian matrix

$$\widetilde{Dg(w)} = \begin{bmatrix} \left[\left. \frac{\partial g_j}{\partial z_k} \right|_w \right]_{j,k} & \left[\left. \frac{\partial g_j}{\partial \overline{z}_k} \right|_w \right]_{j,k} \\ \left[\left. \frac{\partial \overline{g}_j}{\partial z_k} \right|_w \right]_{j,k} & \left[\left. \frac{\partial \overline{g}_j}{\partial \overline{z}_k} \right|_w \right]_{j,k} \end{bmatrix}$$

Therefore, if g is holomorphic, then the differential in the new base system is given by the diagonal matrix

$$\begin{bmatrix} \mathrm{Jac}(g)(w) & 0 \\ 0 & \overline{\mathrm{Jac}(g)(w)} \end{bmatrix}$$

In particular, for a holomorphic function g we have

$$\det\left(Dg(w)\right) = \det\left(\mathrm{Jac}(g)(w)\right) \det\left(\overline{\mathrm{Jac}(g)(w)}\right) = \left|\det\left(\mathrm{Jac}(g)(w)\right)\right|^2 \geq 0$$

Proposition D.5. *Let $g : U \to V$ be a bijective holomorphic map between two open subsets U and V of $\mathbb{C}^n$. Then* $\mathrm{Jac}(g)(w) \neq 0$ *for all $w \in U$. In particular, g is biholomorphic.*

Proof. The proof involves the use of Implicit Function Theorem[6]. For complete proof, see [12, Proposition 1.1.13]. □

Remark D.11. Recall that the product topology on $\mathbb{C}^n = \mathbb{R}^{2n}$ is equivalent to the metric topology, i.e. topology generated by open polydiscs is same as the one generated by open balls. Next, observe that the unit open ball $B(0, 1)$ and unit open polydisc $\Delta(0; 1)$ are diffeomorphic:

[6] For the exact statement and proof, see [10, Theorem I.B.5], [12, Proposition 1.1.11] and [6, Theorem I.7.6]. The proof of the implicit function theorem is a special case of the Weierstrass preparation theorem, discussed in the previous section.

1. $B(0,1)$ is diffeomorphic to $\mathbb{R}^{2n}$ and the diffeomorphism is given by the map

$$\begin{aligned}\phi : B(0,1) &\to \mathbb{R}^{2n}\\ x &\mapsto \frac{x}{\sqrt{1-\|x\|^2}}\end{aligned}$$

2. If $g : (-1,1) \to \mathbb{R}$ is any diffeomorphism, then

$$\begin{aligned}\psi : \Delta(0;1) &\to \mathbb{R}^{2n}\\ (x_1,\ldots,x_{2n}) &\mapsto (g(x_1),\ldots,g(x_{2n}))\end{aligned}$$

   is a smooth map with smooth inverse. Hence $\Delta(0;1)$ is also diffeomorphic to $\mathbb{R}^{2n}$.

However, they are not biholomorphic for $n > 1$ [15, §0.3.2].

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