Sperner's Theorem

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Lemma. *For n a positive integer, the largest of the binomial coefficients*

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$$

is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$$

Proof. Consider the quotient of successive binomial coefficients in the sequence. Let k be an integer with $1 \leq k \leq n$. Then

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k}$$

Hence we have three cases:

Case 1: $\binom{n}{k-1} < \binom{n}{k}$ if $k < n-k+1$

Now, $k < n-k+1$ if and only if $k < (n+1)/2$.
If n is even, then, since k is an integer, $k < (n+1)/2$ is equivalent to $k \leq n/2$. Thus

$$\binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{n/2}$$

If n is odd, then $k < (n+1)/2$ is equivalent to $k \leq (n-1)/2$.

$$\binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{(n-1)/2}$$

Case 2: $\binom{n}{k-1} = \binom{n}{k}$ if $k = n-k+1$

Observe that $k = n-k+1$ if and only if $2k = n+1$. If n is even, $2k \neq n+1$ for any k. Thus, for n even, no two consecutive binomial coefficients in the sequence are equal.
If n is odd, then $2k = n+1$, for $k = (n+1)/2$. For n odd, the only two consecutive binomial coefficients of equal value are

$$\binom{n}{(n-1)/2} \quad \text{and} \quad \binom{n}{(n+1)/2}$$

Case 3: $\binom{n}{k-1} > \binom{n}{k}$ if $k > n-k+1$

Now, $k > n-k+1$ if and only if $k > (n+1)/2$.
If n is even, then, since k is an integer, $k > (n+1)/2$ is equivalent to $k \geq n/2$. Thus

$$\binom{n}{n/2} > \dots > \binom{n}{n-1} > \binom{n}{n}$$

If n is odd, then $k > (n+1)/2$,

$$\binom{n}{(n+1)/2} > \dots > \binom{n}{n-1} > \binom{n}{n}$$

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Now combining the above three cases, if n is even,

$$\binom{n}{0} < \binom{n}{1} < \ldots < \binom{n}{n/2} \qquad \text{and} \qquad \binom{n}{n/2} > \ldots > \binom{n}{n-1} > \binom{n}{n}$$

and if n is odd,

$$\binom{n}{0} < \binom{n}{1} < \ldots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2} \qquad \text{and} \qquad \binom{n}{(n+1)/2} > \ldots > \binom{n}{n-1} > \binom{n}{n}$$

We have following properties of *floor* and *ceiling* functions,

$$\lfloor n/2 \rfloor = \lceil n/2 \rceil = n/2 \quad \text{if } n \text{ is even}$$

and

$$\lfloor n/2 \rfloor = (n-1)/2 \quad \text{and} \quad \lceil n/2 \rceil = (n+1)/2 \quad \text{if } n \text{ is odd}$$

Using these observations about the floor and ceiling functions, for any n,

$$\binom{n}{0} < \binom{n}{1} < \ldots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} \qquad \text{and} \qquad \binom{n}{\lceil n/2 \rceil} > \ldots > \binom{n}{n-1} > \binom{n}{n}$$

Thus, the largest of the binomial coefficients is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$$

□

Definition (Poset)**.** A partially ordered set (also poset) is a set S with a binary relation $\leq$ (sometimes $\subseteq$ is used) such that:

1. $a \leq a$ for all $a \in S$ (reflexivity),
2. if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity),
3. if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).

Definition (Total Order)**.** If for any a and b in S, either $a \leq b$ or $b \leq a$, then the partial order is called a total order, or a linear order.

Definition (Chain)**.** A collection $\mathcal{C}$ of subsets of S is a *chain*, provided that for each pair of subsets in $\mathcal{C}$, one is contained in the other:

$$A_1, A_2 \in \mathcal{C}, A_1 \neq A_2 \quad \text{implies} \quad A_1 \subset A_2 \quad \text{or} \quad A_2 \subset A_1$$

In other words, if a subset of S is totally ordered, it is called a *chain*.
Example: If $n = 5$ and $S = \{1, 2, 3, 4, 5\}$, example of chain is : $\mathcal{C} = \{\{2\}, \{2, 3, 5\}, \{1, 2, 3, 5\}\}$.

Definition (Maximal Chain)**.** If $S = \{1, 2, \ldots, n\}$, a *maximal chain* is a chain with

$$A_0 = \phi \subset A_1 \subset \ldots \subset A_n$$

where $|A_i| = i$ for $i = 0, 1, 2 \ldots, n$.
Example: If $n = 5$ and $S = \{1, 2, 3, 4, 5\}$, then $\mathcal{M} = \{\phi, \{3\}, \{3, 4\}, \{1, 3, 4\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$ is a maximal chain.

Definition (Antichain)**.** Let S be a set of n elements. An *antichain* of S is a collection $\mathcal{A}$ of subsets of S with the property that no subset in $\mathcal{A}$ is contained in another.
In other words, an *antichain* is a set of elements that are pairwise incomparable.
Example: If $S = \{a, b, c, d\}$, then $\mathcal{A} = \{\{a, b\}, \{b, c, d\}, \{a, d\}, \{a, c\}\}$ is an antichain.

Theorem (Sperner's Theorem)**.** *Let S be a set of n elements. Then an antichain on S contains at most* $\binom{n}{\lfloor n/2 \rfloor}$ *sets.*

Proof. Consider the poset of subsets of $S = \{1, 2, \ldots, n\}$. $\mathcal{A} = \{A_1, \ldots, A_m\}$ is an antichain in this poset.

A maximal chain C in this poset will consist of one subset of each cardinality $0, 1, \ldots, n$, and is obtained by starting with the empty set, then any singleton set (n choices), then any 2-subset containing the singleton ($n-1$ choices), then any 3-subset containing the 2-subset ($n-2$ choices), etc. Thus there are $n!$ maximal chains.

Similarly, there are exactly $k!(n-k)!$ maximal chains which contain a given k-subset A of S .

Now we count in two different ways the number β of ordered pairs (A, C) such that A is in $\mathcal{A}$, and C is a maximal chain containing A.

Focusing first on one maximal chain C, since each maximal chain contains at most one subset in the antichain $\mathcal{A}$, β is at most the number of maximal chains; that is, $\beta \leq n!$.

Focusing now on one subset A in the antichain $\mathcal{A}$, we know that, if $|A| = k$, there are at most $k!(n-k)!$ maximal chains C containing A. Let α_k be the number of subsets in the antichain $\mathcal{A}$ of size k so that $|\mathcal{A}| = m = \sum_{k=0}^{n} \alpha_k$. Then

$$\beta = \sum_{k=0}^{n} \alpha_k k!(n-k)!$$

and, since $\beta \leq n!$,

$$\sum_{k=0}^{n} \alpha_k k!(n-k)! \leq n!$$

$$\sum_{k=0}^{n} \alpha_k \frac{k!(n-k)!}{n!} \leq 1$$

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1$$

By above **Lemma**, $\binom{n}{k}$ is maximum when $k = \lfloor n/2 \rfloor$, thus:

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1$$

$$\Rightarrow \sum_{k=0}^{n} \alpha_k \leq \binom{n}{\lfloor n/2 \rfloor}$$

$$\Rightarrow |\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

□

References

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