Tangram

MathToys - 5

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Abstract

In the recent two SUMS lectures[1] the idea of "scissors-congruent" in two and three dimensional euclidean space was introduced. Tangram puzzle illustrates the scissors-cutting congruence of a square.

1 Motivation

Two polygons (polyhedra) in Euclidean 2(3)-space are called scissors-congruent if they can be subdivided into the same finite number of smaller polygons (polyhedra) such that each piece in the first polygon (polyhedron) is congruent to one in the second. If two polygons (polyhedra) are scissors-congruent, then they clearly have the same area (volume). Following theorems illustrate the different consequences of scissors-cutting in polygons and polyhedra:

Theorem (Wallace-Bolyai-Gerwien theorem[2]). *Any two simple polygons of equal area can be dissected into a finite number of congruent polygonal pieces.*

Theorem (Dehn, 1901). *The regular tetrahedron is not scissor-equivalent to any parallelepiped.*

To prove this, for every polyhedron P, Dehn defines a value, now known as the Dehn invariant $D(P)$, with the property that *if P is cut into two polyhedral pieces P_1 and P_2 with one plane cut, then $D(P) = D(P_1) + D(P_2)$*. From this it follows that if P is cut into n polyhedral pieces $P_1, \ldots, P_n$, then $D(P) = D(P_1) + \ldots + D(P_n)$ and in particular, if two polyhedra are scissors-congruent, then they have the same Dehn invariant. He then shows that every cube has Dehn invariant zero while every regular tetrahedron has non-zero Dehn invariant. This settles the matter.

Theorem (Sydler, 1965). *Two polyhedra are scissors-congruent if and only if they have the same volume and the same Dehn invariant.*

We can extend this idea of scissors-congruence to more general figures. In general, a collection of n figures $K_1, K_2, \ldots, K_n$ is called the *set-theoretic decomposition* of K if it satisfy the following conditions:

1. $K_i \cap K_j = \phi$ (no two figures intersect each other)
2. $K = K_1 \cup K_2 \cup \ldots \cup K_n$ (each point in K belongs to one of the figures $K_1, K_2, \ldots, K_n$).

We denote a set-theoretic decomposition of K by

$$K = K_1 + K_2 + \ldots + K_n$$

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[1] SUMS-76: Birth of the notion of Area; SUMS-77: Scissors-Congruent Polygons and Polyhedra

[2] The proof of this theorem is constructive and doesn't require the axiom of choice, even though some other dissection problems (e.g. Tarski's circle-squaring problem, stated on next page) do need it.

Theorem (Banach-Tarski paradox[3]). *For two polyhedra K, L in space, there exist set-theoretic decompositions*

$$K = K_1 + K_2 + \ldots + K_n$$

$$L = L_1 + L_2 + \ldots + L_n$$

such that each K_i is congruent to L_i.

This theorem is called paradox, because there are no conditions imposed on polyhedra K and L. For example, K could be as small as an elementary particle and L could be as big as the sun. Unlike the decomposition into polyhedra, if set-theoretic decomposition is allowed then K and L do not need to have equal volumes. The theorem is called a paradox because we tend to think that the implications of this theorem are contradictory because of our incorrect belief that no matter how complicated figures are, they all should have volume. Actually, the complicated figures which doesn't have volume, do not have pictures. Hence, if one tried to double the the size of a piece of gold by using the Banach-Tarski paradox, they would find it impossible in real life (it is possible only in theory).

> The classical form of the circle-squaring problem of the ancient Greek geometers, to construct a square with the same area as a given circle only with a straightedge and a compass, had been solved negatively in the 19th century. But a new view on the old problem was opened by Alfred Tarski in 1925: *Can a circle be partitioned into sets that can be reassembled to form a square (having the same area)?* This is known as "Tarski's circle-squaring problem" and was proven to be possible by Miklós Laczkovich in 1990; the decomposition makes heavy use of the axiom of choice and is therefore non-constructive.

2 Introduction

The tangram (Chinese word, literally: "seven boards of skill") is a dissection puzzle consisting of seven flat shapes, called tans, which are put together to form shapes. These seven pieces whiach make up the tangram can be cut from a single square, as shown below:

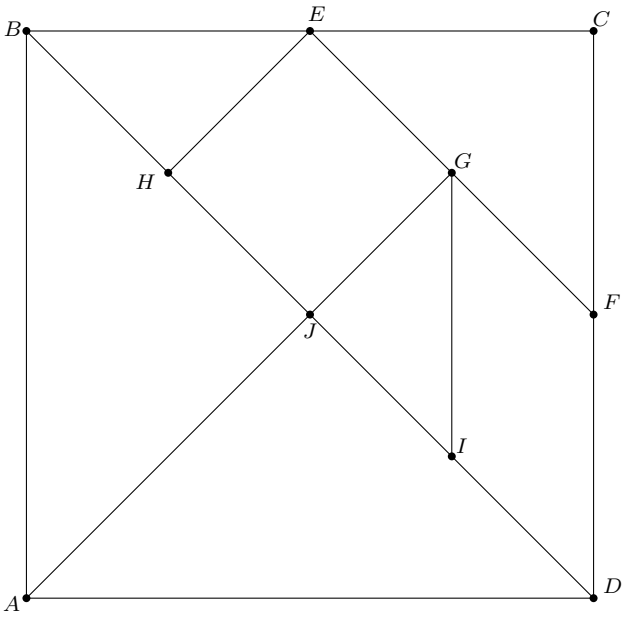

Figure 1: $BE = EC = CF = FD, EG = GF, EH \perp BD, GI \parallel FD$ [Drawn using GeoGebra 4.0.34.0]

[3]Unlike most theorems in geometry, the proof of this result depends in a critical way on the choice of axioms for set theory. It can be proven using the axiom of choice, which allows for the construction of nonmeasurable sets, i.e., collections of points that do not have a volume in the ordinary sense, and whose construction requires an uncountable number of choices.

There are thus two small triangles ($\triangle BHE$ and $\triangle GIJ$), one medium-sized triangle ($\triangle CEF$), and two large triangles ($\triangle ABJ$ and $\triangle ADJ$), in addition to a square ($\square HEGJ$) and a lozenge-shaped[4] piece ($\lozenge DFGI$). The medium-sized triangle ($\triangle CEF$), the square ($\square HEGJ$) and the rhomboid ($\lozenge DFGI$) are all twice the area of one of the small triangles ($\triangle BHE$ or $\triangle GIJ$). Each of the large triangles ($\triangle ABJ$ and $\triangle ADJ$) is four times the area of a small triangle ($\triangle BHE$ or $\triangle GIJ$). All the angles in these pieces are either right angles or angles of 45° or 135°.

3 Counting Tangrams

There are infinite possible arrangements that can be created using the seven pieces of tangram. This can easily be seen by looking at Number 229 in the following figure, for example:

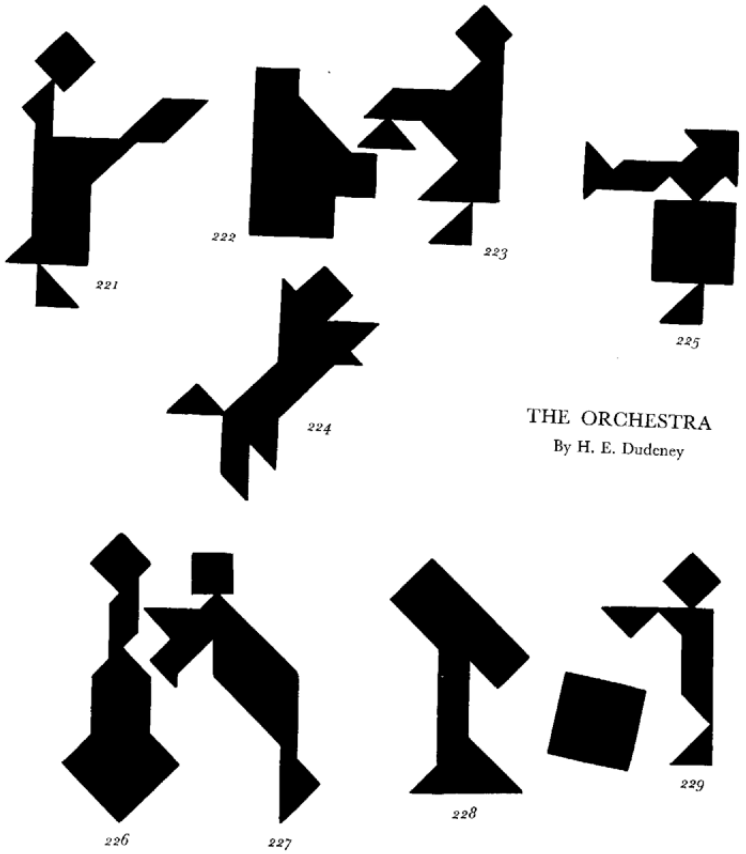

Figure 2: © Ronald C. Read, taken from pp. 50–51 of [2]

The bottom corner of the square piece that represents the head of the drummer-boy can touch the rest of the tangram at an unlimited number of points along the line that represents the shoulder and the outstretched arm. It is true that the different outlines that one would get by putting each piece in the different positions would note be very different from each other, but in the strictest sense they would have to be counted as distinct.

In 1942, two Chinese mathematicians, Fu Traing Wang and Chuan-Chih Hsiung[3], asked and answered the question, "How many convex tangrams are there?".

> The simplest way of seeing difference between a convex figure and one that is not convex is to imagine a piece of string or an elastic band pulled around the figure. If this causes the string to make contact with the figure all the way round its edges, then the figure is convex; but if there are gaps between the string and the edge of the figure, then the figure is not convex.

[4] Often referred to as a diamond, is a form of rhombus. Of these seven pieces, the lozenge is unique in that it has no reflection symmetry but only rotational symmetry, and so its mirror image can be obtained only by flipping it over. Thus, it is the only piece that may need to be flipped when forming certain shapes.

Theorem (Wang-Hsiung, 1942). *By means of the tangram exactly thirteen convex polygons can be formed.*

In particular, four hexagons, two pentagons, six quadrangles, and a triangle are the convex polygons obtained.

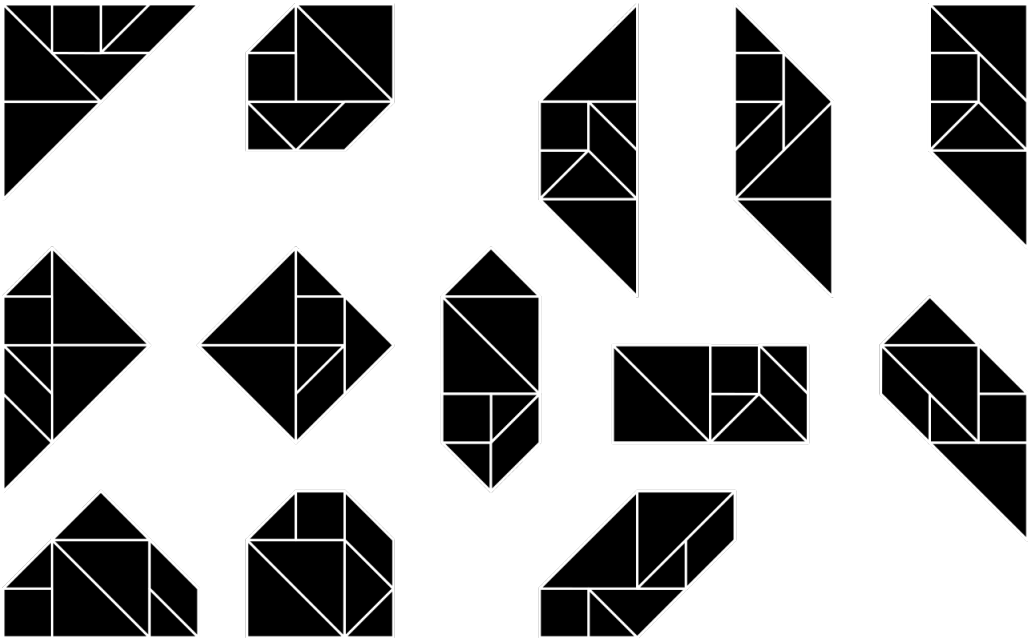

Figure 3: All convex tangram shapes © László Németh, https://upload.wikimedia.org/wikipedia/commons/0/0e/Convex_tangram_shapes.svg

Now a natural question to ask is, "Can we think of any other special kinds of tangrams that would be more numerous than the convex ones, and yet not infinite in number?". Here is the *snug tangram number problem* proposed by Ronald C. Read[2]

> Let us imagine a set of tangram pieces such a size that the equal sides of the small triangles are 1 unit in length. Then the third side of these triangles will be approximately 1.414 units (the square root of 2, to be precise). Now any side of any of the pieces of this set will be one of these lengths, or twice one of these lengths, and we can therefore imagine every side of each of the pieces to be made up of "sections" whose lengths are either 1 unit or 1.414 units. There will be either one or two sections to each side. Imagine now a tangram that has been constructed in such a way that whatever two pieces are in contact at all, they are in contact along a whole section of each, so that the ends of these sections coincide. Moreover, the tangram should be all in one piece. Tangrams which conform to the above restrictions are called "snug" tangrams, because of the close way in which the pieces fit together. It makes sense to ask the questions, "How many snug tangrams are there?" for it can be shown that snug tangrams, unlike tangrams in general, are limited in number.

On next page there is an arrangement illustrating the meaning of snug tangrams. Also the individual pieces are shown. Note that all the convex tangrams are snug.

References

[1] Shiga, Koji and Sunada, Toshikazu. *A Mathematical Gift, III: The interplay between topology, functions, geometry, and algebra*. The American Mathematical Society, 2005 (Translated from Japanese to English by Eiko Tyler)

[2] Read, Ronald C. *Tangrams: 330 Puzzles*. Dover Publications, 1965

[3] Wang, Fu Traing and Hsiung, Chuan-Chih. "A Theorem on the Tangram." *The American Mathematical Monthly* 49, no. 9 (1942): 596-599.

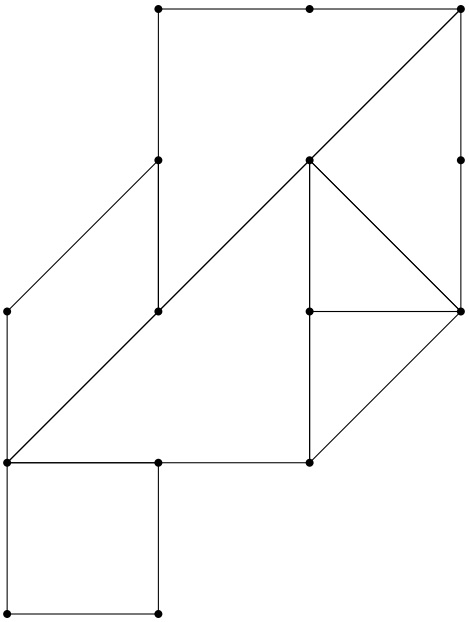

Figure 4: An example of snug tangram with the end of sections indicated by blobs. [Drawn using GeoGebra 4.0.34.0]

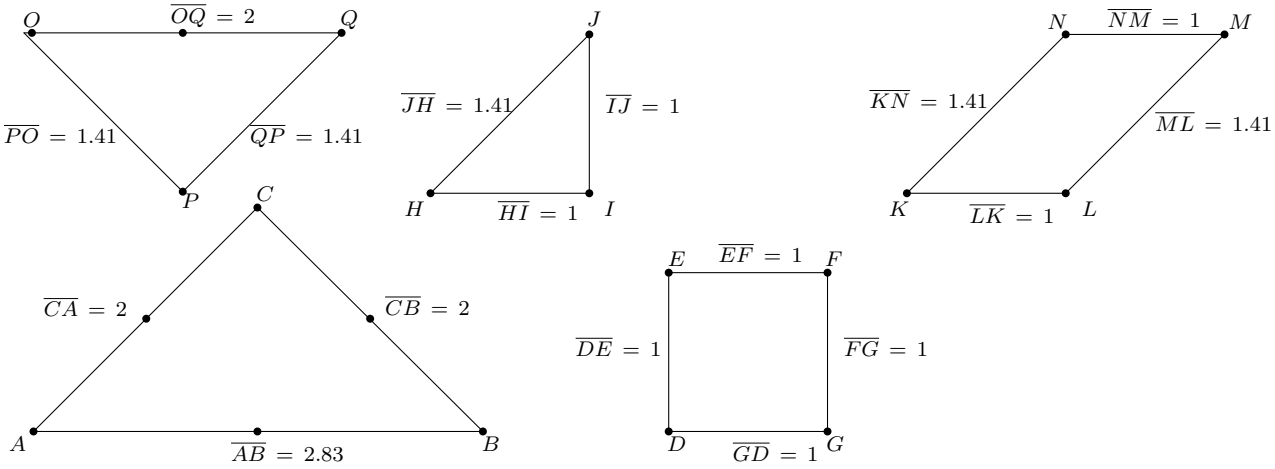

Figure 5: The tangram pieces with the end of sections indicated by blobs. [Drawn using GeoGebra 4.0.34.0]