Lecture 1

Renzo Cavalieri

Enumerative Geometry

Enumerative geometry is an ancient branch of mathematics that is concerned with counting geometric objects that satisfy a certain number of geometric conditions. Here are a few examples of typical enumerative geometry questions:

Q_1**:** How many lines pass through **2** points in the plane?

Q_2**:** How many conics pass through **5** points in the plane?

Q_3**:** How many rational[1] cubics (i.e. having one node) pass through **8** points in the plane?

Q_d**:** How many rational curves of degree d pass through $\mathbf{3d-1}$ points in the plane?

OK, well, these are all part of one big family...here is one of a slightly different flavor:

Q_l**:** How many lines pass through **4** lines in three dimensional space?

Some Observations:

1. I've deliberately left somewhat vague what the ambient space of our geometric objects: for one, I don't want to worry too much about it; second, if you like, for example, to work over funky number fields, then by all means these can still be interesting questions. In order to get nice answers we will be working over the complex numbers (where we have the fundamental theorem of algebra working for us). Also, when most algebraic geometers say things like "plane", what they really mean is an appropriate compactification of it...there's a lot of reasons to prefer working on compact spaces...but this is a slightly different story...

2. You might complain that the questions may have different answers, because I said nothing about how the points are distributed on the plane. Even in Q_1, if you take the two points to coincide, then you actually have

[1] Rational means that "it can be parameterized". I.e. there exists a function from a line to your curve that is generically one-to-one. Alternatively, your curve is the image in the plane of $(f(t), g(t))$, where f and g are polynomials of degree 3.

infinitely many lines going through them... that's why, in enumerative questions like the Q_d's, it is somewhat implicit that the points are taken to be **in general position**. What does this mean exactly? Well, there is a technical definition which I do not want to get into at this point, but think of it this way. If you were to be blinfolded and spun around before tossing each point onto the plane, then with probability one they will land in general position.

In other words, what I am saying is that the disposition of the points is not "too special" (e.g. if the two points don't coincide for Q_1) then there should be one nice finite answer for all the Q_d's.

Problem 1. *What does "too special" mean for Q_2 and Q_3? And if you like this game...try and see if you can say it for general d...*

3. When your points wander around and get "out" of general position often times your number of solutions to an enumerative question jumps to ∞. However, it **should not** (and in fact it doesn't!) jump from a finite number to another finite number. This somewhat heuristic idea is called the **principle of conservation of numbers**, and was used in the old days to solve enumerative questions: if you are able to place your points in any position (even if it is special) in such a way that you are able to find an answer and it is finite, then that is the right answer for your question in general!

   Problem 2 (Challenge)**.** *Use the principle of conservation of numbers to answer Q_l.*

4. Notice that for an enumerative question to have any hope to have a good answer, you have to impose the right number of conditions to your objetcs...in all of the Q_d's, if you ask for incidence to more than $3d-1$ points, then you find no curve at all; if you ask for fewer than $3d-1$ points, then you get infinitely many curves...

   Problem 3. *Using the footnote, figure out why $3d-1$ is the right number of points for the Q_d's.*

Solving Q_2

Any conic in the plane is the zero set of a degree 2 polynomial in x, y:

$$C = \{a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 = 0\}$$

Therefore I can think that the sextuple of numbers $(a_1, \ldots, a_6)$ identifies uniquely a conic in the plane. I.e. there is a function:

$$\left\{ \begin{array}{c} \text{sextuples} \\ (a_1, \ldots, a_6) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{conics in} \\ \text{the plane} \end{array} \right\}$$

There is a lot of redundancy, however, since any two proportional sextuples will identify the same conic. But if we mod out by the equivalence relation

$$(a_1, \dots, a_6) \sim (\lambda a_1, \dots, \lambda a_6),$$

for any $\lambda \neq 0$, then we obtain a bijection:

$$\left\{ \begin{array}{c} \text{sextuples} \\ (a_1, \dots, a_6) \end{array} \right\} / \sim \quad \hookrightarrow \quad \left\{ \begin{array}{c} \text{conics in} \\ \text{the plane} \end{array} \right\}$$

OK, so we have gotten ourselves an "algebraic" way of thinking of our set of conics. Can we translate in this language what it means for a conic to pass through a given point, say, for example $P = (8, 23)$?

It means that:

$$8^2 a_1 + (8)(23) a_2 + 23^2 a_3 + 8a_4 + 23a_5 + a_6 = 0$$

In other words, passing through a point corresponds to satisfying a linear equation in the coefficients $(a_1, \dots, a_6)$.

Our question Q_2 therefore translates to solving a **homogeneous linear system** of five equations in six variables. Linear algebra now tells us that if the rank of the corresponding matrix is 5 (which is precisely our general position condition!) then there is exactly a one parameter homogenous family of solutions - which is precisely a proportionality class of sextuples - which is precisely **one conic**!

Problem 4. *Now that you know what the number is, devise an efficient method to find the solution given* 5 *specific points. I mean, nobody really wants to solve a linear system of rank* 5 *if he has the option not to, right?*

Geometric Interpretation

What we did in the previous paragraph has a geometric interpretation: the moduli space of all conics in the plane is a 5-dimensional projective space (you don't know what that is? Hang on, it is coming up in the next section! For the moment think of $\mathbb{C}^5$). Passing through one point defines a hyperplane - and five general hyperplanes in a five dimensional space intersect in exactly one point.

This suggests a general (geometric) strategy to approach an enumerative question:

1. Understand the moduli space of (all) the geometric objects you are looking for.

2. Translate the geometric conditions you want to satisfy to subvarieties of your moduli spaces.

3. Intersect the above subvarieties.

Thus an enumerative question is really, in disguise, a question about **intersection theory on moduli spaces**.

This new point of view is interesting, fascinating and powerful...however it doesn't mean that we have an easy way to the solution of enumerative questions...for example, for the Q_d's, you can solve Q_3 by generalizing the method of Q_2:

- the space of all cubics is $\mathbb{P}^9$.
- passing through a point is still a hyperplane.
- having a node corresponds to a degree 12 hypersurface in $\mathbb{P}^9$.

And therefore there are 12 rational cubics through 8 points in the plane.

The number of quartics (that incidentally is 620) was obtained in a similar way after a huge amount of hard work, which showed that this was not the right avenue to use in pursuing a solution to the general Q_d.

The key to solve this question, was to change the point of view and consider the "right" moduli spaces for the problem. The problem was solved in the nineties by none the less than M. Kontsevich...but I am getting ahead of myself...we'll come back to this.

Our First Moduli Spaces: $\mathbb{P}^n$

We want projective space $\mathbb{P}^n$ to be the moduli space of lines through the origin in $\mathbb{C}^{n+1}$. We also will see that it is a compactification[2] of affine space $\mathbb{C}^n$. Of course I am going to cheat a bit and present the dimension two case, and over the reals too - and will leave it to you as a useful exercise to generalize to arbitrary dimensions and to complex numbers.

$\mathbb{P}^2$: Take One!

One way to identify a line through the origin in $\mathbb{R}^3$ is to simply give a point on it, provided that the point is different from the origin. This means giving a triple of complex numbers $(z_1, z_2, z_3) \neq (0, 0, 0)$. Again, there is redundancy in this description, since two triples identify the same line if they are proportional to each other.

We therefore need to mod out by the equivalence relation

$$(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3),$$

$\lambda \neq 0$, to obtain:

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \setminus \{(0,0,0)\}}{\sim} \tag{1}$$

[2]This means that $\mathbb{P}^n$ is a compact space and it contains $\mathbb{C}^n$ as a dense open set.

This is a nice very symmetric description to describe $\mathbb{P}^2$ as a set, and it also provides a set of **homogenous coordinates**. However it doesn't tell us much about the structure of $\mathbb{P}^2$.

Problem 5. *Does a polynomial $F(X, Y, Z)$ give a function on $\mathbb{P}^2$?*

$\mathbb{P}^2$: Take Two!

One way to get rid of (most of) the redundancy is, instead of picking any point in $\mathbb{R}^3$ to represent a line, to allow only points that live on a sphere.

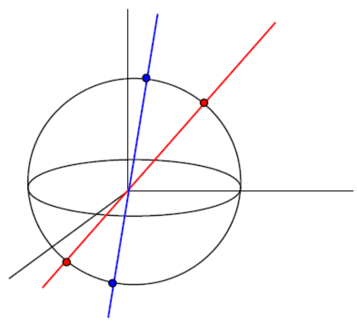

Figure 1: Lines through the origin in $\mathbb{R}^3$ intersect the sphere in two antipodal points.

If we do so, each line corresponds to precisely two antipodal points on the sphere, and therefore

$$\mathbb{P}^2 = \frac{\text{Sphere}}{P \sim -P} \tag{2}$$

This allows us to give a topology to $\mathbb{P}^2$, namely the quotient topology induced from the map from the sphere. Also, since we know the sphere is compact and the image of a compact space via a continuous function is compact, we immediately deduce that $\mathbb{P}^2$ is **compact**.

Problem 6. *Try to prove that $\mathbb{P}^2$ is* ***non-orientable***[3]*. One way to show this is to show that it contains a Mobius strip.*

$\mathbb{P}^2$: Take Three!

Yet another way to parameterize lines through the origin in $\mathbb{R}^3$ is the following: consider the plane $\{z = 1\}$. Most every line hits this plane in precisely one point. Unfortunately we are missing some lines...namely all those that live in

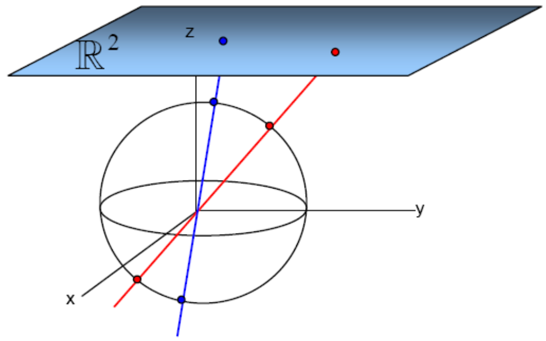

Figure 2: Mapping from lines in $\mathbb{R}^3$ to the plane $z = 1$.

the plane $\{z = 0\}$. Notice that such set of all lines through the origin in a plane is precisely a projective space of dimension 1 (a projective line).

As a set,

$$\mathbb{P}^2 = \mathbb{R}^2 \sqcup \mathbb{P}^1. \tag{3}$$

Problement 7. *Show that with the topology given above, $\mathbb{R}^2$ is an open dense set in $\mathbb{P}^2$. This shows that $\mathbb{P}^2$ **is a compactification** of the plane.*

$\mathbb{P}^2$: Take Three and a Half!

The previous section should leave us slightly unhappy, because the asymmetry of it: the projective plane knows nothing about any particular $\mathbb{R}^2$ being special...therefore our idea is now to consider all possible planes to play the role of screens. This defines an **atlas** for $\mathbb{P}^2$ that allows us to show that $\mathbb{P}^2$ is in fact a smooth differentiable manifold.

Just to be lazy, instead of considering all charts, we choose a minimal atlas consisting of three charts.

$$\begin{array}{cccc} \varphi_z : & U_z = \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x, y) & \mapsto & (x : y : 1) \end{array}$$

$$\begin{array}{cccc} \varphi_y : & U_y = \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x, z) & \mapsto & (x : 1 : z) \end{array}$$

$$\begin{array}{cccc} \varphi_x : & U_x = \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (y, z) & \mapsto & (1 : y : z) \end{array}$$

[3]Here it is essential that we work over $\mathbb{R}$.

Problem 8. *Describe the transition functions and check that they are differentiable on the overlaps.*

Problem 9. *Define a topology on $\mathbb{P}^2$ as follows: a set $U \subset \mathbb{P}^2$ is open if all of its preimages $\varphi_x^{-1}(U)$, $\varphi_y^{-1}(U)$, $\varphi_z^{-1}(U)$ are open sets of the plane with the euclidean topology. Show that this is indeed a topology, that it makes the three φ maps continuous, and that the images $\varphi_x(\mathbb{R}^2)$, $\varphi_y(\mathbb{R}^2)$, $\varphi_z(\mathbb{R}^2)$ become open dense sets of $\mathbb{P}^2$. Show that this coincides with the quotient topology defined before!*

Some more food for thoughts...

Problem 10. 1. *Prove that the following are equivalent definitions for the concept of a **line** in $\mathbb{P}^2$:*

   (a) *the set of solutions of a **homogeneous** degree 1 polynomial in X, Y, Z. I.e. the set of points in $\mathbb{P}^2$ that satisfy an equation of the form:*

$$aX + bY + cZ = 0.$$

   (b) *a line in one of the charts, plus a uniquely determined point in the complement of the chart.*

   (c) *a plane through the origin in $\mathbb{R}^3$.*

2. *Prove that **any** two lines in $\mathbb{P}^2$ intersect in exactly one point.*

3. *In general it makes no sense to ask "where does a polynomial in X, Y, Z vanish in $\mathbb{P}^2$... for example, take the polynomial*

$$f(X, Y, Z) = X + Y + Z^2$$

   *$f(-2, 1, 1) = 0$, and $f(-4, 2, 2) \neq 0$... but $(-2 : 1 : 1)$ and $(-4, 2, 2)$ are **the same point** in $\mathbb{P}^2$.*

   However, in exercise 1, we have defined lines as the solutions of certain polynomial equations...what saved the day in that case? In general, under which conditions are the zeroes of a polynomial a well-defined notion in $\mathbb{P}^2$? Make your guess for what should be a projective algebraic curve of degree d.

4. *Decide whether the following plane conic (ordinary plane, not $\mathbb{P}^2$!) is a parabola, an ellipse or a hyperbola.*

$$x^2 + 4xy + 4y^2 + 342x + 57y - 22 = 0$$

   ***Hint:** think of the plane as one chart for $\mathbb{P}^2$. Using what you discovered in exercise 3, think of how to view this conic in $\mathbb{P}^2$, then ask yourself how do an ellipse, a parabola, a hyperbola intersect the line at infinity (i.e. the complement of the chart).*

5. *Show that the complex projective line is the one point compactification of $\mathbb{C}$, and it is therefore homeomorphic to a sphere.*
6. *Try to generalize all of this to $\mathbb{P}^n$.*

What Do We Want From A Moduli Space?

Let us extrapolate from the previous discussion what are the "qualities" we appreciate in a moduli space $\mathcal{M}$:

m1: Points in the space $\mathcal{M}$ are in bijection with the objects we wish to parameterize.

m2: The moduli space has a natural topology (differentiable structure, algebraic structure ... in general a structure similar to the objects you wish to parameterize). Such topology agrees with the intuitive notion of "small perturbation of the objects".

m3: Families of objects, i.e. a morphism of spaces

$$\begin{array}{c} Y \\ \downarrow \\ X, \end{array}$$

where the preimage of any $x \in X$ is one of our objects, should correspond to functions

$$f : X \to \mathcal{M}.$$

m4: If the moduli space is a compactification of some other natural object, then what you have to add to compactify is some combination of "smaller moduli spaces of the same type". I know, this is kind of vague...but think of how $\mathbb{P}^n$ compactifies $\mathbb{C}^n$ by adding a $\mathbb{P}^{n-1}$. Hopefully we will see more examples of this idea.

$G(k, n)$: Projective Space's Big Brothers

Let us consider an n-dimensional vector space V, and choose once and for all a basis $e_1, \ldots, e_n$. For a fixed $k \leq n$ we call **Grassmannian** the moduli space of linear subspaces of V of dimension k. We denote this space by $G(k, n)$.

Problem 11. *Convince yourself that $G(1, n+1) = \mathbb{P}^n$. Also, $G(n, n+1) \cong \mathbb{P}^n$*

We will try to get an intuition about the following
Fact: *$G(k, n)$ is a smooth compact (in fact projective) manifold of dimension $k(n-k)$.*

But all of this will have to wait until next time...in the mean time, you can start and think about the specific case of $G(2, 3)$ (where you know that the answer should be $\mathbb{P}^2$ AND you can draw pictures!), by trying to unravel the following mystery picture.

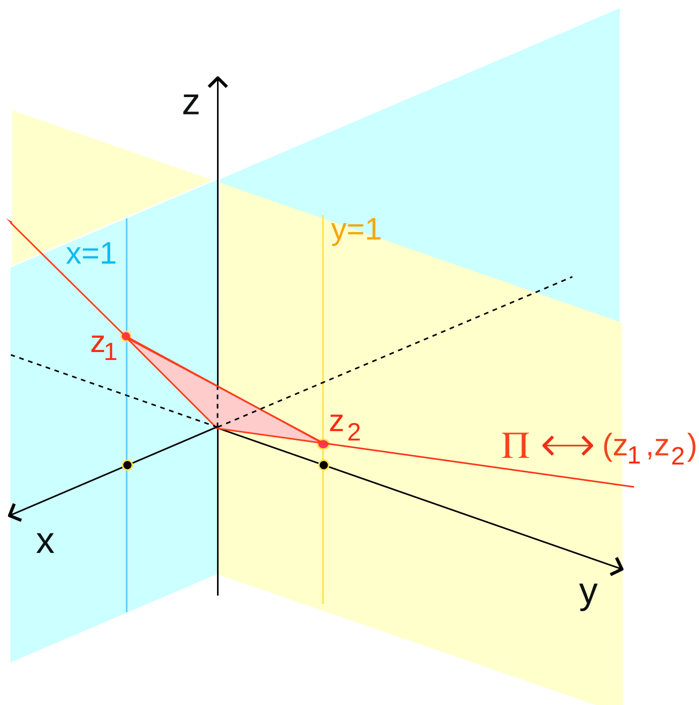

Figure 3: A natural chart for $G(2,3)$.

Lecture 2

Renzo Cavalieri

$G(k,n)$: Projective Space's Big Brothers

Let us consider an n-dimensional vector space V, and choose once and for all a basis $e_1, \ldots, e_n$. For a fixed $k \leq n$ we call **Grassmannian** the moduli space of linear subspaces of V of dimension k. We denote this space by $G(k,n)$.

We now work towards the understanding of the following
Fact: *$G(k,n)$ is a smooth compact (in fact projective) manifold of dimension $k(n-k)$.*

G(2,3): a Motivating Example

As usual we start from a (relatively) simple case to gather intuition and motivation for the general theory. We already know that $G(2,3) \cong \mathbb{P}^2$, because to any plane in $\mathbb{R}^3$ we can associate in a canonical and unique way the perpendicular line through the origin. However, we are going to ignore this fact and find charts for $G(2,3)$ in terms of the planes that $G(2,3)$ parameterizes.

Look at figure 1!

Here we have chosen the two planes $x = 0$ and $y = 0$ as screens. Most every plane in $\mathbb{R}^3$ will intersect each of these two planes in a line. Then we further intersect with the vertical lines $y = 1$ and $x = 1$, and we find two points. A general plane defines these two points, and conversely, given two points on those vertical lines uniquely determines a plane in $\mathbb{R}^3$. But the only free coordinate for those two points is the z coordinate, therefore we have an $\mathbb{R}^2$ worth of generic planes, or, if we prefer, a map:

$$\begin{array}{cccc} \phi_{x,y}: & \mathbb{R}^2 & \to & G(2,3) \\ & (z_1, z_2) & \mapsto & \text{plane through} \\ & & & 0, (1,0,z_1), (0,1,z_2) \end{array}$$

The planes that are not in the image of this chart are those that contain the z axis. There is a projective line worth of them: since one dimension is "taken" by the z axis, all we need to identify one of these planes is the perpendicular direction, therefore reducing our problem to parameterizing lines through the origin in the horizontal plane.

At the end of the day we have recovered what we already knew:

$$G(2,3) = \mathbb{R}^2 \sqcup \mathbb{P}^1 = \mathbb{P}^2$$

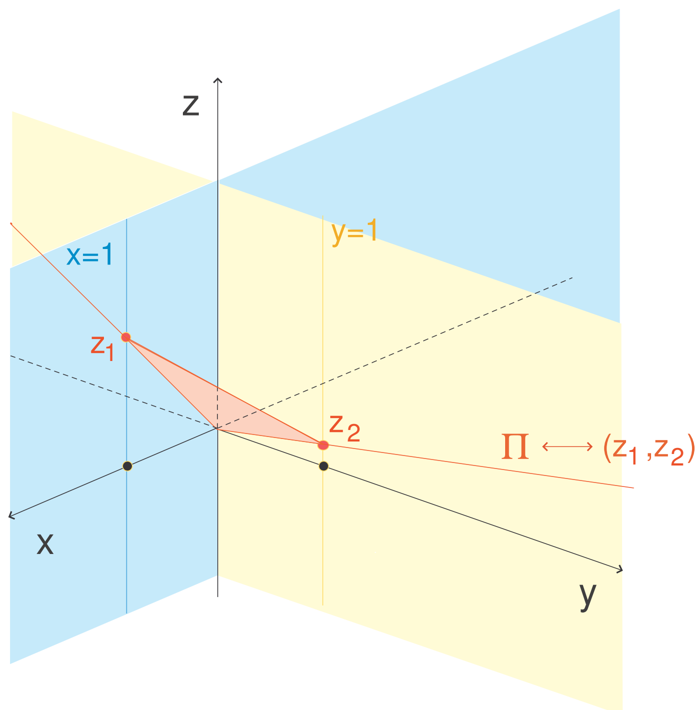

Figure 1: A natural chart for $G(2,3)$.

Problem 1. *Find out the transformation between these charts for $G(2,3)$ and the charts given by associating to a plane its perpendicular line and then using the standard charts for $\mathbb{P}^2$.*

G(k,n) is a Manifold

Let us now try to generalize what we have done. We will do it in two ways.

Geometric Approach

What did we do in the previous section?

1. We chose k linear subspaces $L_1, \dots, L_k$ of V of dimension $n-k+1$. Each of them intersects a general k-subspace of $\mathbb{R}^n$ in line. Let us call these lines $\ell_1, \dots, \ell_k$.

2. Inside each of the L_i, we chose a hyperplane H_i not through the origin. At this point the intersection

$$H_i \cap \ell_i = P_i$$

is one point in an $n-k$ dimensional linear space.

3. The coordinates of the P_i's inside the H_i's define the chart to the Grassmannian. Since we are free to choose k points inside $n-k$ dimensional linear spaces we see that the dimension of $G(k,n)$ is $k(n-k)$.

4. What are the k-subspaces that are not in the image of this chart? Those that intersect any of the subspaces L_i in more than just a line!

Algebraic Approach

If you prefer linear algebra here's another approach. To give a k-subspace of V you can simply give k linearly independent vectors $v_1, \dots, v_k \in V$, or, if you prefer, a $(k \times n)$ matrix of maximal rank:

$$A = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \dots & v_{kn} \end{bmatrix}$$

Of course there is a lot of redundancy in this description, because we can choose to arbitrarily change the basis for our k-subspace. This corresponds to multiplying A on the left by a matrix in $GL(k)$. Now let us choose k columns, for example the first k, just so we don't get a headache with general notation.

If the determinant of the $k \times k$ minor is different from zero, then there is a unique matrix in $\Upsilon \in GL(k)$ (namely the inverse of that minor) such that

$$\Upsilon A = \left[\begin{array}{cccc|ccc} 1 & 0 & \dots & 0 & v_{1(k+1)} & \dots & v_{1n} \\ 0 & 1 & \dots & 0 & v_{1(k+1)} & \dots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & v_{k(k+1)} & \dots & v_{kn} \end{array}\right]$$

We see immediately that the remaining coefficients give local coordinates for the chart, and that they correspond to a copy of $\mathbb{R}^{k(n-k)}$.

What's the Complement of a Chart?

Algebraically, the k-subspaces that are not parameterized in the chart above are those such that the determinant of the chosen $k \times k$ minor vanishes.

For the particular chart above, this corresponds to k-subspaces that intersect the space $\{x_1 = x_2 = \ldots = x_k = 0\}$ in more than just a point. By distinguishing all possible cases, you should now tackle the following

Problem 2 (Challenge)**.**

$$G(k,n) = \mathbb{R}^{k(n-k)} \sqcup G(1,k) \times G(k-1,n-k) \sqcup G(2,k) \times G(k-2,n-k) \sqcup \ldots \sqcup G(k-1,k) \times G(1,n-k)$$

G(k,n) is Projective

We still need to veryfy that $G(k,n)$ is compact. We do so by showing that $G(k,n)$ is a closed subvariety of an appropriate projective space.

Our approach is again to try and reproduce what we did with projective space when we constructed homogeneous coordinates for $\mathbb{P}^n$. As usual, rather than going crazy with indices, let's work with a specific example: $G(2,4)$.

Start from a point $P \in G(2,4)$, that we think as a 2×4 matrix A.

$$A = \left[\begin{array}{cccc} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{array}\right]$$

We want to associate to this matrix a well chosen set of numbers. The problem is, as usual, that A is not the only matrix that represents P. Any matrix ΥA also represents the same point in $G(2,4)$, provided that Υ is an invertible 2×2 matrix. Therefore we try and choose as coordinates numbers that will be as unperturbed as possible by the multiplication by Υ.

Take the determinants of all 2×2 minors of the matrix A:

$$(u_0, \ldots, u_5)(A) = \left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \ldots, \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix} \right)$$

What happens when you multiply on the left by Υ?

$$(u_0, \ldots, u_5)(\Upsilon A) = \left(det(\Upsilon) \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \ldots, det(\Upsilon) \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix} \right)$$

Yes! All numbers are multiplied by the same constant $det(\Upsilon)$! This means that this construction associates to a point in $G(2,4)$ a proportionality class of sextuples - i.e., a point in $\mathbb{P}^5$.

Problem 3. *Show that in general this construction defines a(n injective) map:*

$$H : G(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$$

We know however that $G(2,4)$ is 4 dimensional, so the image of the previous map cannot be all of $\mathbb{P}^5$.

Problem 4. *Show that the points in the image of H must satisfy the homogeneous quadratic equation:*

$$u_0u_5 + u_1u_4 + u_2u_3 = 0.$$

Problem 5. *Generalize all of this for $G(k,n)$*

Moduli Spaces of Points on $\mathbb{P}^1_{\mathbb{C}}$

$M_{0,n}$

A family of moduli spaces of a completely different flavor parameterizes configuration of labelled points on the complex projective line (a.k.a. the Riemann Sphere). To make the problem more interesting than just taking product spaces, we introdce the following rules:

1. We parameterize configurations of n labelled points $(P_1,\dots,P_n)$ on $\mathbb{P}^1_{\mathbb{C}}$.
2. No two points are allowed to coincide.
3. We introduce the following equivalence relation:

$$(P_1,\dots,P_n) \sim (Q_1,\dots,Q_n)$$

   if there exists an automorphism φ of $\mathbb{P}^1_{\mathbb{C}}$ such that

$$\varphi(P_i) = Q_i$$

For reasons that will become apparent later, this moduli space is denoted $\mathbf{M_{0,n}}$.

Problem 6. *In case you are not familiar with automorphisms of $\mathbb{P}_1$, try to convince yourself that an automorphism*

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1$$

can be expressed in any of these three equivalent ways:

- *a pair of homogeneous polynomials of degree* 1.

$$(y_0 : y_1) = (ax_0 + bx_1 : cx_0 + dx_1)$$

- *a Mobius transformation (on one chart):*

$$y = \frac{ax+b}{cx+d}$$

- *a* 2×2 *invertible (!) matrix:*

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

Conclude the following:

1. *There is a* 3 *dimensional group of automorphisms of* $\mathbb{P}^1$.
2. *There is a unique automorphism that sends any three (distinct) points* P_1, P_2, P_3 *to any other three points* Q_1, Q_2, Q_3.

Let us now explore some of these moduli spaces:

$M_{0,3}$**:** by 2. above this moduli space is just one point, because any triple of points can be moved via a unique automorphism to the configuration $0, 1, \infty$.

$M_{0,4}$**:** $\mathbb{P}_1 \smallsetminus \{0, 1, \infty\}$. Spend your automorphism to send the first three points to $0, 1, \infty$, then the fourth point is free to roam on the sphere, except on those three points.

$M_{0,5}$**:** $M_{0,4} \times M_{0,4} \smallsetminus \Delta$. Here Δ means "the diagonal" in the product space.

$M_{0,n}$**:** $M_{0,4}^{n-3} \smallsetminus$ all diagonals. Here by diagonal I mean any subpace of the product space where at least two coordinates are the same.

We have a very good understanding of these moduli spaces. Unfortunately, they are very much not compact. The quest for a good compactification leads us to yet another family of moduli spaces, which is by far more interesting!

$\overline{M}_{0,n}$

The winning idea here is very simple and elegant. We enlarge the class of objects that we wish to parameterize to include some "degenerate" objects. By doing so we construct a compact space that contains $M_{0,n}$ as a dense open set.

Heuristically, when two points (or more) want to come together...we don't allow them to. At the moment in which they would crash together, we make them "jump" on a new sphere attached to the old one at the point of collision. Let us be a little more formal now.

Definition 1. *A* ***tree of projective lines*** *is a connected curve with the following properties:*

1. *Each irreducible component is isomorphic to* $\mathbb{P}^1$.

2. *The points of intersection of the components are ordinary double points.*
3. *There are no closed circuits, i.e., if a node is removed then the curve becomes disconnected.*

Each irreducible component is called a twig. We draw a marked tree as in Figure 2, where each line represents a twig.

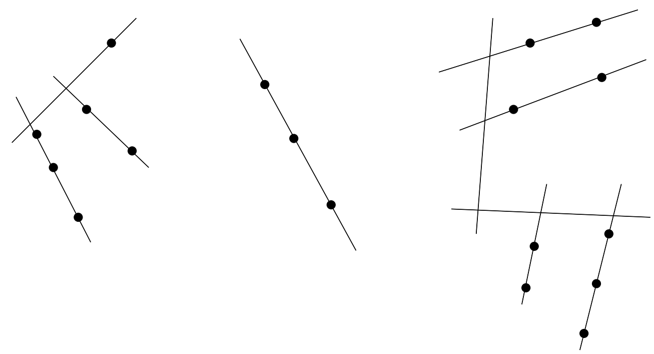

Figure 2: stable marked trees.

Definition 2. *A marked tree is **stable** if every twig has at least three special points (marks or nodes).*

Problem 7. *This stability condition is equivalent to the existence of no non-trivial automorphisms of the tree that fix all of the marks.*

Problem 8. *Show that if we define $\overline{M}_{0,4}$ to be the moduli space of isomorphism classes of four pointed stable trees, we obtain $\overline{M}_{0,4} \cong \mathbb{P}^1$.*

Fact: The moduli space $\overline{M}_{0,n}$ of n-pointed rational stable curves compactifies $M_{0,n}$.

One of the exciting features of this theory is that all these spaces are related to one another by natural morphisms. Consider the map

$$\pi_i : \overline{M}_{0,n+1} \to \overline{M}_{0,n},$$

defined by forgetting the i-th mark. It is obviously defined if the i-th mark does not belong to a twig with only three special points. If it does belong to such a twig, then our resulting tree is no longer stable. In this case, we must perform **contraction**.

Contraction: We need to consider two cases:

1. The remaining two special points are both nodes. We make the tree again stable by contracting this twig so that the two nodes are now one (see Figure 3).

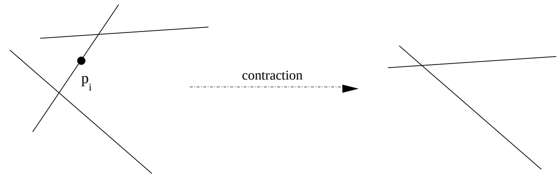

Figure 3: contracting a twig with only two nodes.

2. There is one other mark and one node on the twig in question. We make the tree stable by forgetting the twig and placing the mark where the node used to be (Figure4).

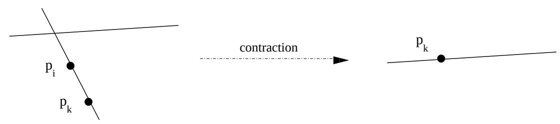

Figure 4: contracting a twig with one node and one mark.

The boundary

The **boundary** is all we have added to make our space compact, i.e. the complement of $M_{0,n}$ in $\overline{M}_{0,n}$. It consists of all nodal stable curves.

Fact: the boundary is a union of irreducible components, corresponding to the different possible ways of arranging the marks on the various twigs.

Problem 9. *Show that the codimension of a boundary component equals the number of nodes in the curves in that component.*

Problem 10. *Notice that the irreducible components of the boundary are isomorphic to products of moduli spaces of rational stable curves with strictly fewer number of points.*

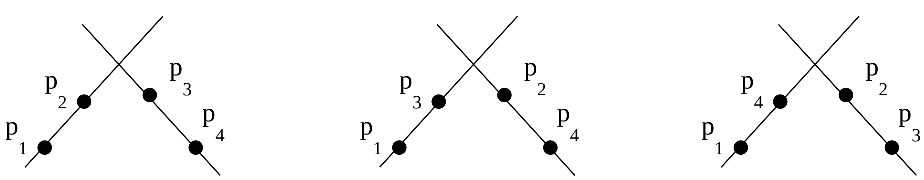

Figure 5: irreducible components of the boundary of $\overline{M}_{0,4}$

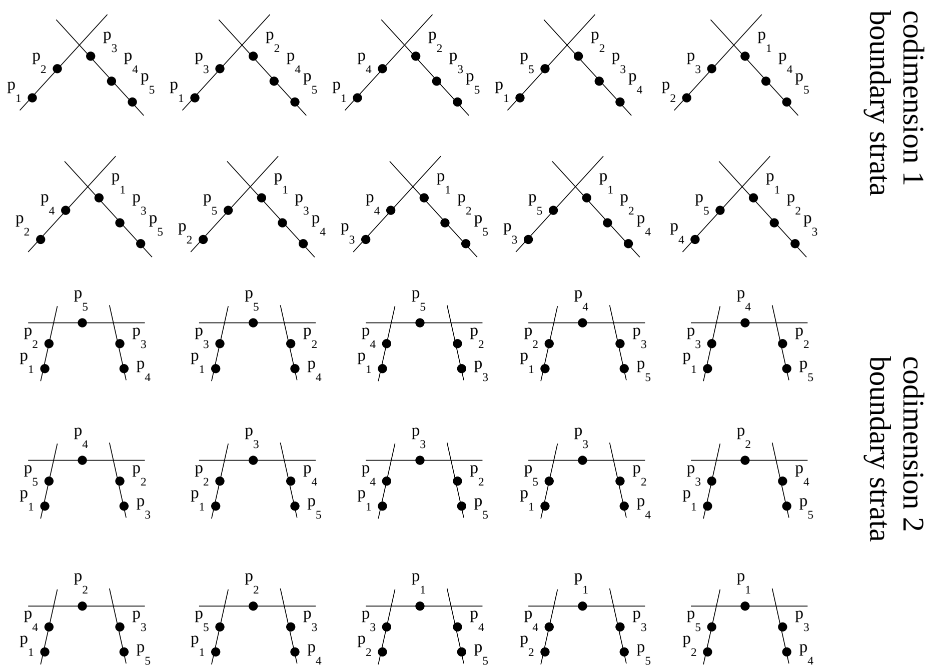

Figure 6: boundary cycles of $\overline{M}_{0,5}$

In Figures 5 and 6 we draw all boundary strata for $\overline{M}_{0,4}$ and $\overline{M}_{0,5}$.

There is plenty more to be said about the spaces $\overline{M}_{0,n}$, their relationships, and their boundaries, but time is short and we need to get to Kontsevich's proof by next week, so I'll stop here. The book [KV99] is an **excellent** reference for beginners.

Moduli Spaces of Curves

We are going to wrap up the day with a whirlwind tour of higher genus...this will be unbelievably fast and imprecise, but hopefully it will make you want to know more about these topics!

Recall that a projective variety is the zero set of a bunch of homogeneous polynomials in some projective space $\mathbb{P}^n$.

Definition 3. *A* ***projective curve*** *is a projective variety of dimension* 1*.*

Problem 11. *Show that* $\mathbb{P}^1$ *is a projective curve.*

A large class of examples is given by plane curves: zero sets of one homogeneous polynomial in x, y, z, considered as varieties in $\mathbb{P}^2$.

If we work over the complex numbers, then curves are really "surfaces". It is not hard to see that projective curves actually are complex manifolds (well, smooth curves are, singular curves are...almost everywhere). The complex structure forces orientability, and being a closed subset of $\mathbb{P}^n$ tells us that they are compact.

Fact: topologically, projective curves are all "doughnuts". The number of holes, called the **genus**, is the unique discrete topological invariant.

Let's point out, at the cost of being boring, that $\mathbb{P}^1$ is THE smooth curve of genus 0.

Why did I write "THE"? It was not a typo, nor a language mishap. In fact it is true that any smooth curve of genus 0 is isomorphic to $\mathbb{P}^1$. That is why moduli spaces of genus 0 curves are not very interesting by themselves, and we had to "spice them up" by adding mark points!

Let me point out another feature, that I find quite amazing. While we are not too surprised that an algenraic curve is a complex manifold (after all polynomials are holomorphic functions), it is kind of amazing that the converse is true as well, i.e. any compact complex manifold of complex dimension 1 is in fact algebraic. And more is true in general.

Fact. The following mathematical structures on a topological surface of genus g are equivalent:

1. A structure of an algebraic curve.
2. A complex structure.
3. (for $g \geq 2$) A hyperbolic structure (i.e. a metric with constant negative curvature) up to isotopy.

While $\mathbb{P}^1$ is the unique curve of genus 0, for higher genera there are lots and lots of curves (or of complex structures, or of hyperbolic metrics).

In fact there is a $3g-3$ dimensional moduli space of smooth curves of genus g, called $\mathbf{M_g}$. It is homeomorphic to the quotient of a ball by a finite group, called the mapping class group...but this is another story.

Again, M_g is very much non compact, because curves can degenerate. Again, the solution is to allow nodal curves to enter the picture.

Definition 4. *A stable curve of genus g is:*

1. *A connected nodal curve.*
2. *The genus of the curve is g. For nodal curves the genus is counted in the following way:*
   (a) *add the genus of all irreducible components.*
   (b) *add the number of loops created by the irreducible components.*
3. *Each component of genus 0 must have at least three nodes.*
4. *Each component of genus 1 must have at least one node.*

Problem 12. *Make sense of the above definition by drawing some pictures!*

The moduli space of stable curves of genus g, denoted $\overline{\mathbf{M}}_{\mathbf{g}}$ is a compactification of M_g.

Of course, if you want, you can do all of the above with marked points as well. The corresponding moduli space is denoted $\overline{\mathbf{M}}_{\mathbf{g,n}}$ and it is a moduli space of dimension $3g-3+n$.

Final Observation: look at the boundary of $\overline{M}_g$. Its components are isomorphic to products of moduli spaces of marked curves! So, even if you only care about unmarked curves, studying moduli spaces of marked curves is essential in order to understand the boundary.

I have quickly told you the existence of a huge network of interesting geometric spaces connected by a ton of natural maps. I hope I conveyed the fact that we just scratched the very tip of a huge iceberg. Let me leave you with some reading reccommendations, for anyone eager to know a little more. The book [HM98] is certainly a pleasant and innstructive read. The survey by Ravi Vakil (*The moduli space of curves and Gormov-Witten theory* - available on the ArXiv), starts basic and gets steep quickly, but it's very well done. Realistically, the first 10 pages may be accessible to a general audience, but that would still be a lot of good stuff.

References

[HM98] Joe Harris and Ian Morrison. *Moduli of Curves.* Springer, 1998.

[KV99] Joachim Kock and Israel Vainsencher. *Kontsevich's formula for rational plane curves.* IMPA, 1999.

Lecture 3

Renzo Cavalieri

Goal of the Day

The goal of today is to find an answer for our old friend

$\mathbf{Q_d}$: *What is the number of rational curves of degree d through $3d-1$ points in the plane?*

We will tackle this question by introducing moduli spaces of stable maps, and we will sketch the proof of Kontsevich using Gromov-Witten invariants. Before we do so though, I want to go back to Q_3, where I told you the answer was 12, and present a classical proof of this fact. Hopefully the amount of cleverness needed for this proof will convince you of the need for a new idea to approach the general question.

Sketch of Classical Proof for 12 Rational Cubics

Since we know that passing through 8 points corresponds to 8 linear conditions, we need to show that being a rational (aka nodal) cubic cuts a hypersurface of degree 12 in the $\mathbb{P}^9$ parameterizing cubics in $\mathbb{P}^2$.

We therefore consider a general line (with coordinate t) in the space of cubics: it has the form

$$f(x,y) + tg(x,y) = 0, \tag{1}$$

where f and g are polynomials of degree 3. Figure 1 illustrates the situation. On the right hand side we (attempted to) draw the total space $\mathcal{S}$ of the family over the t-line. This means, we consider the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ cut out by equation (1). Or, another way to think of it, the fiber over a particular point $\bar{t}$ is precisely the cubic $\{f(x,y) + \bar{t}g(x,y) = 0\}$ living in the $\mathbb{P}^2$-plane $t = \bar{t}$.

We now compute the Euler characteristic of the total space of $\mathcal{S}$ in two different ways, and use this to compute the number of nodal cubics in this family.

Global description: $\mathcal{S}$ is "almost" equal to $\mathbb{P}^2$, because, for any point $P \in \mathbb{P}^2$ different from the 9 points of intersection of f and g, there is exactly one cubic in the family containing P. Those 9 points, on the other hand, are

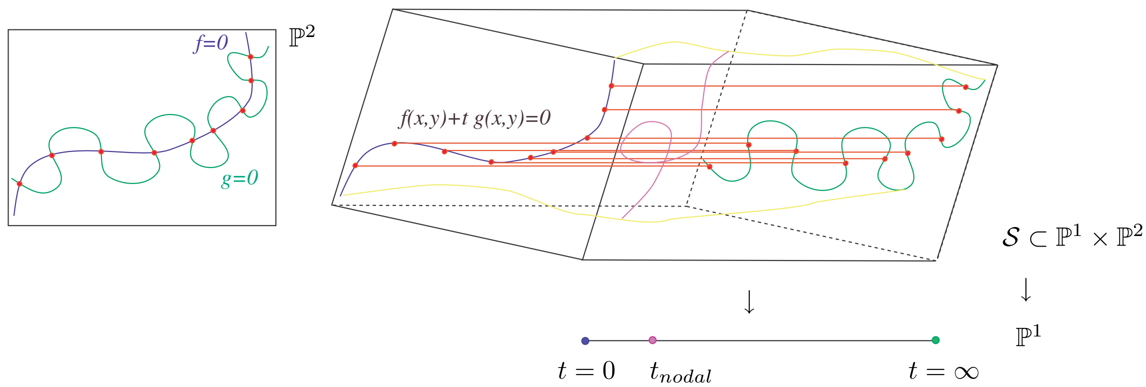

Figure 1: A general line in the space of cubics obtained as the linear span of $f = 0$ and $g = 0$. On the right hand side, $\mathcal{S}$ is the total space of the family. Notice that this surface contains 9 "horizontal" lines.

contained in every single cubic of the family, giving rise to the 9 horizontal lines drawn in the picture. We therefore see that:

$$\mathcal{S} = \mathbb{P}^2 \smallsetminus 9\text{points} \sqcup 9\mathbb{P}^1$$

(those with a little bit of experience in algebraic geometry will have recognized $\mathcal{S}$ as the blow-up of $\mathbb{P}^2$ at the 9 points above). Therefore

$$\mathcal{X}(\mathcal{S}) = 3 - 9 + 18 = 12 \tag{2}$$

Fiberwise description: now consider the family S fiber by fiber. The general fiber is a smooth cubic, which is a torus and has Euler Characteristic 0. There are a number n_{nod} of nodal cubics, which contribute 1 to the Euler Characteristic. I.e.

$$\mathcal{X}(\mathcal{S}) = n_{nod} \tag{3}$$

And equating (2) and (3) gives precisely what we want: there are 12 nodal cubics in the family!

Moduli Spaces of Rational Stable Maps

Seen how much cleverness was required to solve Q_3 this way, we are going to radically change our point of view. Instead of thinking of a rational curve of degree d as of a curve of degree d that happens to have enough nodes as to be rational, we think of it as the image of a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ of degree d.

Problem 1. *Describe the moduli space of maps $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ of degree d*[1] *. Find that its dimension is $3d - 1$!*

Problem 2. *Introduce marks in the picture. Realize that each mark increases the dimension by* 1*.*

As usual, this moduli space is not very interesting, and further it is not compact. And, as usual, it is the compactification that makes things a lot more interesting.

Definition 1. *An **n-pointed rational stable map** is a map $\varphi : C \to \mathbb{P}^2$, where:*

1. *C is a n-marked tree of projective lines.*
2. *Every twig in C mapped to a point must have at least three special points on it.*

Problem 3. *Realize that condition* 2 *is equivalent to asking that the map has only finitely many automorphisms. Since I haven't told you what an automorphism of a map is, this might be a bit tricky...however I will leave as part of the exercise figuring out what the natural concept of an automorphism might be in this case.*

Fact/Definition: *The **moduli space of rational stable maps of degree** d **to** $\mathbb{P}^2$ **with** n **marks*** (in short $\overline{M}_{0,n}(\mathbb{P}^2, d)$) is a smooth[2] compactification of the moduli spaces of n-pointed maps from a smooth $\mathbb{P}^1$.

Natural Maps

There are natural maps between moduli spaces of stable maps:

evaluation maps: there are as many of these maps as there are marks.

$$\begin{array}{rccc} ev_i : & \overline{M}_{0,n}(\mathbb{P}^2, d) & \to & \mathbb{P}^2 \\ & (C, \varphi, P_1, \dots, P_n) & \mapsto & \varphi(P_i) \end{array}$$

forgetting points:

$$\begin{array}{rccc} \mathrm{forg}_i : & \overline{M}_{0,n}(\mathbb{P}^2, d) & \to & \overline{M}_{0,n-1}(\mathbb{P}^2, d) \\ & (C, \varphi, P_1, \dots, P_n) & \mapsto & (C, \varphi, P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n) \end{array}$$

[1] A given geometric map can have more than one algebraic expression! This introduces an equivalence relation that you have to keep in account when answering this question.

[2] This is special to genus 0 and the target being a "convex" variety.

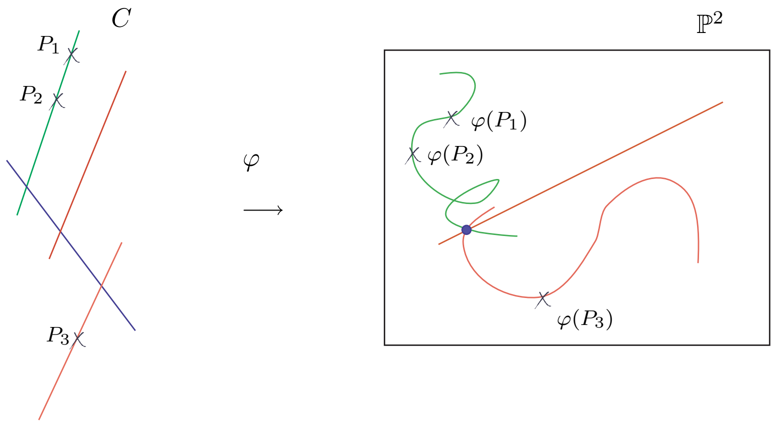

Figure 2: A rational stable map of degree d to $\mathbb{P}^2$

forgetting the map:

$$\begin{array}{cccc} f: & \overline{M}_{0,n}(\mathbb{P}^2, d) & \to & \overline{M}_{0,n} \\ & (C, \varphi, P_1, \ldots, P_n) & \mapsto & (C, P_1, \ldots, P_n) \end{array}$$

Problem 4. *What I just wrote is true generically, but there are cases in which you need to contract twigs and such to make things well defined. Make all of this rigorous.*

The boundary

The boundary can be described in terms of moduli spaces of maps of smaller degree. But in this case, we can't just take products, as we want to make sure that the points corresponding to the node "end up" in the same place on the target (see Figure 3). Therefore we have to take a fiber product with respect to the appropriate evaluation morphisms.

In the example of Figure 3, the boundary stratum is isomorphic to:

$$B \cong \overline{M}_{0,2\cup\{\bullet\}}(\mathbb{P}^2, d_1) \times_{ev_\bullet \times ev_\star} \overline{M}_{0,1\cup\{\star\}}(\mathbb{P}^2, d_2)$$

Remark. Recall that taking a fiber product is equivalent to intersecting the ordinary product with the pullback of the diagonal, i.e. :

$$\overline{M}_{0,2\cup\{\bullet\}}(\mathbb{P}^2, d_1) \times_{ev_\bullet \times ev_\star} \overline{M}_{0,1\cup\{\star\}}(\mathbb{P}^2, d_2) = \overline{M}_{0,2\cup\{\bullet\}}(\mathbb{P}^2, d_1) \times \overline{M}_{0,1\cup\{\star\}}(\mathbb{P}^2, d_2) \cap (ev_\bullet \times ev_\star)^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2})$$

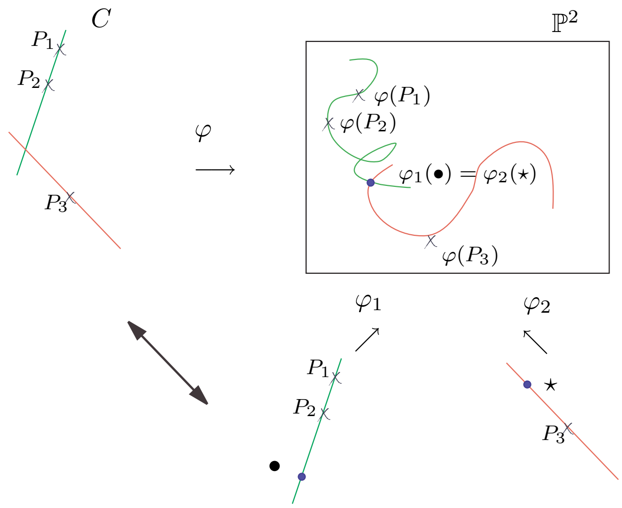

Figure 3: A boundary stratum.

Gromov-Witten Invariants

Finally we are ready to define our heroes: **Gromov-Witten invariants**. These are simply top intersections of special classes on moduli spaces of stable maps: take a closed subvariety α of the target space, and consider:

$$ev_i^*(\alpha).$$

I.e., all maps from pointed curves such that the i-th mark lands in α! We call this is a Gromov-Witten class.

Problem 5. *Show that a Gromov-Witten class has codimension in the moduli space of stable maps equal to the codimension of α in the target space.*

We define a **Gromov-Witten invariant** to be an intersection of Gromov-Witten classes that consists of a finite number of points. We denote it:

$$\langle \alpha_1 \dots \alpha_n \rangle_{0,d}^{\mathbb{P}^2} := \int_{\overline{M}_{0,n}(\mathbb{P}^2,d)} ev_1^*(\alpha_1) \cap \dots \cap ev_n^*(\alpha_n),$$

where the integral sign simply represents "counting the number of such points". The invariant is 0 if the intersection of the classes is either empty or of positive dimension.

Some properties of Gromov-Witten Invariants

Here are some basic properties of Gromov-Witten invariants.

Degree 0: the only (possibly) nonzero degree 0 invariants are those with exactly 3 mark points and sum of the codimensions of the three classes equal to the dimension of the target. In that case.

$$\langle \alpha_1\alpha_2\alpha_3\rangle_{0,0}^X = \alpha_1 \cap \alpha_2 \cap \alpha_3$$

Fundamental class insertions: any Gromov-Witten invariant containing a fundamental class insertion vanishes, unless it is of degree 0 and three pointed, in which case:

$$\langle \alpha_1\alpha_2 1\rangle_{0,0}^X = \alpha_1 \cap \alpha_2$$

Writing what we just said in a formula:

$$\langle \alpha_1\alpha_2 \dots \alpha_{n-1} 1\rangle_{0,d}^X = 0$$

Divisor equation: if one of the insertions is a hypersurface D of degree e, then

$$\langle D\alpha_2 \dots \alpha_{n-1} 1\rangle_{0,d}^X = de\langle \alpha_2 \dots \alpha_{n-1} 1\rangle_{0,d}^X$$

Kontsevich's Proof

Believe it or not, we know enouhg about Gromov-Witten invariants to answer our question Q_d. Throughout this section, we call P (the class of) a generic point in $\mathbb{P}^2$, ℓ (the class of) a generic line in $\mathbb{P}^2$, 1 the fundamental class of $\mathbb{P}^2$. Also, we denote N_d the answer to Q_d, i.e.

$\mathbf{N_d}$: number of rational curves of degree d through $3d-1$ points in $\mathbb{P}^2$.

We can interpret N_d as a Gromov-Witten invariant:

$$N_d = \langle \underbrace{P \dots P}_{3d-1 \text{ times}} \rangle_{0,d}^{\mathbb{P}^2}$$

So what? We still do not know how to compute it...well, wait just one more second. Kontsevich's genius was to...break the symmetry a bit, and break one of the points into two lines, so as to consider:

$$\mathfrak{C} = ev_1^*(\ell) \cap ev_2^*(\ell) \cap ev_3^*(P) \cap \dots \cap ev_{3d}^*(P)$$

Counting dimensions, we see that $\mathfrak{C}$ is a curve in $\overline{M}_{0,3d}(\mathbb{P}^2, d)$. We are now going to intersect this curve with two equivalent hypersurfaces, and extract from equating the result a recursion that computes N_d.

WDVV

Recall our forgetful morphisms from a while ago...now we are going to use them. We are going to forget a bunch of marks (all of them minus 4), and we are going to forget the map. All together we obtain:

$$F : \overline{M}_{0,3d}(\mathbb{P}^2, d) \longrightarrow \overline{M}_{0,4} = \mathbb{P}^1$$

We consider the hypersurface $F^{-1}(\text{point}) \subset \overline{M}_{0,3d}(\mathbb{P}^2, d)$. Since any two points in $\mathbb{P}^1$ are equivalent, we can really choose any point we want. We are going to choose two special points, corresponding to the boundary divisors in Figure 4. By doing so, we obtain:

$$\mathfrak{C} \cap F^{-1}(Q_1) = \mathfrak{C} \cap F^{-1}(Q_2) \tag{4}$$

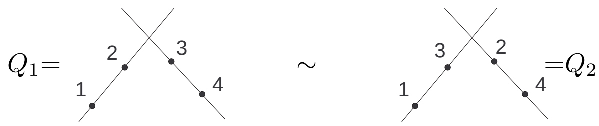

Figure 4: Two equivalent points in $\overline{M}_{0,4}$

All we have left to do is now interpret what (4) means. On the left hand side we have to restrict our attention to boundary divisors that have the first two marks on one twig, the third and fourth on the other. On the right hand side, 1 and 3 are together, and so are 2 and 4.

Recall the structure of the boundary: we have to take fiber products over the evaluation morphisms of two moduli spaces of maps of degrees adding to d, where our original set of marks has been partitioned in two, and then we have to add one mark on each twig that will become the node.

By mentioning the fact that $\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}$ is equivalent to $P \times 1 + \ell \times \ell + 1 \times P$, we finally can write (4) as follows:

left hand side:

$$\sum_{d_1+d_2=d} \Big(\langle \ell\ell *** 1 \rangle^{\mathbb{P}^2}_{0,d_1} \langle P *** PP \rangle^{\mathbb{P}^2}_{0,d_1} + \langle \ell\ell *** \ell \rangle^{\mathbb{P}^2}_{0,d_1} \langle \ell *** PP \rangle^{\mathbb{P}^2}_{0,d_1} +$$

$$+ \langle \ell\ell *** P \rangle^{\mathbb{P}^2}_{0,d_1} \langle 1 *** PP \rangle^{\mathbb{P}^2}_{0,d_1} \Big)$$

right hand side:

$$\sum_{d_1+d_2=d} \Big(\langle \ell P *** 1 \rangle^{\mathbb{P}^2}_{0,d_1} \langle P *** \ell P \rangle^{\mathbb{P}^2}_{0,d_1} + \langle \ell P *** \ell \rangle^{\mathbb{P}^2}_{0,d_1} \langle \ell *** \ell P \rangle^{\mathbb{P}^2}_{0,d_1} +$$

$$+ \langle \ell P *** P \rangle^{\mathbb{P}^2}_{0,d_1} \langle 1 *** \ell P \rangle^{\mathbb{P}^2}_{0,d_1} \Big)$$

Here, we put $***$ to mean that one needs to distribute the remaining marks in all possible ways.

This looks like a huge combinatorial mess, but in fact it is not that bad, because a lot of the terms vanish. In fact, it is much more convenient to tackle the question by analyzing what are the terms that do not vanish!

First observe that of all the terms that contain a 1, there is only one that is non-zero, and it contributes precisely N_d. What are left are the terms with no 1. Notice that we can pull out the ℓ's with the divisor axiom. Now, for those guys not to vanish the only possibility is that the number of points on both sides be the "right one" (i.e. $3d_i - 1$ on each side). At the end of the day, and I am more than glad to leave the actual derivation as a good exercise, one gets the recursive equation:

$$N_d = \sum_{d_1+d_2=d, d_1,d_2>0} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right]$$

Finally, by inputting $N_1 = 1$, we obtain $N_2 = 1$, $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$ ...