What is cryptography?

SIAM Mini-Conference 2025

Gaurish Korpal

University of Arizona

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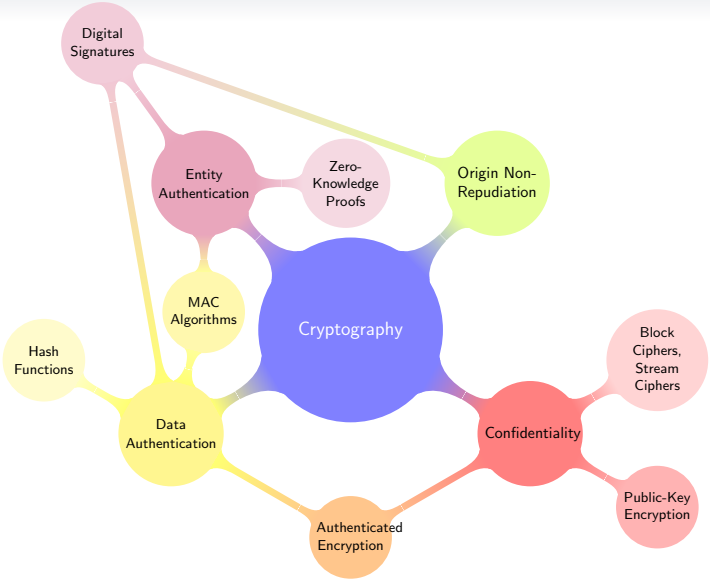
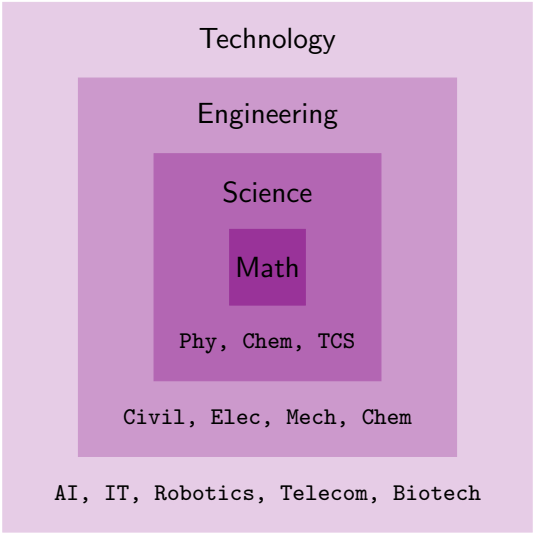
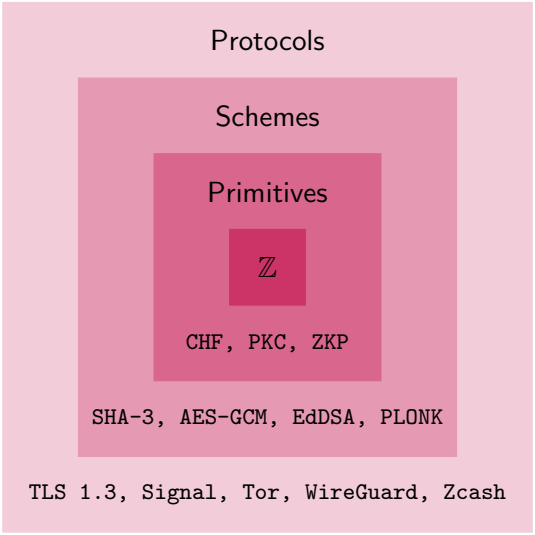


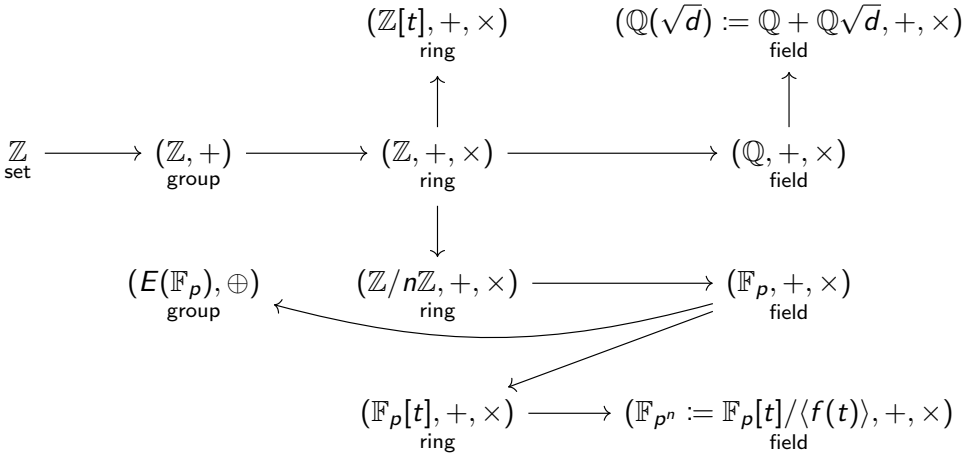
Introduction

Propaganda


Technology
Engineering
Science
Math
Phy, Chem, TCS
Civil, Elec, Mech, Chem
AI, IT, Robotics, Telecom, Biotech


Protocols
Schemes
Primitives
$\mathbb{Z}$
CHF, PKC, ZKP
SHA-3, AES-GCM, EdDSA, PLONK
TLS 1.3, Signal, Tor, WireGuard, Zcash

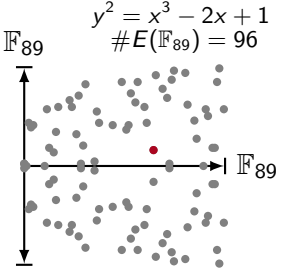
Algebra primer



Pairing-based cryptography

Elliptic curves

- $E : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{F}_p$ and $4a^3 + 27b^2 \neq 0$.
- Points $E(\mathbb{F}_p)$ form a group with $\mathcal{O}_E$ as identity.
- $P \in E(\mathbb{F}_p)[r]$, that is $\underbrace{P \oplus \cdots \oplus P}_{r\text{-times}} = [r]P = \mathcal{O}_E$.
- ECDLP: Given $Q = [m]P$, find m.



For r prime to p there exists a non-degenerate distorted bilinear map:

$$e_r : E(\mathbb{F}_p)[r] \times E(\mathbb{F}_p)[r] \to \mathbb{F}_{p^u}^{\times}$$

u is called the embedding degree of E w.r.t. r.

- $e_r(aP, bQ) = e_r(P, Q)^{ab}$
- $e_r(Q, Q) \neq 1$
- If $e_r(Q_1, Q_2) = 1$ for all $Q_1 \in E(\mathbb{F}_p)[r]$ then $Q_2 = \mathcal{O}_E$.

Scout's honor!

Short Signature

- Supports non-interactive aggregation property: given a collection of signatures $(\sigma_1, \ldots, \sigma_n)$, anyone can produce a short signature (σ) that authenticates the entire collection.
- BLS short signature (2001) is relies on pairing-friendly curves.
- Ethereum blockchain uses BLS signatures.

Polynomial Commitment Scheme

- Allows one party to prove to another the correct evaluation of a polynomial at some set of points, without revealing any other information about the polynomial.
- KZG polynomial commitment (2010) relies on pairing-friendly curves.
- Irrespective of the degree of the polynomial, KZG commitment size is constant.

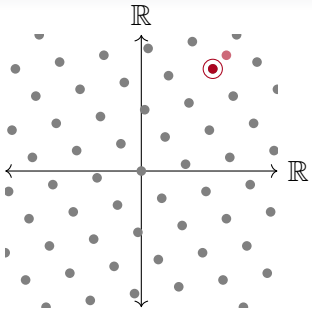
datatracker.ietf.org/doc/draft-boneh-bls-signature/

https://risencrypto.github.io/Kate/

Lattice-based cryptography

Lattices

- A lattice Λ is a discrete subgroup of $\mathbb{R}^n$. Given basis matrix $B \in \mathbb{R}^{m \times n}$, $\Lambda = \{B\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n\}$
- CVP: Find a vector closest to given vector $\mathbf{v} \in \Lambda$.
- LWE: Given $A \in \mathbb{F}_p^{m \times n}$ and $\mathbf{b} \in \mathbb{F}_p^m$ such that $A \cdot \mathbf{s} + \mathbf{e} = \mathbf{b}$ find $\mathbf{s} \in \mathbb{F}_p^n$ for unknown error $\mathbf{e} \in \mathbb{F}_p^m$.
- LWE$\leftrightarrow$CVP: The lattice vector $A \cdot \mathbf{s}$ with distance $\mathbf{e}$ is almost always a vector closest to $\mathbf{b}$.



Let $f \in \mathbb{Z}[t]$ be a monic polynomial of degree n and consider the ring $R \coloneqq \mathbb{Z}[t]/f$ and ideal $I \subset R$.

$$(R, +) \longleftrightarrow (\mathbb{Z}^n, +) \quad \text{and} \quad I \longleftrightarrow \Lambda$$

Multiplicative closure property of ideal lattice results in bonus geometric symmetries.

- $I \subseteq R$ is called an ideal if it is a subgroup of $(R, +)$ that absorbs multiplication by elements of R.
- If $I = \alpha R$ then I is principal ideal.
- $\mathbb{Z}[t]/f = \{g \mod f \mid g \in \mathbb{Z}[t]\}$ where $\deg(g \mod f) < \deg(f)$.

Bend, don't break!

Post-Quantum Cryptography

- Symmetric cryptography do not rely on mathematics vulnerable to quantum computers.
- Security of common key exchange and digital signature schemes rely on hardness of factorization and discrete logarithm, vulnerable to Shor's quantum algorithm.
- Cryptographic Suite for Algebraic Lattices (CRYSTALS) is one of the first standardized PQC scheme (2024).

https://pq-crystals.org/index.shtml

Fully Homomorphic Encryption

- Homomorphic refers to homomorphism in algebra: $\varphi(a \oplus b) = \varphi(a) \otimes \varphi(b)$
- Allows computations to be performed on encrypted data without first having to decrypt it.
- Gentry constructed the first ever FHE scheme using ideal lattices (2009).
- All known fully-homomorphic encryption schemes with compact ciphertexts use lattice techniques.

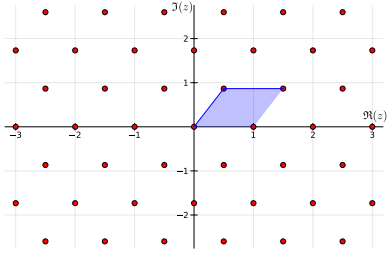
https://fhe.org/history/

Class-group-based cryptography

Imaginary quadratic orders

- Let $D < 0$ be such that $D \equiv 0, 1 \pmod 4$. Then the ring $\mathbb{Z}[\omega] = \mathbb{Z} + \mathbb{Z}\omega$ where $\omega = \frac{D+\sqrt{D}}{2}$ is called an imaginary quadratic order of discriminant D.
- The field of fractions is $\mathbb{Q}(\sqrt{D})$.
- A fractional ideal of $\mathbb{Z}[\omega]$ is a subset $J \subset \mathbb{Q}(\sqrt{D})$ such that aJ is an ideal of $\mathbb{Z}[\omega]$ for some $a \in \mathbb{N}$.
- J is invertible if there is fractional ideal J' such that $JJ' := \{\sum_{i=1}^{n} a_i b_i \mid a_i \in J, b_i \in J'\} = \mathbb{Z}[\omega]$.

$\mathbb{Z}\left[\frac{-3+\sqrt{-3}}{2}\right]$ in $\mathbb{C}$



- The class group $\mathrm{Cl}(D)$ of $\mathbb{Z}[\omega]$ is the quotient group of invertible fractional ideals by principal ideals with ideal multiplication.
- It is a composite order group of unknown order with a subgroup of known order where the DL is easy.

DDH for $\mathrm{Cl}(D)$ can be characterized as a HSM since it is hard to determine if a given element is a member of $\mathrm{Cl}(D)$.

Rest assured!

Multi-Party Computation

- Allows a group of mutually distrustful parties to compute a joint function on their inputs without revealing any information beyond the result of the computation.
- Class groups were first proposed as an alternative to ECC, but CL attack broke it.
- Ideas from the CL attack make class groups well-suited for MPC protocols that require a one-time transparent setup with minimal interaction among parties.

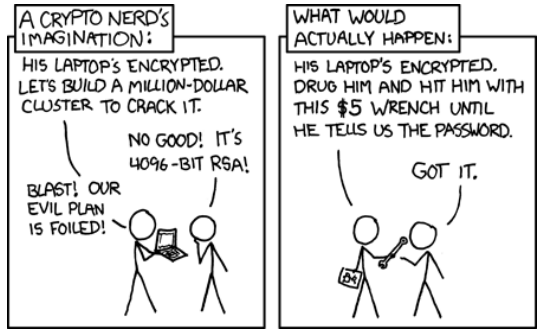
Verifiable Delay Function

- Allows a prover to show a verifier that a certain amount of time running a function was spent, and do it in a way that the verifier can check the result quickly.
- Groups of unknown order are great candidates for VDF construction.
- Class groups are one of the most popular choice because the can be generated without trusted setup (Wesolowski, 2018)

https://eprint.iacr.org/2022/1466.pdf

https://eprint.iacr.org/2018/712.pdf

Thank you!



Randall Munroe, Security. https://xkcd.com/538/.