CHAPTER 2

Bruhat-Tits Theory and Buildings

JIU-KANG YU (PURDUE UNIVERSITY)

Introduction

This article originated from slides used for a mini-course on Bruhat-Tits theory in a Fields institute workshop in 2004, which were slightly edited and expanded shortly after the workshop, then again in 2008. The informal style of the original lectures is perserved. Some imprecisions and mistakes were fixed but some more probably remain.

Ahead of the workshop we recommended that the participants study Tits' summary of Bruhat-Tits theory in the Corvallis proceedings [Tit79]. This summary has been the portal to Bruhat-Tits theory for most people, and it will continue to be the best user guide. It is very well-written and precise. It does an excellent job of hiding the technicalities and of describing everything in terms of elegant abstract characterizations.

The goal of my lectures is to provide more hints and help to people who want to use Bruhat-Tits theory. I tried to be as orthogonal to Tits' summary as possible. Therefore, I do not give a systematic account of the theory itself. In the first lecture, I provide some background materials that are probably most useful if you learn them before starting to read Tits' article. In the second lecture, I give diverse complements to Tits' article. From my experience, these should be helpful to someone who is currently studying Tits' article. In particular, I try to explain how to go between [Tits] and [BT1-5]. For the third lecture, I discuss more recent developments in representation theory and "functoriality" of buildings that go beyond Tits' article.

2.1 Lecture 1

2.1.1 History and Literature.

Prehistory

— O. Goldman and N. Iwahori: *The theory of p-adic norms*, Acta Math. **109** (1963), 41 pages [GI63].
— N. Iwahori and H. Matsumoto: *On some Bruhat decomposition and the structure of the Hecke ring of* $\mathfrak{p}$*-adic Chevalley groups*, Publ. IHES. **25** (1965), 44 pages [IM65].
— H. Hijikata: *On the arithmetic of* $\mathfrak{p}$*-adic Steinberg groups*, Mimeographed notes at Yale University (1964) [Hij64].

The beginning

It all started from Bruhat's article in the Proceedings of the Boulder conference (1965) "Algebraic groups and discontinuous subgroups"[1]

p-adic Groups

BY

FRANÇOIS BRUHAT

1. **Bounded subgroups.** If G is a real connected Lie group, then the following two statements are well known:

(1) Any compact subgroup of G is contained in a maximal compact subgroup of G.

(2) Two maximal compact subgroups are conjugate by an inner automorphism.

[skipping to the end of the article]

Added in November 1965. During the conference, considerable progress was made towards an affirmative solution of the conjectures above. It also appeared that the properties thus established have interesting applications; for instance, they provide a simplified approach to Kneser's theorem on H^1 of simply connected groups over the $\mathfrak{p}$-adics. A joint paper on this subject is in preparation, by F. Bruhat and J. Tits.

These results were exposed orally by J. Tits at the conference. The precise form on which they are given in the mimeographed notes of his talk must however be somewhat modified; in particular, it is not true that minimal k-parahoric subgroups of a group G—as defined in these notes—are conjugate by elements of G_k. In fact, the notion of k-parahoric subgroup given there does not appear to be "the good one" when G does not split over an unramified extension of k.

On the other hand, the methods sketched there turn out to give further results. For instance, it can be shown that the Conjecture (II) (iv) above is essentially a consequence of the other parts of that conjecture and, in particular, is true in the split case.

BIBLIOGRAPHY

1. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des representations des groupes $\mathfrak{p}$-adiques*. Bull. Soc. Math. France **89** (1961), 43–75.

Announcements by Bruhat-Tits

- *Groupes algébriques simples sur un corps local*, Proceedings of a Conference on Local Fields (Driebergen, 1966), 14 pages [BT67].
- *BN-paires de type affine et données radicielles*, C.R. Acad. Sci. Paris **263** (1966), 4 pages [BT66a].
- *Groupes simples résiduellement déployés sur un corps local*, ibid., 3 pages [BT66b].
- *Groupes algébriques simples sur un corps local*, ibid., 4 pages [BT66c].
- *Groupes algébriques simples sur un corps local: cohomologie galoisienne, décompositions d'Iwasawa et de Cartan*, ibid., 3 pages [BT66d].

Tits' summary

Tits It is [Tit79] in the Corvallis proceedings (1977) [BC79]. A must read.

[1]Reprinted from [Bru66], with permission.

The canon

BT1 *Groupes réductifs sur un corps local I*, Publ. Math. I.H.E.S. **41** (1972), 247 pages [BT72].

BT2 *Groupes réductifs sur un corps local II*, Publ. Math. I.H.E.S. **60** (1984), 180 pages [BT84a].

BT3 *Schémas en groupes et immeubles des groupes classiques sur un corps local*, Bulletin Soc. Math. France **112** (1984), 43 pages [BT84b].

BT4 *Schémas en groupes et immeubles des groupes classiques sur un corps local II: groupes unitaires*, Bulletin Soc. Math. France **115** (1987), 55 pages [BT87a].

BT5 *Groupes algébriques sur un corps local*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 28 pages [BT87b].

Remark 2.1.1.1 (i) Officially, [BT1], [BT2], [BT5] are labelled as Chapter I, Chapter II and Chapter III. (ii) Somehow [BT3], [BT4], [BT5] are relatively unknown to users of Bruhat-Tits theory. Some of the results there have been rediscovered years later by other authors, repeatedly. Why?

Comments

- The writing took over 20 years (1965–1987).
- The canon all together is over 550 pages.
- The formulation evolved during the period.
- Tits' summary, which is the portal for most users of Bruhat-Tits theory, was written way before most of the canon.
- [BT] (=[BT1–5]) always tried to pursue maximal generalities. Some of these are not necessary for representation theory. But it is amazing that essentially everything in their theory became quite useful to certain part of mathematics later.

Some work on Bruhat-Tits theory itself

- I.G. Macdonald: *Spherical functions on a group of p-adic type* (1971) [Mac71].
- G. Rousseau's thesis (1977), unpublished [Rou77].
- P. Garret: *Buildings and classical groups* (1997), about 400 pages [Gar97].
- W.T. Gan and J.-K. Yu: *Schémas en groupes et immeubles des groupes exceptionnels sur un corps local, Première partie: Le groupe G_2*, Bulletin SMF, 55 pages [GY03].
- W.T. Gan and J.-K. Yu: *Schémas en groupes et immeubles des groupes exceptionnels sur un corps local, Deuxième partie: Les groupes F_4 et E_6*, Bulletin SMF, 43 pages [GY05].
- J.-L. Kim and A. Moy: *Involutions, classical groups and buildings*, J. Algebra (2000), 17 pages [KM01].
- E. Landvogt, *A compactification of the Bruhat-Tits building*, Springer LNM (1995), 156 pages [Lan96].
- E. Landvogt, *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. 518 (2000), 29 pages [Lan00].
- A. Moy: *Displacement functions on the Bruhat-Tits building*, The mathematical legacy of Harish-Chandra (Baltimore 1998), 17 pages [Moy00].
- G. Prasad: *Galois-fixed points in the Bruhat-Tits building of a reductive group*, Bulletin SMF [Pra01].
- PY G. Prasad and J.-K. Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. (2002) [PY02].

— J.-K. Yu: *Smooth models associated to concave functions in Bruhat-Tits theory*, preprint (2003), 34 pages [Yu03].

Remark 2.1.1.2 Several articles by G. Prasad and S. Raghunathan were written before [BT2]. They worked out many results and formulas independently. Therefore, these articles can be used as alternative references to some extent. Their Annals articles [PR84] about central extensions (which is the precursor of Moy-Prasad theory) are particularly useful.

Applications to representation theory (The list is short and incomplete, concentrating on more recent ones.)

— J.D. Adler: *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math. **185** (1998), 32 pages [Adl98].
— J.D. Adler and S. DeBacker: *Some applications of Bruhat-Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group*, Michigan Math. J. **50** (2002), 19 pages [AD02].
— D. Barbasch and A. Moy: *A new proof of the Howe conjecture*, Journal of the AMS **13** (2000), 12 pages [BM00].
— S. DeBacker: *Some applications of Bruhat-Tits theory to Harmonic analysis on a reductive p-adic group*, Michigan Math. J. (2002), 21 pages [DeB02b].
— S. DeBacker: *Parametrizing nilpotent orbits via Bruhat-Tits theory*, Ann. of Math. **156** (2002), 38 pages [DeB02c].
— S. DeBacker: *Homogeneity results for invariant distributions of a reductive p-adic group*, Ann. Sci. Ecole Norm. Sup. **35** (2002), 32 pages [DeB02a].
— S. DeBacker and M. Reeder: *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math, to appear [DR].
— A. Moy and G. Prasad: *Unrefined minimal K-types for p-adic groups*, Inv. Math. **116**, 393–408 (1994) [MP94].
— A. Moy and G. Prasad: *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helvetici **71**, 96–121 (1996) [MP96].
— P. Schneider and U. Stuhler: *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. **85**, 97–191 (1997) [SS97].
— J.-K. Yu: *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14**, 579–622 (2001) [Yu01].

Comments For reasons commented above, Bruhat-Tits theory is not easy to read for many people. In particular, people in representation theory only want to learn it as a foundation. For that, going through 550 pages would be a lot. Tits' summary [Tits] is the portal into Bruhat-Tits theory for most people, and is highly recommended.

Macdonald's little book is also quite accessible, but it only sketches the case of simply connected groups. Now Landvogt's book is a usable replacement to many results of [BT1] and [BT2]. However, it doesn't treat groups associated to concave functions which have become increasingly useful now (this theory is not covered in Tits' summary or Macdonald's book either). My article on smooth models [Yu03] can replace most of the algebro-geometric part of [BT2], at least in the case of discrete valuations.

Garret's book describes the buildings of classical groups as a simplicial complex, in a rather concrete way. However, it is not clear that the building in his book is the building described in Bruhat-Tits theory. In fact, Bruhat and Tits realized this kind of matching is not obvious at all, so they wrote [BT3] and [BT4] to match their buildings and the theory of Goldman-Iwahori. Today it is fairly easy to match the building in Garret's book with that in Bruhat-Tits theory. A general strategy was developed in the two articles of Gan and myself ([GY03], [GY05]), and then was applied to exceptional groups. I sketched the case of classical groups in a lecture at Banff. Notes of that lecture can be found on my webpage `http://www.math.purdue.edu/~jyu`.

In addition, I will give a few suggestions, which are of course very subjective.

About learning to use Bruhat-Tits theory

— Don't try to swallow all proofs.
— Learn a bit about symmetric spaces.
— Learn a bit about spherical buildings of a reductive group.
— Learn the case of GL_n really well.
— Learn a bit about affine root systems.
— Learn a bit about BN-pairs (a.k.a. Tits systems).
— Draw/play with the 2-dimensional apartments.
— Understand the case of split/quasi-split groups.
— (Learn the theory of schemes).

Remark 2.1.1.3 It may look like I am recommending a lot or things not directly related to the Bruhat-Tits theory; you may suspect that this is overloading on top of something which is already complicated. However, most users of Bruhat-Tits theory want to learn the facts and to get the feeling, not going through details and proofs. It is best to achieve this through analogies. We only need a little bit of everything from above and I believe that in the end you will find the experience rewarding.

A few things that I do not recommend

— Don't try to picture a building of dimension > 1.
— It is not necessary to learn a lot about (poly-)simplicial complexes.
— It is not necessary to learn a lot about general buildings.
— Don't think of Bruhat-Tits theory as just a case of BN-pairs (even though the building can be described using a BN-pair, the B and the N are not easy to specify; moreover, there are many features in Bruhat-Tits theory that are not in the theory of BN-pairs).
— Although one can say a lot by just talking about chambers (a.k.a. alcoves) and apartments, don't be afraid of talking about the whole building.

2.1.2 Symmetric spaces of real reductive groups.

References

— T.A. Springer, *Reductive groups*, the Corvallis proceedings (A. Borel and W. Casselman, Eds.), v.1., 3–27 (1977) [Spr79].
— W.T. Gan and J.-K. Yu: *Schémas en groupes et immeubles des groupes exceptionnels sur un corps local, Première partie: Le groupe G_2*, Bulletin SMF, 55 pages [GY03].

Notation

— G: a connected reductive group over $\mathbb{R}$;
— K: a maximal compact subgroup of $G(\mathbb{R})$;

— $\mathfrak{g} := \operatorname{Lie} G$.

The symmetric space $\mathscr{S} = \mathscr{S}(G)$ of G (or of $G(\mathbb{R})$) is essentially $G(\mathbb{R})/K$. But we want to talk about this more intrinsically without singling out K.

If G is semisimple, we can say that $\mathscr{S}$ is the set of maximal compact subgroups of $G(\mathbb{R})$ (since all of them are $G(\mathbb{R})$-conjugate to K).

But I prefer to use the bijections

$$\begin{aligned} &\{\text{maximal compact subgroups of } G(\mathbb{R})\} \\ \leftrightarrow\ &\{\text{Cartan involutions on } G(\mathbb{R})\} \\ \leftrightarrow\ &\{\text{Cartan involutions on } \mathfrak{g}\} \end{aligned}$$

to say that $\mathscr{S}$ is the set of Cartan involutions on $\mathfrak{g}$.

Recall: A Cartan involution on $\mathfrak{g}$ is an order 2 automorphism θ of $\mathfrak{g}$ such that if $\mathfrak{k}$ and $\mathfrak{p}$ are the (-1)- and $(+1)$-eigenspaces of $\mathfrak{g}$ under θ, then $\mathfrak{k} + i\mathfrak{p}$ is a compact form of $\mathfrak{g}$ (and compactness of a real form can be detected by the positivity of the Killing form). A Cartan involution θ induces a unique automorphism on $G(\mathbb{R})$, again denoted by θ. The subgroup $K_\theta := G(\mathbb{R})^\theta$ is then a maximal compact subgroup of $G(\mathbb{R})$ and all maximal compact subgroups can be obtained this way.

Remark 2.1.2.1 We will deal with all reductive groups later.

Example 2.1.2.2 Let $G = \mathrm{SL}_n$. Then the standard corresponding Cartan involution is $X \mapsto -{}^tX$ on $\mathfrak{g} = \mathfrak{sl}_n$, and $g \mapsto ({}^tg)^{-1}$ on $G(\mathbb{R})$.

Let us consider the case $G = \mathrm{SL}(V)$ more intrinsically. A maximal compact subgroup of $G(\mathbb{R})$ is $\mathrm{SO}(q)$ for a positive definite quadratic form q on V. The Cartan involution is again $g \mapsto ({}^tg)^{-1}$, and now ${}^tg$ is the adjoint of g relative to the symmetric bilinear form $\langle -,-\rangle = \langle -,-\rangle_q$ associated to q:

$$\langle g.v, w\rangle = \langle v, {}^tg.w\rangle.$$

Of course, $\sqrt{q}$ is then a Euclidean norm on V. Therefore,

$\mathscr{S}(\mathrm{SL}(V))$ is the space of norms on V (up to constant multiples).

In general, for a connected reductive group $G/\mathbb{R}$, we write $\mathscr{S}_{\mathrm{red}}(G)$ for the space of Cartan involutions on $\mathfrak{g}$ (or on G).

Example 2.1.2.3 Let $G = \mathbb{G}_{\mathrm{m}}^d$ be an d-dimensional split torus. Then $\mathscr{S}_{\mathrm{red}}(G)$ consists of one element, which is $X \mapsto -X$ on $\operatorname{Lie} G$ and $g \mapsto g^{-1}$ on $G(\mathbb{R})$. On the other hand, $\mathscr{S}(G) = (\mathbb{R}^\times_{>0})^d \simeq (\mathbb{R}^\times/\{\pm 1\})^d$ (see below for the definition of $\mathscr{S}(G)$ for non-semisimple G).

If G is anisotropic, then $G(\mathbb{R})$ is compact and $\mathscr{S}_{\mathrm{red}}(G)$ consists of only one point, which is $X \mapsto X$ on $\operatorname{Lie} G$ and $g \mapsto g$ on $G(\mathbb{R})$.

More generally, if G is anisotropic modulo its center, then $\mathscr{S}_{\mathrm{red}}(G)$ consists of one point only.

Almost every statement about $\mathscr{S}(G)$ that has a counterpart in Bruhat-Tits theory can be deduced from the following (together with a few standard facts):

Theorem 2.1.2.4 *Let $G \subset H$ be connected reductive groups over $\mathbb{R}$. Then*

(i) *If $\theta' \in \mathscr{S}_{\mathrm{red}}(H)$ stabilizes $G(\mathbb{R})$, then $\theta'|G(\mathbb{R}) \in \mathscr{S}_{\mathrm{red}}(G)$.* "restriction"

(ii) *For any* $\theta \in \mathscr{S}_{\mathrm{red}}(G)$, *the set*

$$\{\theta' \in \mathscr{S}_{\mathrm{red}}(H) : \theta'|G(\mathbb{R}) = \theta\}$$

Can do even if θ' doesn't stabilize $G(\mathbb{R})$?

is non-empty, and is permuted transitively by the real points of $Z_H(G)$, *the centralizer of* G *in* H.

so is a $Z_H(G)$-torsor.

Apartments For each maximal $\mathbb{R}$-split torus S of G, let $A_{\mathrm{red}}(S) = \{\theta \in \mathscr{S}_{\mathrm{red}}(G) : \theta(S(\mathbb{R})) \subset S(\mathbb{R})\}$. We call $A_{\mathrm{red}}(S)$ the *apartment* of S. We notice that $\theta \in A_{\mathrm{red}}(S)$ if and only if θ extends the only Cartan involution of $S(\mathbb{R})$.

depends on choice of max'l $\mathbb{R}$-split torus.

Example 2.1.2.5 Let $G = \mathrm{GL}(V)$. To give a maximal $\mathbb{R}$-split torus S of G amounts to giving a decomposition of V into 1-dimensional subspaces: $V = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_n$.

The quadratic form q lies in $A_{\mathrm{red}}(S)$ if and only if the form is diagonalized with respect to the basis $v_1, \ldots, v_n$:

$$q(x_1v_1 + \cdots + x_nv_n) = a_1x_1^2 + \cdots + a_nx_n^2, \qquad a_i > 0.$$

We know that only 1 Cartan.

Remarks

- If we give $\mathscr{S}_{\mathrm{red}}$ the structure of a Riemannian symmetric space, then the apartments are the maximal flat subspaces.
- *All apartments are conjugate by* $G(\mathbb{R})$. Since all the S's are conjugate.
- *Each point lies in an apartment.* The unique Cartan involution of $S(\mathbb{R})$ always extends to a Cartan involution of $G(\mathbb{R})$ by (ii) of the theorem. Therefore, at least one Cartan involution lies on an apartment. It follows that any Cartan involution lies on an apartment.
- $S(\mathbb{R})$ *acts on* $A_{\mathrm{red}}(S)$ *transitively.* If θ_1 and θ_2 both extend the only Cartan involution of S, then $\theta_2 = z.\theta_1$ are conjugate by some z of $Z(\mathbb{R})$ by (ii) of the theorem, where $Z = Z_G(S)$. We can write $z = sk$, where $z \in S(\mathbb{R})$ and k is in the maximal compact subgroup of $Z(\mathbb{R})$ (Cartan decomposition). Since $k \in K_{\theta_1}$, we have $K_{\theta_2} = zK_{\theta_1}z^{-1} = sK_{\theta_1}s^{-1}$. Hence $\theta_2 = s.\theta_1$.
- *The dimension of* $A_{\mathrm{red}}(S)$ *is* $\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, Z(G)$. Let $\theta \in A_{\mathrm{red}}(S)$. The stabilizer of θ in $G(\mathbb{R})$ is the same as the normalizer of K_θ in $G(\mathbb{R})$, and is simply $Z_G(\mathbb{R}).K_\theta$, where $Z_G = Z(G)$ is the center of G. In particular, the stabilizer of θ in $S(\mathbb{R})$ is $S(\mathbb{R}) \cap Z_G(\mathbb{R}).K_\theta$ is the maximal $\mathbb{R}$-split torus of $Z(G)$.
- *Any two points* $x, y \in \mathscr{S}_{\mathrm{red}}(G)$ *lie on some apartment* $A_{\mathrm{red}}(S)$. Suppose that $x \in A_{\mathrm{red}}(S)$. Then we have the *Cartan decomposition* $G(\mathbb{R}) = K_xS(\mathbb{R})K_x$. Suppose that $K_y = gK_xg^{-1}$ with $g = ksk'$, where $k, k' \in K_x$ and $s \in S$. Then it is clear that $K_y = (ks)K_x(ks)^{-1}$ and y lies on the apartment associated to $(ks)S(ks)^{-1} = kSk^{-1}$. It is also clear that x lies in kSk^{-1} also since $\theta_x(kSk^{-1}) = \theta_x(k)\theta_x(S)\theta_x(k)^{-1} = kSk^{-1}$. Thus x, y both lie on $A_{\mathrm{red}}(kSk^{-1})$.

Remark 2.1.2.6 In fact, the above statement is equivalent to the Cartan decomposition. Notice that when $G = \mathrm{GL}(V)$, the above statement is the following familiar fact in linear algebra: *any two positive definite symmetric real matrices can be diagonalized simultanenously.*

Why? Do they commute?

- K_x *permutes apartments containing* x *transitively.* Suppose that $x \in A_{\mathrm{red}}(S)$ and $x \in A_{\mathrm{red}}(S')$. We may assume that $S' = gSg^{-1}$. The points of $\mathscr{S}_{\mathrm{red}}(G)$

on $A_{\text{red}}(S')$ are precisely $gS(\mathbb{R})g^{-1}.(gx) = gS(\mathbb{R}).x$. Therefore, there exists $s \in S(\mathbb{R})$ such that $x = gs.x$. Let $k = gs$, then we have $k \in K_x$ and $kSk^{-1} = S'$. Therefore, $A_{\text{red}}(S') = k.A_{\text{red}}(S)$.

— *Another description of the apartments containing x.* Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition relative to θ_x. Then there is a bijection between the apartments containing x and the maximal abelian sub-algebras of $\mathfrak{p}$. The bijection is $A_{\text{red}}(S) \mapsto \operatorname{Lie} S$. Therefore the previous statement is the well-known fact that K_x permutes transitively the abelian sub-algebras of $\mathfrak{p}$.

In general, let $V = V(G)$ be the maximal vector subgroup in the center of $G(\mathbb{R})$, and we define $\mathscr{S} = \mathscr{S}_{\text{red}} \times V$. We can put a natural action of $G(\mathbb{R})$ on $\mathscr{S}$ to reinstate the property that the stabilizer of points on $\mathscr{S}$ are precisely the maximal compact subgroups. (Here I am following the definition of $V(G)$ in the literature. One can also define $V(G)$ to be the maximal connected abelian quotient of $G(\mathbb{R})$. In fact this makes the action of $G(\mathbb{R})$ on $V(G)$ more transparent, and makes a better analogy with the p-adic case).

We also define the extended apartment $A(S)$ as $A_{\text{red}}(S) \times V(G)$. In particular, $\dim A(S) = \dim S$. See the appendix of Gan-Yu [GY03] for more details.

The point of these discussions is: the symmetric space has many properties analogous to those of the Bruhat-Tits building. But they are a lot easier to derive from basic facts in Lie groups.

The analogous results in Bruhat-Tits theory are often related to analogous structure theory and analogous decompositions. But the arguments are often more convoluted. Therefore, learning the real case helps us to get a picture, and pinpoint some basic ingredients.

2.1.3 The building of a p-adic GL_n.

References

— Goldman-Iwahori [GI63].
— [BT1], 10.2, *Note ajoutée sur épreuves.*
— [Tits], 2.9
— [BT3]: this is the definitive treatment.
— Gan-Yu, the G_2 article [GY03].

Motivation For a real vector space V,

$$\mathscr{S}(\mathrm{GL}(V)) \textit{ is the space of Euclidean norms on } V.$$

Notation Let K be a non-archimedean local field, and V a finite-dimensional vector space over K.

Definition 2.1.3.1 A *(classical) norm* on V is a function $||-||_\alpha : V \longrightarrow \mathbb{R}_{\geq 0}$ satisfying:

- $||x+y||_\alpha \leq \max\{||x||_\alpha, ||y||_\alpha\}$, for all $x, y \in V$;
- $||\lambda x||_\alpha = |\lambda| \cdot ||x||_\alpha$, for $\lambda \in K$ and $x \in V$;
- $||x||_\alpha = 0$ if and only if $x = 0$.

Definition 2.1.3.2 An *(additive) norm* on V is a function $\alpha : V \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfying:

- $\alpha(x+y) \geq \min\{\alpha(x), \alpha(y)\}$, for all $x, y \in V$;
- $\alpha(\lambda x) = \operatorname{ord}(\lambda) + \alpha(x)$, for $\lambda \in K$ and $x \in V$;
- $\alpha(x) = \infty$ if and only if $x = 0$.

Clearly, $||-||_\alpha$ is a classical norm $\iff \alpha(-) = \log||-||_\alpha$ is an additive norm. From now on, *"norm" means "additive norm"*.

Definition 2.1.3.3 The Bruhat-Tits building of $\mathrm{GL}(V)$, $\mathscr{B} := \mathscr{B}(\mathrm{GL}(V))$, is the set of all norms on V.

We let $\mathrm{GL}(V)$ act on $\mathscr{B}$ by

$$g.\alpha = \alpha \circ g^{-1} : v \mapsto \alpha(g^{-1}.v).$$

It is clear that $g.\alpha$ is also a norm on V.

Later, we will put a metric on $\mathscr{B}$ so that $\mathscr{B}$ becomes a topological space.

Example 2.1.3.4 (i) Say $V = K.v$ is 1-dimensional. Pick $c \in \mathbb{R}$ and put $\alpha(\lambda.v) = \mathrm{ord}(\lambda) + c$. Then α is a norm on V.

Clearly, all norms on $V = K.v$ is of this form.

(ii) If $V = V_1 \oplus V_2$, and α_i is a norm on V_i. Put $\alpha(v_1 \oplus v_2) = \min\{\alpha_1(v_1), \alpha_2(v_2)\}$, then α is a norm on V.

(iii) Pick a basis $v_1, \ldots, v_n$ of V, and real numbers $c_1, \ldots, c_n$. Put

$$\alpha\Big(\sum \lambda_i v_i\Big) = \min\Big\{\mathrm{ord}(\lambda_i) + c_i\Big\}.$$

Then α is a norm on V.

Definition 2.1.3.5 We say that $\{v_1, \ldots, v_n\}$ is a *splitting basis* for α if (iii) holds.

Fact 2.1.3.6 Any norm on V admits a splitting basis.

Compare. Any quadratic form on a real vector space can be diagonalized.

Fact 2.1.3.7 Any two norms on V have a common splitting basis.

Compare. Any two positive definite quadratic forms on a real vector space can be diagonalized simultaneously.

We now interprete Fact 2.1.3.7 in a special case. In the description of a splitting norm, if $c_i = 0$ for all i, then

$$\alpha(v) = \max\{m : v \in \pi^m L\},$$

where L is the lattice $\mathscr{O}\langle v_1, \ldots, v_n\rangle$, $\mathscr{O} = \mathscr{O}_K$, π is a prime in $\mathscr{O}$.

Such a norm is a called a *hyperspecial norm*, or a *hyperspecial point* on $\mathscr{B}$. Now Fact 2.1.3.7 for hyperspecial points means the following:

> Let L, M be lattices in V. Then there is an $\mathscr{O}$-basis $\{v_1, \ldots, v_n\}$ of L and integers $e_1 \leq \cdots \leq e_n$ such that $\pi^{e_1}v_1, \ldots, \pi^{e_n}v_n$ is a basis of M.

This is the fundamental theorem of finitely generated modules over a PID/DVR (theorem of elementary divisors). It is also equivalent to the *Cartan decomposition for* $\mathrm{GL}_n(K)$:

$$\mathrm{GL}_n(K) = \mathrm{GL}_n(\mathscr{O}).A.\,\mathrm{GL}_n(\mathscr{O}),$$

where A consists of diagonal matrices $\mathrm{diag}(\pi^{e_1}, \ldots, \pi^{e_n})$ with $e_1 \leq \cdots \leq e_n$.

The metric

We now use Fact 2.1.3.7 to put two metrics on $\mathscr{B}$ as follows: if α, β are two norms with common splitting basis $\{v_1, \ldots, v_n\}$, we put

$$d(\alpha, \beta) = \left(\sum_i \Big(\alpha(v_i) - \beta(v_i) \Big)^2 \right)^{1/2},$$
$$d'(\alpha, \beta) = \max\big\{ |\alpha(v_i) - \beta(v_i)| : i \big\}.$$

Challenge. Show that d is well-defined (independent of the splitting basis; I don't know any simple proof of this). Show that d and d' satisfy the triangle inequality. It is easy to show: d' is well-defined, and if d is also well-defined, d and d' define the same topology on $\mathscr{B}$.

Observation 2.1.3.8 If $\{v_1, \ldots, v_n\}$ is a basis of V, $c \in \mathbb{R}^n$, and α_c is the norm $\alpha_c(v_i) = c_i$. Then $c \mapsto \alpha_c$ is an isometry $\mathbb{R}^n \to \mathscr{B}$ for the metric d.

Definition 2.1.3.9 The image of this isometry is called the *apartment* associated to the split torus S for which $\{v_1, \ldots, v_n\}$ is an eigen-basis for all $g \in S(K)$. We denote this apartment by $A(S)$.

Interpretation of Fact 2.1.3.6. Every point of $\mathscr{B}$ lies on an apartment.

Interpretation of Fact 2.1.3.7. Every two points lie on a common apartment.

Observation. $S \mapsto A(S)$ is a bijection.

Observation 2.1.3.10 The action of $S(K)$ on $\mathscr{B}$ stabilizes $A(S)$. But it doesn't act on $A(S)$ transitively. For any $x \in A(S)$, $S(K).x$ looks like a set of lattice points on the Euclidean space $A(S)$.

Observation 2.1.3.11 $A(S)$ is isometric to a Euclidean space. But there is no natural base point. $A(S)$ is an affine space, **not** a vector space. The space of translations can be identified with $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$.

Observation 2.1.3.12 Let N be the normalizer of S. Then the elements of $N(K)$, represented by matrices with respect to the basis $\{v_1, \ldots, v_n\}$ are simply the group of monomial matrices. The quotient $N(K)/S(K)$ is the Weyl group of (G, S). The action of $N(K)$ on $\mathscr{B}$ also stabilizes $A(S)$, and there the action is given by affine transformations.

Simplicial structure The topological space $\mathscr{B}$ carry a canonical poly-simplicial structure. For simplicity I want to get rid of the prefix "poly". So I define $\alpha \sim \beta$ if $\alpha(v) = \beta(v) + c$ for some $c \in \mathbb{R}$ and put $\mathscr{B}_{\text{red}} = \mathscr{B}/\sim$. This is the *reduced building* of $\mathrm{GL}(V)$, or the building of $\mathrm{PGL}(V)$ or $\mathrm{SL}(V)$. Now $\mathscr{B}_{\text{red}}$ is truely a simplicial complex, as I will describe presently (*Remark.* A poly-simplex is a direct product of simplices, and a poly-simplicial structure on a topological space is a division into poly-simplices).

The vertices are the hyperspecial points, which now correspond to lattices modulo the equivalence $L \sim \pi^n L$ for all $n \in \mathbb{Z}$. So they are really *lattice classes.*

A set of $k+1$ vertices form a k-simplex if and only if there are lattices $L_0, \ldots, L_k$ representing the corresponding lattice classes, such that

$$L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_k \supsetneq \pi L_0.$$

Now we have seen a few essential feautures of the Bruhat-Tits building associated to a p-adic reductive group, namely

- $\mathscr{B}(G)$ is a complete metric space (with an affine structure);
- $\mathscr{B}(G)$ is a (poly-)simplicial complex;
- $G(K)$ acts isometrically on $\mathscr{B}(G)$ by (poly-)simplicial automorphisms;
- $\mathscr{B}(G)$ has a collection of distinguished subsets, known as apartments, which are indexed by the maximal split tori of G. Each apartment is isometric to a Euclidean space.

Remark 2.1.3.13 If you are familiar with the work of Bushnell-Kutzko, you would recognize a simplex on the Bruhat-Tits building of $\mathrm{SL}(V)$ as *a lattice chain.* It is possible to rephrase the notion of "norm" in a lattice-theoretical language. Bruhat-Tits did this and called the notion "a graded lattice chain". The same notion has been rediscovered again and again. For example, they correspond to "filtrations" in Moy-Prasad theory, and "lattice sequences" in Bushnell-Kutzko's work on semisimple types of GL_n.

Therefore, we review here the translation between the language of norms, and that of filtration/lattice sequences/graded lattice chains. We will adopt the terminology of "filtration".

Definition 2.1.3.14 A *filtration* on V is a decreasing family of lattices $\{L_r\}_{r\in\mathbb{R}}$ in V, indexed by real numbers r such that $L_{r+1} = \pi L_r$.

Observation 2.1.3.15 There is a bijection

$$\{\text{norms on } V\} \leftrightarrow \{\text{filtrations on } V\}.$$

The construction is: $\alpha \mapsto \{L_r\}_{r\in\mathbb{R}}$ such that $L_r = \{v \in V : \alpha(x) \geq r\}$.

2.2 Lecture 2

2.2.1 Characterization of the apartment.

We first proceed with the construction of the space A; the set Φ_a and the X_α's will be defined in §§1.6 and 1.4. The relations (5) show us the way. The group $X^*(Z)$ of K-rational characters of Z can be identified with a subgroup of finite index of X^*. Let $\nu\colon Z(K) \to V$ be the homomorphism defined by

$$\chi(\nu(z)) = -\ \omega(\chi(z)) \quad \text{for } z \in Z(K) \text{ and } \chi \in X^*(Z), \tag{1}$$

and let Z_c denote the kernel of ν. Then, $\Lambda = Z(K)/Z_c$ is a free abelian group of rank $\dim S = \dim V$. The quotient $\tilde{W} = N(K)/Z_c$ is an extension of the finite group ${}^{v}\tilde{W}$ by Λ. Therefore, there is an affine space A $(= A(G, S, K))$ under V and an extension of ν to a homomorphism, which we shall also denote by ν, of N in the group of affine transformations of A. If G is semisimple, the system (A, ν) is canonical, that is, unique up to unique isomorphism. Otherwise, it is only unique up to isomorphism, but one can, following G. Rousseau **[19]**, "canonify" it as follows: calling $\mathscr{D}G^\circ$ the derived group of G° and S_1 the maximal split torus of the center of G°, one takes for A the direct product of $A(\mathscr{D}G^\circ, G^\circ \cap S, K)$ (which is canonical) and $X_*(S_1) \otimes \boldsymbol{R}$ .The affine space A is called the *apartment of* S (relative to G and K). The group $N(K)$ operates on A through $\tilde{W}$.

This is one of the questions I was asked most often regarding Tits' article. What does this paragraph mean?[2] It is not difficult, but not entirely obvious either. The detail can be found in Landvogt's book.

Explanation The principle is this: suppose that you have an extension of groups

$$1 \to \Lambda \to N \to W \to 1,$$

where Λ is a free abelian group of finite rank, normal in N, and W is finite. Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then we can pull out the above diagram with $\Lambda \hookrightarrow V$ and get

$$1 \to V \to N' \to W \to 1.$$

If you represent the first extension by a class in $H^2(W, \Lambda)$, the second extension is represented by the image of that class in $H^2(W, V)$. But $H^2(W, V) = 0$, so the second extension is trivial.

We notice that the action map $a : W \to \mathrm{Aut}_{\mathrm{group}}(V)$ is induced from $W \to \mathrm{Aut}_{\mathrm{group}}(\Lambda) = \mathrm{GL}_{\mathbb{Z}}(\Lambda)$, hence factors through $\mathrm{GL}_{\mathbb{R}}(V)$. Thus it is obvious that there exists a pair (A, f) such that A affine space under V (*i.e.*, a principal homogeneous space of V), $f : N \to \mathrm{Aut}_{\mathrm{affine}}(A)$ such that $f(\lambda)$ is translation by λ for all $\lambda \in \Lambda$, and $d\big(f(n)\big) = a(\bar{n})$ for all $n \in N$, where $\bar{n}$ is the image of n in W.

Now assume that $V^W = 0$. Suppose that both the pairs (A, f) and (A', f') have the above property. We claim that there is a unique isomorphism $(A, f) \simeq (A', f')$. Indeed, a simple analysis shows that the obstruction to the existence of such an isomorphism lies in $H^1(W, V) = 0$, and the obstruction to its uniqueness lies in $V^W = H^0(W, V) = 0$.

This, when applied to the context in [Tits, 1.2], shows that for semisimple G, $A(G, S, K)$ is uniquely characterized as an affine space (up to a unique isomorphism). For reductive G, the reduced apartment has the same uniqueness.

2.2.2 Extended building versus reduced building. [Tits] deals exclusively with the extended building (a.k.a. enlarged building, reductive building). But throughout [BT], "building" usually mean the reduced building (a.k.a. semisimple building).

We recall that the reduced building is really canonically defined. The extended building is canonical in the sense that we can "canonify" the definition. But this is somewhat artificial and hence the behavior is not ideal. The main reason to favor the extended building is this: *when the center of G is of split rank > 0, the stabilizer of points on $\mathscr{B}_{\mathrm{red}}$ is no longer a compact subgroup of $G(K)$.*

We now recall that the construction of $\mathscr{B}_{\mathrm{ext}}$ from $\mathscr{B}_{\mathrm{red}}$. It is completely analogous to going from $\mathscr{S}_{\mathrm{red}}$ to $\mathscr{S} = \mathscr{S}_{\mathrm{ext}}$. We put

$$G(K)^1 = \{g \in G(K) : \operatorname{ord}\chi(g) = 0 \quad \forall \chi \in \mathrm{Hom}_K(G, \mathbb{G}_{\mathrm{m}})\}.$$

Then $G(K)/G(K)^1$ is a finite generated abelian group, and there is an isomorphism

$$\Big(G(K)/G(K)^1\Big) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R},$$

where Z is the center of G. We let $G(K)$ acts on the vector space $X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ by translations via the above isomorphism, and define $\mathscr{B}_{\mathrm{ext}}$ as the product of two $G(K)$-sets: $\mathscr{B}_{\mathrm{ext}} = \mathscr{B}_{\mathrm{red}} \times (X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R})$.

There is another way of dealing with this, and this viewpoint is also often used in [BT]: *instead of using $G(K)$ and $\mathscr{B}_{\mathrm{ext}}$, we use $G(K)^1$ and $\mathscr{B}_{\mathrm{red}}$.*

[2] Reprinted from [Tit79], with permission.

In fact, since most results in [BT] are stated using $\mathscr{B}_{\rm red}$, it is often necessary to do the above. Notice that this means that the results in [BT] often actually apply to $G(K)^1$ instead of $G(K)$. Therefore, one gets decomposition theorems *etc.* for $G(K)^1$. Then one does a little bit more work to get the results for $G(K)$.

2.2.3 The maximal bounded subgroups. I want to bring attention to the fact that "maximal bounded subgroups", "stabilizer of vertices", and "maximal parahoric subgroups" are all different concepts. Although Tits' article is rather clear about this, this is still a common misconception (probably because people tend to extrapolate the easier case of BN-pairs). A few examples should impress you about this. For simplicity, assume that K is a locally compact non-archimedean field.

It is a consequence of Bruhat-Tits fixed-point theorem that every maximal compact subgroup of $G(K)$ is the stabilizer of some point on the building of G. But not all such stabilizers are maximal compact subgroups.

If x is a vertex (for the canonical polysimplicial structure on $\mathscr{B}$), then the stabilizer $G(K)^x$ of x is a maximal compact subgroup. If G is semi-simple and simply connected, this gives precisely all the maximal compact subgroup.

This fails for non-simply connected groups. A simple and instructive example is $G = \mathrm{PGL}_2(K)$. Recall that the lattice classes represented by the lattices $L = \mathscr{O}.e_1 + \mathscr{O}.e_2$, $M = \mathscr{O}.e_1 + \pi\mathscr{O}.e_2$ corresponds to vertices x, y on the building which are connected by a 1-simplex $\overline{xy}$. For most point z on the interior of $\overline{xy}$, $G(K)^z$ is $G(K)^x \cap G(K)^y = \left\{\begin{pmatrix} a & \pi b \\ c & d\end{pmatrix}\right\}$, and is equal to the Iwahori subgroup $\mathscr{I}$ associated to $\overline{xy}$.

But there is one exception: when z is the midpoint of $\overline{xy}$, $G(K)^z \smallsetminus \mathscr{I}$ contains an extra element which is represented by $\begin{pmatrix} 0 & 1 \\ \pi & 0\end{pmatrix}$. Notice that this switches x and y and hence fixes z. In this case, $[G(K)^z : \mathscr{I}] = 2$ and $G(K)^z$ is another maximal compact subgroup.

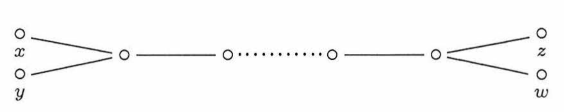
Here is another example.

Example 2.2.3.1 Let $G = \mathrm{SO}(2n)$, $n \geq 5$. For simplicity let us assume G split. Write $G = \mathrm{SO}(V)$, and let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a Witt basis of V (so the quadratic form is $q(\sum x_i e_i + \sum y_i f_i) = \sum x_i y_i$).

For $0 \leq i \leq n$, let L_i be the lattice with basis

$$\pi e_1, \ldots, \pi e_i, e_{i+1}, \ldots, e_n, f_1, \ldots, f_n.$$

According to Bruhat's Boulder conference article [Bru66], each $G(K) \cap \mathrm{GL}(L_i)$ is a maximal compact subgroup of $G(K)$, and there are exactly $n+1$ conjugacy classes of maximal compact subgroups. There are $n+1$ vertices on a chamber. Are the maximal compacts simply the stabilizer of vertices? This seems reasonable. But if you look at the local Dynkin diagram (whose vertices correspond to vertices of a chamber) of this group, you may be puzzled: how can we assign these lattices to the vertices on the local Dynkin diagram reasonably?



The answer is: no, there are two maximal compacts (up to conjugacy) which are **not** stabilizer of vertices. There are 4 "terminal vertices" on the local Dynkin diagram. Let's call them x, y, z, w so that x, y is on one end and z, w on the other end. Then the stabilizer of x is $G(K)$-conjugate to that of y, and the stabilizer of z is $G(K)$-conjugate to that of w. Therefore, the stabilizers of vertices only give $n-1$ conjugacy classes of maximal compacts. However, the stabilizer of the midpoint of $\overline{xy}$ (resp. $\overline{zw}$) is also a maximal compact. This accounts for the $n+1$ maximal compacts.

2.2.4 The parahoric subgroups. It is quite interesting to notice what Tits said about Iwahori subgroups and parahoric subgroups in his summary.[3]

We recall that if G is semisimple and simply connected, the group $\tilde{W} = N(K)/Z(K)$ coincides with the Weyl group W of the affine root system Φ_{af}. As before, we set $\mathscr{B} = \mathscr{B}(G, K)$.

3.1.1. *Suppose that* $\tilde{W} = W$. *Then* $G(K)^F = G(K)^x$ *for every facet* F *of* $\mathscr{B}$ *and every* $x \in F$. *Furthermore, if* C *is a chamber of* $A = A(G, S)$, *the pair* $(G(K)^C, N(K))$ *is a BN-pair (or Tits system: cf.* [5], [23]) *in* $G(K)$ *with Weyl group* W. In that case, the groups $G(K)^x$ for $x \in \mathscr{B}$ are called the *parahoric subgroups* of $G(K)$ (cf. [8]), but we shall avoid using that terminology here in order not to prejudge of its most suitable extension to the nonsimply connected case. An alternative construction of the building $\mathscr{B}$ starting from the above BN-pair (which can be defined independently of the building, as we shall see) and using the parahoric subgroups defined by means of that BN-pair is given in [8, §2].

3.3. *Various decompositions.* Let C be a chamber of $A = A(G, S)$. We identify A with the vector space V in such a way that 0 becomes a special point contained in the closure of C; in particular, $G(K)^0$ *is a special subgroup* of $G(K)$. Set $D = \mathbf{R}_+^* \cdot C$ (a "vector chamber") and $B = G(K)^C$; if $\bar{K}$ is finite or, more generally, if G is residually quasi-split, and if G is simply connected, B is an *Iwahori subgroup* of $G(K)$ (cf. §3.7). Let U^+ be the group generated by all U_a for which $a|_C$—and hence $a|_D$—is positive and let Y be the "intersection of V and $\tilde{W}$", that is, the group of all translations of A contained in $\tilde{W}$; thus, Y is the image of $Z(K)$ by the homomorphism ν of §1.2. Set $Y_+ = Y \cap \bar{D}$ (closure of D) and $Z(K)_+ = \nu^{-1}(Y_+)$, a subsemigroup of $Z(K)$.

3.3.1. *Bruhat decomposition.* One has $G(K) = BN(K)B$ and the mapping $BnB \mapsto \nu(n)$ $(n \in N(K))$ is a bijection of the set $\{BgB \mid g \in G(K)\}$ onto $\tilde{W}$.

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[3]Reprinted from [Tit79], with permission.

3.7. *Iwahori subgroups; volume of maximal compact subgroups.* In this section, we suppose G residually quasi-split; remember that that is no assumption if the residue field $\bar{K}$ is finite (1.10.3).

To every chamber C of the building $\mathscr{B}$, we associate as follows a subgroup $\mathrm{Iw}(C)$ of $G(K)$, called the *Iwahori subgroup* corresponding to C: if $\bar{G}_C^\circ$ denotes the neutral component of the algebraic group $\bar{G}_C$ (cf. 3.4), $\mathrm{Iw}(C)$ is the inverse image in $\mathscr{G}_C(\mathfrak{o}) = G(K)^C$ of the group $\bar{G}_C^\circ(\bar{K})$ under the reduction homomorphism $\mathscr{G}_C(\mathfrak{o}) \to \bar{G}_C(\bar{K})$. Clearly, *all Iwahori subgroups of* $G(K)$ *are conjugate.* From 3.5.2, it follows that $\bar{G}_C^\circ$ is a solvable group, hence is the semidirect product of a torus $\bar{T}$ by a uni-

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Tits was indeed very careful in stating everything precisely. But still, in the literature people sometimes quote these results incorrectly.

We saw that Tits was undecided about what parahoric subgroups should be (though he was firm about the case of Iwahori subgroups). Today there should be no ambiguity. In [BT2], Bruhat-Tits defined parahoric subgroups in the same way Tits defined the Iwahori subgroups. Namely, the definition has to involve the smooth group schemes constructed by Bruhat-Tits and their neutral components.

Therefore, the parahoric subgroups are **not** the stabilizer/fixer of a facet on the building. However, we caution you that in the literature, some people use conventions inconsistent with Bruhat-Tits.

Bruhat-Tits' choice has several advantages:

— There is a bijection between facets and parahoric subgroups, $F \mapsto G(K)_F$. The bijection is order reversing: $G(K)_F \subset G(K)_{F'} \iff F' \subset \overline{F}$. [This is just like the case of a BN-pair].

— The parahoric subgroups contained in a fixed parahoric subgroups $G(K)_F$ are in bijection with the parabolic subgroups of $\mathscr{G}$ mod π, where $\mathscr{G}$ is the neutral component of the smooth group scheme associated to $G(K)_F$. In other words, they are in bijection with facets on the associated spherical building.

— The smooth group scheme $\mathscr{G}$ associated to a parahoric subgroup is connected by definition. We can apply representation theory of $\mathscr{G}(\mathscr{O}/\pi)$. Most theories (*e.g.*, work of Lusztig) do require $\mathscr{G}$ mod π to be connected.

Caution. Be careful about the following statement regarding the disconnection of stabilizers.[4] The proof cannot be found in [BT] and the statement is incorrect.

If G is simply connected, the group $\bar{G}_\Omega$ is connected. In general, the group of components of $\bar{G}_\Omega$ is easily computed when one knows the group $\Xi_1 = \Xi(G, K_1)$ (cf. §2.5), where K_1 is the maximal unramified extension of K. Here, we shall give the result only in the case of a facet.

3.5.3. *The group of components of* $\bar{G}_F$ *is canonically isomorphic with the intersection of the stabilizers of the orbits* $O(v)$ *with* $v \in I_F$ *in the group* Ξ_1. *A component is defined over* $\bar{K}$ *if and only if the corresponding element of* Ξ_1 *is centralized by* $\mathrm{Gal}(K_1/K)$. *If* $\bar{K}$ *is finite, every component of* $\bar{G}_F$ *which is defined over* $\bar{K}$ *has a* $\bar{K}$-*rational point (by Lang's theorem).*

[4]Reprinted from [Tit79], with permission.

Example 2.2.4.1 If $G = T$ is a torus and K is strictly henselian, and x any point on the building, then a result of Kottwitz in [Kot97] (which was made explicit by Rapoport[5]) says that the component group is the torsion subgroup of the group of co-invariants $X_*(T)_{\mathrm{Gal}(\bar{K}/K)}$.

Remark 2.2.4.2 E. Kushnirsky has found a formula for computing the order of the component group [Kus04]. A more recent preprint (*On parahoric subgroups*, 2008) by Haines and Rapoport [HR08] gives more information on the component groups.

2.2.5 Valuation of root datum. A basic feature of Bruhat-Tits theory is that if U_a is a root subgroup, then $U_a(K)$ is filtred. For example, for split groups, $U_a(K)$ is (isomorphic to) the additive group of K, and is filtred by the subgroups $\pi^n \mathcal{O}_K$.

For Bruhat-Tits theory, it is important that you index the filtrations on $U_a(K)$, for varying a's, in a coherent way. In fact, this point is almost the whole theory. But the languages of doing this in [Tits] and [BT] are different.

- In [Tits], $U_a(K)$ is filtred by the groups X_α, where α varies over affine functions on A with vector part a. The set of such functions form a real line, of course.
- In [BT], $U_a(K)$ is filtred by $U_a(K)_{\varphi,r}$, a family of groups indexed by the real line $\mathbb{R}$ directly [the notation φ is there to indicate the choice of a "valuation of root datum"].

The approach of [Tits] is elegant, since it does not depend on any anxiliary choice. In contrast, the filtration in [BT] depends on the choice of φ, "the valuation of root datum". The precise definition is somewhat long, and it is difficult to show that such an object exists. In fact, a very large part of [BT] is devoted to its existence.

Once the existence is known, the whole theory of Bruhat-Tits buildings can be developed. In fact, it then follows that to give a valuation of root datum is to give a point on the building. Therefore, the notion of valuation of root datum is indeed very natural, and it can be considered as a remote analog of Cartan involution.

In [Tits], this filtration of $U_a(K)$ indexed by affine functions is always well-defined. What is not clear is that this filtration has many wonderful properties. If you read [BT], you don't see any attempt to prove these properties from the definition in [Tits]. Instead, all the effort is putting into proving the existence of a valuation of root datum. It will follow, however (since there is a translation between the two languages), that all the nice properties are valid for the groups X_α.

In order to access the results in [BT] not summarized in [Tits], one needs to know the translation, which we give now.

First, today "root datum" usually means the gadget (consisting of a dual pair of lattices and a dual pair of root systems) classifying a split reductive group, as defined in SGA3 (see also, Springer's Corvallis article [Spr79]).

[5]Rapoport originally wrote a short note about this which was not published. Now this result is incorporated into his article with Haines [HR08].

However, in [BT], "root datum" is a gadget in abstract group theory. For us, the "root datum" for a reductive group over K is the datum $(\{U_a(K)\}_{a\in\Phi(G,S)}, Z(K))$, where S is a maximal K-split torus in G, U_a the root subgroup, and $Z = Z_G(S)$.

Then, the set of valuations on the root datum $(\{U_a(K)\}_{a\in\Phi(G,S)}, Z(K))$ is in bijection with points on $A_{\text{red}} = A_{\text{red}}(G, S, K)$. Indeed, by fixing $x \in A_{\text{red}}$, we can put a filtration $\{U_a(K)_{x,r}\}_{r\in\mathbb{R}}$ on $U_a(K)$ by setting

$$U_a(K)_{x,r} = \bigcup_{\substack{d\alpha=a\\ \alpha(x)\geq r}} X_\alpha.$$

Conversely, if these filtration groups are known, then we can recover the group X_α's by

$$X_\alpha = U_a(K)_{x,\alpha(x)}.$$

Notice that I have adopted the convention that a point on the building *is* a valuation of a root datum. This completes the translation.

2.2.6 Figuring out the filtration. As we have mentioned, [Tits, 1.4][6] defined the filtration $\{X_\alpha\}_\alpha$ on $U_a(K)$ by an elegant recipe. But it is quite hard to prove the good properties of this filtration ([Tits, 1.5]).

1.4. *Filtration of the groups $U_a(K)$.* Let $a \in \Phi$ and $u \in U_a(K) - \{1\}$. It is known (cf. [3, §5]) that the intersection $U_{-a}uU_{-a} \cap N$ consists of a single element $m(u)$ whose image in ${}^v\tilde{W}$ is the reflection r_a associated with a, from which follows that $r(u) = \nu(m(u))$ is an affine reflection whose vector part is r_a. Let $\alpha(a, u)$ denote the affine function on A whose vector part is a and whose vanishing hyperplane is the fixed point set of $r(u)$ and let Φ' be the set of all affine functions whose vector part belongs to Φ. For $\alpha \in \Phi'$, we set $X_\alpha = \{u \in U_a(K) \mid u = 1 \text{ or } \alpha(a, u) \geqq \alpha\}$. The following results are fundamental.

1.4.1. *For every α as above, X_α is a group.*

1.4.2. *If $\alpha, \beta \in \Phi'$, the commutator group (X_α, X_β) is contained in the group generated by all $X_{p\alpha+q\beta}$ for $p, q \in N^*$ and $p\alpha + q\beta \in \Phi'$.*

[Tits] also provided several examples of computing the filtration X_α. However, in my experience this is not a very pleasant drill, in particular for the exceptional groups.

Therefore, in practice, to figure out the filtration, it is easier to follow the strategy in [BT], which is to go through two descent steps: descending from the split case to the quasi-split case, and descending from the quasi-split case through an unramified extension.

I do not have the time to go through either steps. So I will just recast the starting point: the split case (which was already emphasized by [Tits] in the beginning, Section 1.1) in the language of valuation of root datum. You will see that it is very simple. The quasi-split case is also fairly simple, and can be found in either [BT2], Landvogt's book [Lan96], or Prasad-Raghunathan's article [PR88] on central extensions.

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[6] Reprinted from [Tit79], with permission.

So let $\mathscr{G}$ be a Chevalley scheme over $\mathscr{O}$. This means that $\mathscr{G}$ is a smooth group scheme over $\mathscr{O}$, such that both the generic fiber G and the special fiber are split reductive groups.

Then $\mathscr{G}$ has a maximal split torus $\mathscr{S}$ over $\mathscr{O}$, with respect to which there is a root decomposition of $\mathrm{Lie}\,\mathscr{G}$, and root subgroups $\mathscr{U}_a$, $a \in \Phi(G,S)$, $S = \mathscr{S}_K$. There are isomorphisms $x_a : \mathbb{G}_\mathrm{a} \to \mathscr{U}_a$ over $\mathscr{O}$. Then we put

$$U_a(K)_{\varphi,r} = (x_a \otimes_{\mathscr{O}} K)(\{\lambda \in K : \mathrm{ord}(\lambda) \geq r\}).$$

It can be shown that this is indeed a valuation of root datum, and hence determines a point on $A(G,S,K) \subset \mathscr{B}(G,K)$. This point, of course, is the hyperspecial point whose stabilizer is $\mathscr{G}(\mathscr{O}_K)$.

2.3 Lecture 3

2.3.1 Notation for the third lecture.

- K: non-archimedean local field
- $\tilde{K}$: the maximal unramified extension of K
- $\mathscr{O} = \mathscr{O}_K$, $\tilde{\mathscr{O}} = \mathscr{O}_{\tilde{K}}$, π a uniformizer of K, $\kappa = \mathscr{O}/\pi$, $\bar{\kappa} = \tilde{\mathscr{O}}/\pi$
- G: connected reductive group over K
- S: maximal K-split torus in G
- $\Phi = \Phi(G,S)$: relative root system
- Φ^{aff}: relative affine root system
- $\{U_a\}_{a\in\Phi}$: root subgroups
- $Z = Z_G(S)$: centralizer of S, also denoted by U_0

Theorem 2.3.1.1 (A theorem of Steinberg) *$G \otimes_K \tilde{K}$ is quasi-split. Therefore, if T is a maximal $\tilde{K}$-split torus in $G \otimes_K \tilde{K}$, then $Z_G(T)$ is a torus.*

Theorem 2.3.1.2 (Key result of Bruhat-Tits' "Étale Descent") *There exists a torus T over K, containing S, such that $T \otimes_K \tilde{K}$ is a maximal $\tilde{K}$-split torus.*

Consequence. Bruhat-Tits theory behaves very well respect to unramified extensions. For example, $\mathscr{B}(G \otimes_K \tilde{K})^\Gamma = \mathscr{B}(G)$, where $\Gamma = \mathrm{Gal}(\tilde{K}/K) \simeq \mathrm{Gal}(\bar{\kappa}/\kappa)$.

Convention. Today I will always assume: G is quasi-split over K and in addition, we can take $S = T$. This is for simplicity of exposition. The extension to the general case is fairly easy.

2.3.2 Background about integral models.

Remark 2.3.2.1 For the discussion here, G can be any linear algebraic group over K.

Definition 2.3.2.2 A *(integral) model* of G is an *affine* group scheme $\mathscr{G}$ (of finite type) over $\mathscr{O}$ with generic fiber

$$\mathscr{G}_K := \mathscr{G} \otimes_{\mathscr{O}} K = G.$$

Concretely, you try to rewrite the defining equations and the polynomials defining multiplication and inverse using polynomials over $\mathscr{O}$. A model is just such a system of $\mathscr{O}$-equations.

Remark 2.3.2.3 A model carries more structure than the original algebraic groups. Examples:

- $\mathscr{G}(\mathscr{O})$: *groups of integral points*, *i.e.*, solutions to the equations with coordinates in $\mathscr{O}$;
- $\mathscr{G}_\kappa$: *special fiber of* $\mathscr{G}$ or *reduction of* $\mathscr{G}$: take the defining equations/polynomials mod π, we get an "algebraic group" over κ (more properly: a group scheme over κ, which is not necessarily smooth).
- $\mathrm{Lie}\,\mathscr{G}$: *Lie algebra of* $\mathscr{G}$, a lattice in $\mathrm{Lie}\,G$, closed under Lie bracket.

For simplicity, assume that $K = \tilde{K}$ for a while.

One may consider all subgroups of $G(K)$ of the form $\mathscr{G}(\mathscr{O})$ privileged. Such a subgroup is quite special, *e.g.*, it is open and bounded (and more). Here, we call such a subgroup *schematic*.

If H is schematic subgroup of $G(K)$, there are many $\mathscr{G}$'s such that $\mathscr{G}(\mathscr{O}) = H$. But one of them is *canonical*, and is characterized by either

- it is the initial object in the category of models $\mathscr{G}$ such that $\mathscr{G}(\mathscr{O}) = H$; or
- it is smooth over $\mathscr{O}$.

This follows from the theory of *group smoothening* (due to Néron, Raynaud, etc.). This canonical model will be denoted by $\mathscr{G}^H$.

Remark 2.3.2.4 Bruhat-Tits showed that many groups occuring in Bruhat-Tits theory are schematic.

This theory attached to schematic subgroups H extra structures which are quite non-trivial, *e.g.*,

- the *congruence subgroups*
$$\Gamma(\pi^n; \mathscr{G}^H) = \left\{ g \in \mathscr{G}^H(\mathscr{O}) \;\middle|\; g \bmod \pi^n = 1 \text{ in } \mathscr{G}^H(\mathscr{O}/\pi^n\mathscr{O}) \right\},$$
- the special fiber of $\mathscr{G}^H$,
- the Lie algebra of $\mathscr{G}^H$.

Remark 2.3.2.5 If $K \neq \tilde{K}$, and H is a schematic subgroup of $G(\tilde{K})$ which is stable under $\mathrm{Gal}(\tilde{K}/K)$, then $\mathscr{G}^H$, which is defined over $\tilde{\mathscr{O}}$ a priori, is actually defiend over $\mathscr{O}$.

2.3.3 The filtration on U_a revisited. We recall that $U_a(K)$ is filtred by the $\{X_\alpha\}_{d\alpha=a}$, or by the $\{U_a(K)_{x,r}\}_{r\in\mathbb{R}}$.

Now according to my convention $S = T$, the meaning of "U_a" doesn't change when you go from K to $\tilde{K}$. So we can also talk about $U_a(\tilde{K})_{x,r}$ etc. This group is $\mathrm{Gal}(\tilde{K}/K)$-stable.

Fact 2.3.3.1 The group $U_a(\tilde{K})_{x,r}$ is schematic. Let $\mathscr{U}_{a,x,r}$ be the associated smooth model over $\mathscr{O}$. Then $\mathscr{U}_{a,x,r}(\mathscr{O}) = U_a(K)_{x,r}$.

Fact 2.3.3.2 $\Gamma(\pi^n; \mathscr{U}_{a,x,r}) = U_a(K)_{x,r+n}$.

Fact 2.3.3.3 The special fiber of $\mathscr{U}_{a,x,r}$ is a product of additive groups $\mathbb{G}_\mathrm{a}$.

2.3.4 The filtration on $\mathrm{Lie}\,U_a$. It is natural to consider the Lie algebra of $\mathscr{U}_{a,x,r}$. Although this is already considered in [BT], Moy-Prasad seems to be the first to consider the whole family $\{\mathrm{Lie}\,\mathscr{U}_{a,x,r}\}_{r\in\mathbb{R}}$ together.

Fact 2.3.4.1 Let $\mathfrak{u}_{a,x,r} = \mathrm{Lie}\,\mathscr{U}_{a,x,r}$. Then the family $\{\mathfrak{u}_{a,x,r}\}$ is a filtration on $\mathrm{Lie}\,U_a$ in the sense we discussed in the first lecture (of the kind called "lattice sequence" by Bushnell-Kutzko; *i.e.*, $\mathfrak{u}_{a,x,r+1} = \pi\mathfrak{u}_{a,x,r}$).

Remark 2.3.4.2 The definition here is not the one given by Moy-Prasad. They made more *ad hoc* use of the fact that G is quasi-split and specific knowledge of the structure of U_a. There was no direct connection to $U_{a,x,r}$. It was just an analogy.

The definition here is more natural, and it shows that $\mathfrak{u}_{a,x,r}$ is attached to $\mathscr{U}_{a,x,r}$ or $U_a(\tilde{K})_{x,r}$ canonically. It also makes the following transparent:

Theorem 2.3.4.3 (Moy-Prasad isomorphism) *We have*

$$U_a(K)_{x,r}/U_a(K)_{x,r+} \simeq \mathfrak{u}_{a,x,r}/\mathfrak{u}_{a,x,r+},$$

where

$$U_a(K)_{x,r+} = \bigcup_{s>r} U_a(K)_{x,s},$$

etc.

2.3.5 The filtration on $U_0(K)$ and $\operatorname{Lie} U_0$. We recall that we apply the convention $U_0 = Z_G(S)$. Also notice that U_0 is the last part of Bruhat-Tits' "root datum" $\big((U_a)_{a\in\Phi}, U_0\big)$.

It is interesting to observe that from Chapter I of [BT], it is clear that Bruhat-Tits wanted to consider a filtration on $U_0(K)$. But somehow this was not done in Chapter II or any subsequent chapters. Ten years later, Moy-Prasad (Inventiones 1994, [MP94]) and Schneider-Stuhler (Publ. IHES 1997, [SS97]) supplied the (same) definition (Schneider-Stuhler acknowledged help from Tits).

Definition 2.3.5.1 Let Z be any torus over K. Let L/K be an extension over which Z is split. Define $Z(L)_0$ to be the maximal bounded subgroup of $Z(L)$ and for $r > 0$,

$$Z(L)_r := \{z \in Z(L) : \operatorname{ord}_K(\chi(z-1)) \geq r \ \forall \chi \in \operatorname{Hom}_L(Z, \mathbb{G}_m)\}.$$

Here $\operatorname{ord}_K : L^\times \to \mathbb{Q}$ is the valuation extending the normalized valuation $\operatorname{ord}_K : K^\times \twoheadrightarrow \mathbb{Z}$. We then define $Z(K)_0$ to be the Iwahori subgroup of $Z(K)$, and for $r > 0$,

$$Z(K)_r := Z(K)_0 \cap Z(L)_r.$$

Let $\mathfrak{z} = \operatorname{Lie} Z = \operatorname{Lie} U_0 = \mathfrak{u}_0$. Moy-Prasad also defined

$$\mathfrak{z}_r = \mathfrak{u}_{0,r} = \{x \in \mathfrak{z} : \operatorname{ord}_K(d\chi(x) \geq r \ \forall \chi \in \operatorname{Hom}_L(Z, \mathbb{G}_m)\}.$$

One can show that $\{Z(K)_r\}$ satisfies the conditions prescribed in Chapter I of [BT]. However, there are some problems with these definitions. The most serious one is that there is no Moy-Prasad isomorphism (similar to the one stated above, but with $a = 0$) in general. But the Moy-Prasad isomorphism is used in a fundamental way in the proof of Moy-Prasad theory.

We remark that the Moy-Prasad isomorphism is valid in many cases, for example when Z becomes an induced torus over a tamely ramified extension. Therefore, the first paper of Moy-Prasad is free from this issue since they deal with simply connected groups, and for such groups, Z is an induced torus.

There are now two ways to resolve the situation.

(i) DeBacker has given alternative arguments to the main results in Moy-Prasad theory, without using the Moy-Prasad isomorphism. But some features of the theory is lost [DeB02b].

(ii) I have given a different definition of $\{Z(K)_r\}$ and $\{\mathfrak{z}_r\}$. For this definition, the Moy-Prasad isomorphism is valid and hence the rest of the Moy-Prasad theory can be used without any modification [Yu03].

But the new definition involves a lot more algebraic geometry, and is more difficult to compute. I will not give it here. However, if Z becomes an induced torus over a tamely ramified extension, the two definitions agree.

For the rest of this lecture, I will use my definition, or assume that Z becomes an induced torus over a tamely ramified extension.

2.3.6 The Moy-Prasad filtration.

Definition 2.3.6.1 Let $x \in \mathscr{B}(G), r \geq 0$. The Moy-Prasad filtration group associated to (G, x, r) is the subgroup of $G(K)$ generated by $U_a(K)_{x,r}$ for all $a \in \Phi \cup \{0\}$, where we put $U_0(K)_r = Z(K)_r$.

Similarly, $\mathfrak{g}_{x,r}$ is the (direct) sum of $\mathfrak{u}_{a,x,r}$ for all $a \in \Phi \cup \{0\}$, where $\mathfrak{u}_{0,x,r} = \mathfrak{u}_{0,r} = \mathfrak{z}_r$.

Theorem 2.3.6.2 (The Moy-Prasad isomorphism) *For $r > 0$,*

$$G(K)_{x,r}/G(K)_{x,r+} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}.$$

Theorem 2.3.6.3 (Main theorem in Moy-Prasad theory) *Let (π, V) be an irreducible admissible representation of $G(K)$. The number*

$$\rho = \inf\{r \in G(K) : \exists x \in \mathscr{B}(G) \text{ s.t. } V^{G(K)_{x,r+}} \neq 0\}$$

is a rational number and the infimum can be achieved for some $x \in \mathscr{B}(G)$. It is called the depth *of π.*

Moreover, if $V^{G(K)_{x,\rho+}} \neq 0$, the irreducible constituents of this space as a representation of $G(K)_{x,r}/G(K)_{x,r+}$ enjoy a certain nice non-degeneracy characterization (which relies on the Moy-Prasad isomorphism).

The depth of a representation is preserved by the Jacquet functor and parabolic induction.

I would like to mention a nice interpretation of the depth.

Theorem 2.3.6.4 *Let $G = \mathrm{GL}_n$, or a tamely ramified torus. Let π be an irreducible smooth representation of G and $\phi : W'_K \to {}^LG$ the corresponding Langlands parameter. Then*

$$\mathrm{depth}(\pi) = \inf\{r : \phi(P^{r+}) = \mathrm{id}\}.$$

Here, $P \subset W_K \subset W'_K$ is the wild inertia group and the filtration is the upper number filtration by ramification groups.

The case of GL_n can be proved as follows. One first reduces to the case of a supercuspidal representation. In this case the depth can be related to conductoral exponent and the standard L-function of π by a paper of Bushnell [Bus87]. Since this L-function agrees with the Artin L-function of ϕ, and the depth of ϕ is related to its conductoral exponent and the Artin L-function in a similar manner, one can derive the result.

For the case of a tamely ramified torus, see my article on the local Langlands correspondence for tori in Chapter 7.

Remark 2.3.6.5 We suspect that this relation is valid fairly generally. Therefore, the depth should be preserved by Langlands functoriality.

For example, it should be preserved by the Generalized Jacquet-Langlands / Deligne-Kahzdan-Vigneras correspondence between representations of GL_n and those of division algebras. (See Theorem 3.1.2 or Theorem 4.5) Recently, Lansky and Raghuram [LR03] proved that it is so in some cases.

2.3.7 More group schemes.

Theorem 2.3.7.1 (Yu) *The groups $G(\tilde{K})_{x,r}$ are schematic. Let $\mathscr{G}_{x,r}$ be the associated smooth model over $\mathscr{O}$, then* $\mathrm{Lie}\,\mathscr{G}_{x,r} = \mathfrak{g}_{x,r}$. *Moreover,*

$$\Gamma(\pi^n, \mathscr{G}_{x,r}) = G(K)_{x,r+n}.$$

It is interesting to remark that the definition of Moy-Prasad filtrations and Schneider-Stuhler filtrations are rather involved. But the theory of smooth models now allow us to give a very short and conceptual description of the Schneider-Stuhler filtration $U_F^{(e)}$ associated to facet F:

> Let $\mathscr{U}_F$ be the canonical model for $\mathrm{Stab}(G,F)$, and let $\mathscr{U}_F^+$ be the dilatation on $\mathscr{U}_F$ of $R_{\mathrm{u}}\left((\mathscr{U}_F)_\kappa^\circ\right)$. Then $U_F^{(e)} = \Gamma(\pi^e; \mathscr{U}_F^+)$ for all $e \in \mathbb{Z}_{\geq 0}$.

2.3.8 An interpretation of the Moy-Prasad filtration.

Fact 2.3.8.1 The lattices $\{\mathfrak{g}_{x,r}\}_{r\in\mathbb{R}}$ form a filtration on the vector space $\mathfrak{g}$ corresponding to a norm on $\mathfrak{g}$, *i.e.*, $\mathfrak{g}_{x,r+n} = \pi^n \mathfrak{g}_{x,r}$.

Denote this norm on $\mathfrak{g}$ by α_x. Recall that we can regard α_x as a point on $\mathscr{B}(\mathrm{GL}(\mathfrak{g}))$. Therefore, the theory of Moy-Prasad filtrations for $\mathfrak{g}$ is a map

$$\iota : \mathscr{B}(G) \to \mathscr{B}(\mathrm{GL}(\mathfrak{g})).$$

Remark 2.3.8.2 Moy-Prasad also defined a filtration on $\mathfrak{g}^*$. From our point of view, this is simply obtained by composing the above map with the natural isomorphism $\mathscr{B}(\mathrm{GL}(\mathfrak{g})) \simeq \mathscr{B}(\mathrm{GL}(\mathfrak{g}^*))$. The latter map is the one induced by the identification $\mathrm{GL}(\mathfrak{g}) = \mathrm{GL}(\mathfrak{g}^*)$ and is described explicitly in [BT3].

For simplicity, assume that G is adjoint, so $G \hookrightarrow \mathrm{GL}(\mathfrak{g})$. Then the map $\iota : \mathscr{B}(G) \to \mathscr{B}(\mathrm{GL}(\mathfrak{g}))$ enjoys the following properties.

(i) it is $G(K)$-equivariant;
(ii) it is continuous and injective;
(iii) it is *toral* in the sense that if S is a maximal K-split torus of G, there exists a maximal K-split torus S' of $G = \mathrm{GL}(\mathfrak{g})$ such that $S \subset S'$ and $\iota A(G,S) \subset A(G',S')$ and the restriction $\iota|A(G,S) \to A(G',S')$ is affine;
(iv) it is compatible with base field extensions.

This is an example of descent maps between buildings, and in this case it is canonically defined. The theory of descent maps can be considered as the functoriality of the building. Below we first review the case of symmetric spaces (see [GY03] for details).

2.3.9 Functoriality of symmetric spaces.

Functoriality
Assume that G, H are reductive groups over $\mathbb{R}$ such that $G \subset H$. Suppose that $x \in \mathscr{S}(G)$, $y \in \mathscr{S}(H)$ are such that the associated maximal compact subgroups $G_x \subset G(\mathbb{R})$, $H_y \subset H(\mathbb{R})$ satisfy $G_x \subset H_y$, then we have a $G(\mathbb{R})$-equivariant map $\mathscr{S}(G) \to \mathscr{S}(H)$, sending $g.x$ to $g.y$. It is easy to see that $H_y \cap G(\mathbb{R}) = G_x$, hence this equivariant map is always injective.

However, the Cartan involution θ_y associated to y may not stabilize G.

Example 2.3.9.1 Let $G = \mathrm{SL}_2$, $H = \mathrm{SL}_4$, and define the inclusion $\iota : G \hookrightarrow H$ by $g \mapsto \left[\begin{smallmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{smallmatrix}\right]$. The standard maximal compact subgroup K of $G(\mathbb{R})$ is the stabilizer of the quadratic form represented by the 2×2 identity matrix I_2.

It is easy to see that ιK stabilizes the quadratic form q represented by the 4×4 matrix $\left[\begin{smallmatrix} aI_2 & bI_2 \\ bI_2 & cI_2 \end{smallmatrix}\right]$. When q is positive definite (for example, $a = c = 1$ and b sufficiently small), the stabilizer K_q of Q_q in $H(\mathbb{R})$ is a maximal compact subgroup such that $\iota K \subset K_q$. But the Cartan involution θ_q associated to K_q does not stabilize $\iota G(\mathbb{R})$ as long as $b \neq 0$.

Definition 2.3.9.2

(i) We say that the map $\mathscr{S}(G) \to \mathscr{S}(H)$, $g.x \mapsto g.y$ is a *descent map* if θ_y stabilizes $G(\mathbb{R})$ (and hence θ_y descends to θ_x: $\theta_y|_{G(\mathbb{R})} = \theta_x$).

(ii) We say that $\mathscr{S}(G) \to \mathscr{S}(H)$ is a *toral map* if there exists a maximal $\mathbb{R}$-split torus S of G, a maximal $\mathbb{R}$-split torus T of G such that $x \in A(S)$, $y \in A(T)$, and $S \subset T$.

Proposition 2.3.9.3

(i) *A descent map is a toral map.*

(ii) *A toral map is a descent map.*

(iii) *If $\mathscr{S}(G) \to \mathscr{S}(H)$ is a descent map then $\theta_{y'}|G(\mathbb{R}) = \theta_{x'}$ for all $x' \in \mathscr{S}(G)$, where y' is the image of x'.*

Base change
Denote $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(G \otimes \mathbb{C})$ by $G_{\mathbb{C}}$. Then there is a canonical map $\mathscr{S}_{\mathrm{red}}(G) \to \mathscr{S}_{\mathrm{red}}(G_{\mathbb{C}})$. It is constructed as follows: let $\theta \in \mathscr{S}_{\mathrm{red}}(G)$, and let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of θ on $\mathfrak{g} = \mathrm{Lie}\, G$. Then a Cartan involution $\theta_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathrm{Lie}\, G_{\mathbb{C}}$ is defined by $\theta_{\mathbb{C}}(k + p) = \bar{k} - \bar{p}$ for all $k \in \mathfrak{k} \otimes \mathbb{C}$, $p \in \mathfrak{p} \otimes \mathbb{C}$.

Now the base change map $\mathscr{S}_{\mathrm{red}}(G) \to \mathscr{S}_{\mathrm{red}}(G_{\mathbb{C}})$ is $\theta \mapsto \theta_{\mathbb{C}}$. It is easy to see that this map is $G(\mathbb{R})$-equivariant.

We also have an obvious inclusion $V(G) \hookrightarrow V(G_{\mathbb{C}})$. Combining with the previous construction, we get a base change map

$$\mathscr{S}(G) \to \mathscr{S}(G_{\mathbb{C}}),$$

which is clearly a $G(\mathbb{R})$-equivariant descent map.

Proposition 2.3.9.4 *The image of the base change descent map $\mathscr{S}(G) \to \mathscr{S}(G_{\mathbb{C}})$ is $\mathscr{S}(G_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$.*

Proposition 2.3.9.5 *Assume that $G \subset H$ is an embedding of complex groups in the sense that there are complex reductive groups $G' \subset H'$ such that $G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} G'$ and $H = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} H'$. Then any equivariant map $\mathscr{S}(G) \to \mathscr{S}(H)$ determined by an inclusion $G_x \subset H_y$ is a descent map.*

Descent maps and base change
Let $\iota : G \subset H$ be an inclusion of real reductive groups. Assume $x \in \mathscr{S}(G), y \in \mathscr{S}(H)$ are such that $G_x \subset H_y$ and denote by ι_* the equivariant map $\mathscr{S}(G) \to \mathscr{S}(H)$, $g.x \mapsto g.y$.

Proposition 2.3.9.6 *The map ι_* is a descent map if and only if ι_* extends to a $G_{\mathbb{C}}(\mathbb{R})$-equivariant map $\mathscr{S}(G_{\mathbb{C}}) \to \mathscr{S}(H_{\mathbb{C}})$.*

Finite group actions
Suppose that we have a finite group F acting on H. Then $G = (H^F)^\circ$ is also reductive. We denote the inclusion $G \to H$ by ι.

Theorem 2.3.9.7 *Let X be the set of descent maps $\iota_* : \mathscr{S}(G) \to \mathscr{S}(H)$ satisfying $\iota_*(\mathscr{S}(G)) \subset \mathscr{S}(H)^F$. Then,*

(i) *X is non-empty.*
(ii) *$\iota_*(\mathscr{S}(G)) = \mathscr{S}(H)^F$ for all $\iota_* \in X$.*
(iii) *X is a principal homogeneous space of $V(G)$.*

2.3.10 Functoriality of buildings. Unfortunately, this is still not well understood and there is no adequate literature. The main reference is [BT1] and

— E. Landvogt, *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. 518 (2000), 29 pages [Lan00].

Nothing is mentioned about the analogy with the case of symmetric spaces. But I think that this analogy provides very good insights and motivations. The two references also talk in totally different languages. [BT1] talks about analogues of the descent maps, namely they want to consider a compatibility condition regarding valuation of root data. In Landvogt's article, he considers the analogoue of the toral maps.

In an article which I am guilty of not finishing, I showed that these two concepts are the same, and so are several analogues of the above-mentioned results in the symmetric space case. The analogue of the theorem about finite group action (with a tameness assumption) is now a Theorem of Gopal Prasad and myself [PY02].

The most important result in this direction is the main theorem of Landvogt in the above mentioned paper, which asserts the existence of descent maps compatible with unramified base change. However, for some applications we do want to know the existence of descent maps compatible with ramified finite extension of base field. I can show the existence if the residue characteristic is not 2.

Remark 2.3.10.1 Proposition 2.4.1 in [Landvogt] ([Lan00, Prop. 2.4.1]) can be proved easily from Proposition 1.3 [Prasad-Yu] ([PY02, Prop. 13]). This would save a few pages of arguments.

Remark 2.3.10.2 The last main theorem (2.7.4 and 2.7.5) in [Landvogt] appears to be incorrect in both the statement and the proof. The case $G =$ split SO_3 and $H = \mathrm{SL}(3)$ in residue characteristic 2 provides a simple counterexample. However, Gopal Prasad has shown me a different argument for 2.7.4 when the residue characteristic is 0.

Remark 2.3.10.3 It is often asked whether for an embedding $G \hookrightarrow H$ and descent map $\iota : \mathscr{B}(G) \to \mathscr{B}(H)$, we have compatibility of Moy-Prasad filtrations:

$$G(K)_{x,r} = H(K)_{\iota x,r} \cap G(K)?$$

Again, this is true if the residue characteristic is sufficiently large. But it fails in some cases.

2.3.11 Concave functions.

Definition 2.3.11.1 A function $f : \Phi \cup \{0\} \to \mathbb{R}$ is called *concave* if $f(a+b) \leq f(a) + f(b)$ whenever $a, b, a+b \in \Phi \cup \{0\}$.

For such a function, we define $G(K)_{x,f}$ to be the subgroup of $G(K)$ generated by $U_a(K)_{x,f(a)}$ for all $a \in \Phi \cup \{0\}$.

Example 2.3.11.2 If $f(a) = r$ for all a, then f is concave and the associated $G(K)_{x,f}$ is the Moy-Prasad $G(K)_{x,r}$.

These groups are used more and more frequently in representation theory. Unfortunately they are not mentioned in Tits' article. To prove many basic facts in Moy-Prasad theory, one needs to use results about $G_{x,f}$ in [BT] but not in [Tits]. We will just mention two results here.

Theorem 2.3.11.3

(i) *$G(K)_{x,f}$ is bounded and open.*
(ii) *If $f(0) > 0$, then the multiplication map*

$$\prod_{a \in \Phi \cup \{0\}} U_a(K)_{x,f(a)} \to G(K)_{x,f}$$

is a bijection (the map sends $(u_a)_a$ to the product of the u_a's for a suitable order).

Theorem 2.3.11.4 *$G(K)_{x,f}$ is schematic.*

This result is proved in [BT2] when $f(0) = 0$. The proof there doesn't seem to extend to the general case. So I proved the above result in a preprint using the theory of group smoothening [Yu03]. It turns out that this new approach is substantially simpler, and can be used as a subtitute for a large part of [BT2] if one only concerns the case of discrete valuations.

For more properties of $G(K)_{x,f}$, see [BT1, 6.4].