On Abel's Hyperelliptic Curves

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Summary. In this note we discuss a class of hyperelliptic curves introduced by Abel in an 1826 paper. After some indications of the context in which he introduced them and a description of his main result we give some results on the moduli space of such curves.

In particular we compute the dimension of it at each of its points as well as giving a combinatorial formula for the number of components.

In his paper [1] Abel takes up a very special case of the problem of deciding when a rational differential form is the logarithmic differential of a rational function. Even though it is easy to imagine that the problem later led Abel to his famous Paris dissertation, the solution that Abel proposed is very special and quite different from the approach he chose later. Nevertheless, the present article is concerned with investigating the problem of [1].

A characteristic, and for the time quite unusual, feature of all of Abel's work is his insistence on treating general cases rather than special examples. It thus seems entirely fitting to study the moduli problem of all the solutions to his proposed problem and we shall indeed consider a number of aspects of the moduli spaces (or more precisely stacks) that classify his solutions.

We start by introducing a (small) number of variants of an attempt to formulate Abel's condition as a moduli problem. In particular Abel's solution of his problem in terms of a polynomial type Pell's equation appears not as the moduli problem that

O.A. Laudal et al. (eds.), *The Legacy of Niels Henrik Abel*

has been chosen as the central moduli problem of this article but as a chart for it. We then study the relations between these variants, the end result being that they are indeed closely related.

After a discussion of the case of genus 0 and 1 we give a reformulation of the moduli problem in terms of families of maps between genus zero curves. We then proceed to make an infinitesimal study of the moduli problem which allows us to conclude that it is smooth in characteristic 0. We then go on to study a Lyashko–Looijenga type map and show that it is a covering map. This allows us to give a topological covering space type description of the moduli stack which in particular gives us a combinatorial description of the set of components of the moduli space.

We shall, except for the last section, adopt a purely algebraic approach. Apart from reasons of taste there are some arguments in favour of such a choice. The reader's attention should be particularly directed to Theorem 6.2, where we shall discover that some naturally defined "equi-ramification strata" turn out to be non-reduced. It seems likely that the multiplicity with which those strata appear is significant.

We shall also use the language of algebraic stacks. This may seem unnecessary particularly as our stacks are very close to being spaces (cf. Proposition 8). However, I claim that it is the technically most convenient as well as most intuitive way of doing things. In particular when defining maps between solutions to moduli problems, representing these solutions as stacks means that in order to define maps between them one may often follow the path of first deciding what the map should do on points and then verify that this pointwise construction is natural enough so that it makes sense for families of objects. This is in fact what we shall do most of the time. Sometimes, however, we shall discover that some choices that were made in the point case can not be made in the case of a family and we shall then have to incorporate those choices in the definition of the moduli problem. This will lead for instance to the three slightly different versions of Abel's hyperelliptic curve.

Conventions. By a *monic polynomial* we shall mean a polynomial in one variable whose highest degree coefficient is equal to 1. Such a polynomial will be said to be *normalised* if its next to highest degree coefficient is equal to 0.

As we shall deal extensively with stacks it seems natural to use the term 'scheme' to denote an algebraic space and hence by 'locally' mean 'locally in the étale topology'.[1] Though we shall do so, this is not strictly necessary, however, and then 'locally' may at times be interpreted as 'locally in the Zariski topology,' though consistently using the étale topology will always work.

We have made only a token attempt at formulating our results in arbitrary characteristics.[2] Starting with Sect. 3 all our schemes and stacks will be over Spec $\mathbb{Z}[1/2]$ and starting with Sect. 6 we shall work exclusively in characteristic zero, this will

[1] Note that in practice the only difference between ordinary schemes and algebraic spaces is that for the latter the Zariski topology is not available.

[2] It will be clear that if the characteristic is large enough with respect to the degree n, the situation will be similar to that of characteristic 0.

also be true at the end of the preliminary Sect. 1 and at points in Sect. 3 (which will be explicitly spelled out).

As usual a *multiset* is a set whose members are counted with certain multiplicities, formally it is a set provided with a multiplicity function from it to the integers ≥ 1. If S is a multiset, we shall use $\underline{S}$ to denote the domain of the multiplicity function and μ_S for the multiplicity function itself. We shall use set-theoretic notation when dealing with multisets:

- $S := \{1, 1, 1, 2, 2, 3\}$ will denote the multiset for which $\underline{S} = \{1, 2, 3\}$ and $\mu_S(n) = 4 - n$.
- $\sum_{s \in S} s^2$ should be interpreted as $\sum_{s \in \underline{S}} \mu_S(s) s^2$, i.e., $3 \cdot 1^2 + 2 \cdot 2^2 + 3^2$.
- Similarly, $\{\lfloor s/2 \rfloor \mid s \in S\}$ should be interpreted as $\{0, 0, 0, 1, 1, 1\}$.

A multiset S is finite if $\underline{S}$ is and then its *cardinality*, $|S|$, equals $\sum_{s \in S} 1$. A multiset S is said to be a submultiset of the multiset T if $\underline{S} \subseteq \underline{T}$ and $\mu_S(s) \leq \mu_T(s)$ for all $s \in \underline{S}$.

1 Preliminaries

We shall sometimes speak about the universal object over a stack which classifies some type of geometric object. Note that, contrary to the case when the moduli problem of classifying such objects is representable by a scheme, this is somewhat ambiguous and is not quite as strong. Firstly, for a family of objects over S, the family may not be the pullback of the universal family but is so only locally on S. Secondly, the universal object is not unique; two such objects are only locally isomorphic. Thirdly, a universal object may in fact not even exist over the stack itself, but only locally. It would be more proper to speak about the stack of universal objects, but we shall allow ourselves the luxury of not doing that. The first phenomena are shown quite clearly in the case of the classifying stack, BG, of a finite group G. A universal object is given by the trivial G-torsor, and a non-trivial G-torsor over S is of course not the pullback of the trivial one. In fact, any G-torsor over the base is universal and there may very well be non-trivial G-torsors over the base.

Assume that $X \to S$ is a scheme and $X \to \mathbb{P}^1 \times S$ an S-morphism. Let C and D be the schematic inverse images of $0 \times S$ and $\infty \times S$ and assume C and D are Cartier divisors. We shall repeatedly use the (obvious) fact that such a morphism is the same thing as an isomorphism $\mathcal{O}_X(C) \xrightarrow{\sim} \mathcal{O}_X(D)$.

Let S be a scheme. A line bundle $\mathcal{L}$ and a trivialisation φ of $\mathcal{L}^2$ will be called an *involutive line bundle*. Consider further $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}) \to S$, the projective bundle on the vector bundle $\mathcal{O}_S \oplus \mathcal{L}$, the two sections ∞ and $\mathbf{0}$ associated to the two projections of $\mathcal{O}_S \oplus \mathcal{L}$ and the involution σ of $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}) \to S$ defined as the composite of the map $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}) \to \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_S)$ that switches the two factors, the standard identification and distributivity $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_S) = \mathbb{P}((\mathcal{L} \oplus \mathcal{O}_S) \bigotimes \mathcal{L}) = \mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L})$ and φ applied to the first factor $\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}) \to \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$. We shall call the data $(\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}), \mathbf{0}, \infty, \sigma)$ the *involutive projective bundle* associated to the involutive line bundle $(\mathcal{L}, \varphi)$ and denote it $\mathbb{P}_{\mathcal{L}}$.

Example 1. If $\mathcal{L} = \mathcal{O}_S$ and $\varphi = \lambda \in \mathcal{O}_S^\times$, then the involutive bundle is $(\mathbb{P}^1 \times S, 0 \times S, \infty \times S, x \mapsto \lambda/x)$. Locally this is the general situation.

Note also that the fixed point locus of σ is a double covering of the base that is isomorphic to the double cover associated to $\mathcal{L}$ and φ; something which is seen for instance by using the local description just given. We shall call it the *involutive locus*.

Similarly to the remark above, an S-morphism $X \to \mathbb{P}_{\mathcal{L}}$ such that the inverse images of ∞ and $\mathbf{0}$ are Cartier divisors C and D is the same thing as an isomorphism $\mathcal{O}_X(C) \xrightarrow{\sim} \mathcal{O}_X(D) \otimes \mathcal{L}$.

The involutive bundle will be said to be *split* if one is given a trivialisation of $\mathcal{L}$ for which φ becomes the identity. Then the involutive projective bundle is identified with $\mathbb{P}^1 \times S$ in such a way that 0 corresponds to the zero section, ∞ to the section at infinity and the involutive locus is given by $\{(s\colon t) \mid s^2 = t^2\}$ which when 2 is invertible is $\{(\pm 1 : 1)\}$.

We may explicitly construct the quotient of $\tau\colon \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}) \to S$ by the action of σ in the following way. We define an S-map $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}) \to \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$ by giving $\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(2)$ as a quotient of $\tau^*(\mathcal{O}_S \oplus \mathcal{L})$. By adjunction giving such a map is the same as giving a map $\mathcal{O}_S \oplus \mathcal{L} \to \tau_*\mathcal{O}(1) = S^2(\mathcal{O}_S \oplus \mathcal{L})$. We do this by mapping 1 of the $\mathcal{O}$-factor to $1 \otimes 1 \oplus \varphi(1)$ in $S^2\mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \subset S^2(\mathcal{O}_S \oplus \mathcal{L})$ and the $\mathcal{L}$-factor to $\mathcal{O}_S \otimes \mathcal{L} \subset S^2(\mathcal{O}_S \oplus \mathcal{L})$ through $1 \otimes id$. In the local normal form above – homogenised – this map is given by $(x : y) \mapsto (x^2 + \lambda y^2 : xy)$ which evidently has no base points, i.e., it is surjective and hence gives a map $\pi\colon \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}) \to \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$. As σ locally has the form $(x\colon y) \mapsto (\lambda y\colon x)$ it is clear that π is equivariant with trivial action on the target. Using again the local form it is easily verified that it is the quotient map. We shall speak of it as the *involutive quotient map* associated to $\mathcal{L}$ and φ. Note that the involutive locus maps to the image under $x \mapsto 2x$ of itself. For this reason, starting with Sect. 3, we shall instead use $(x\colon y) \mapsto (x^2 + \lambda y^2\colon 2xy)$ as quotient map so that the involutive locus is mapped to itself.

Seen from the point of view of its target the involutive quotient map π is a double covering. Restricting ourselves to the case when 2 is invertible we may describe this covering as follows. We get a map $\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-1) \to \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}$: By adjunction it corresponds to a map $\pi^*\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}$ and by construction we have $\pi^*\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-1) = \mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-2)$ so that such a map corresponds to a section of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(2)$, i.e., a section of $S^2(\mathcal{O}_S \oplus \mathcal{L})$ and we choose $\otimes 1 \oplus -\varphi(1)$. In the local form above the section $\otimes 1 \oplus -2\varphi(1)$ corresponds to $1/2(x^2 - \lambda y^2)$. From that it is easily verified that the map $\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-1) \to \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}$ is injective and has as image the -1-eigenspace of σ. The double cover π is now determined by the square map $\mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})}$, i.e., a section of $S^2(\mathcal{O}_S \oplus \mathcal{L})$. A local calculation shows that that section is $1 \otimes 1 \oplus -\varphi(1)$.

We shall have need of the following technical result on Cartier divisors.

Proposition 1. *Let $\pi : X \to S$ be a smooth, proper map of schemes with connected fibres and $D \subset X$ a relative (wrt to π) effective Cartier divisor, and let n be an integer such that $n\mathcal{O}_S = \mathcal{O}_S$ (i.e., n is invertible in $\mathcal{O}_S$).*

The functor that to a $T \to S$ associates the set of relative effective Cartier divisors $D \subset X \times_S T$ such that $nD = C \times_S T$, is representable by a closed subscheme of S.

In particular, if S is reduced and if for every geometric fibre of $X \to S$, C is n times an effective Cartier divisor, C itself is n times an effective Cartier divisor.

Proof. The functor is clearly representable by some S-scheme $V \to S$ locally of finite type, and we may, by a standard limit argument, assume that S is noetherian. What needs to be proven is that $V \to S$ is proper, injective on geometric points and unramified. For the properness we may use the valuative criterion so that S is the spectrum of a discrete valuation ring and we assume given a D_η over the generic point with $nD_\eta = nC_\eta$. We then let D be the schematic closure of D_η, which is a Cartier divisor, as X is regular and is relative as it does not have any horisontal components.

As for injectivity on geometric points we may assume that S is the spectrum of an algebraically closed field and then the uniqueness of D is clear as X is regular and thus the group of Cartier divisors is torsion free.

Finally, to prove that $V \to S$ is unramified, it is enough to show that it is formally unramified, so we may assume that $S = \operatorname{Spec} R$, where $(R, \mathfrak{m}_R)$ is a local Artinian ring, $0 \neq \delta \in R$ annihilates $\mathfrak{m}$ and we assume that a D exists over the closed subscheme defined by δ. We then are to prove that there is at most one lifting of D to $X \to S$. Now, a Cartier divisor E is given by specifying a line bundle $\mathcal{L}$ and an injective $\mathcal{O}$-homomorphism $\mathcal{O} \to \mathcal{L}$. As $X \to S$ is flat, the injectivity follows from injectivity over the special fibre and is hence automatic in our situation. Furthermore, if D is given by $s\colon \mathcal{O} \to \mathcal{L}$, then nD is given by $s^{\otimes n}\colon \mathcal{O} \to \mathcal{L}^{\otimes n}$. In our situation we assume a pair $(\mathcal{M}, t)$ over X representing C and two pairs $(\mathcal{L}, s)$ and $(\mathcal{L}', s')$ whose n'th powers are isomorphic to $(\mathcal{M}, t)$ and whose reductions modulo δ are isomorphic. Now the kernel and cokernel of the reduction $\operatorname{Pic}(X) \to \operatorname{Pic}(X/\delta)$ are $\mathcal{O}_S$-modules, so that multiplication by n is by assumption bijective on them, which shows that $\mathcal{L}'$ and $\mathcal{L}$ are isomorphic, and we may assume them to be equal. Hence s' is of the form $s + \delta w$ with w a section of $\overline{\mathcal{L}} = \mathcal{L}/\mathfrak{m}$ by the flatness of $X \to S$. By assumption their n'th powers are isomorphic so that $(1 + \delta\lambda)s^n = (s + w)^n = s^n + ns^{n-1}\delta w$ for some $\lambda \in R$. This gives $\overline{s}^{n-1}(\lambda\overline{s} + nw) = 0$, where $\overline{s}$ is the reduction of s modulo $\mathfrak{m}$. As $\overline{s}$ is a non-zero divisor this gives $w = -\lambda/n\overline{s}$, i.e., $\delta w = -\lambda/n\delta s$, which gives $s' = (1 - \lambda/n\delta)s$, so that the pairs $(\mathcal{L}, s)$ and $(\mathcal{L}', s')$ are isomorphic.

The last statement follows immediately from the previous ones as under its assumptions V has the same topological space as S. □

The following result is no doubt well known but I do not know of a reference.

Proposition 2. *i) Let $f\colon C \to D$ be a separable non-constant map of smooth proper curves over a field $\mathbf{k}$ and consider the deformation functor whose values on a nil-thickening of* $\operatorname{Spec} \mathbf{k}$ *are isomorphism classes of deformations of C and the map f. The map that maps such deformations to similar deformations of the formal completions of C resp. D along the ramification resp. branch locus is an isomorphism.*

ii) Let $\mathbf{k}$ be a field, n an integer invertible in $\mathbf{k}$ and $f\colon \operatorname{Specf} \mathbf{k}[[x]] \to \operatorname{Specf} \mathbf{k}[[t]]$ be the map $t \mapsto x^n$. Then $t \mapsto x^n + \sum_{0 \le i < n-1} a_i x^i$, where the a_i are power series variables, is a miniversal deformation of f.

Proof. The first part can be proved by noticing that outside of the ramification/branch loci the map is unramified and hence extends uniquely along any nil-thickening. This shows that the deformation problem is the same as that for the localisation along the ramification/branch loci. The comparison between the deformation for the localisations and the completions is also clear as when one inverts generators for the ramification/branch loci then the map is étale. This means that the map is specified by choosing a lattice in the ring of functions in the source over the ring of functions of the target. Specifying such a lattice is the same in the localisation as in the completion.

Alternatively one can use deformation theory. If $R \to S$ is a small extension of local Artinian algebras with residue field $\mathbf{k}$, small meaning that the kernel K is killed by the maximal ideal of R, then the liftings of a deformation over S to one over R is in bijection with $H^0(C, f^*T_D/T_C) \otimes_{\mathbf{k}} K$. Indeed, if the deformation of C is kept fixed, then liftings of deformations of f, given one, are in bijection with $H^0(C, f^*T_D) \otimes K$. Taking into account the possibility of varying also deformations of C, we have to divide out by the action of liftings of automorphisms of the deformation of C, i.e., sections of $T_C \otimes K$. This action is given by addition composed with the map $T_C \to f^*T_D$ and hence the full problem is in bijection with $H^0(C, f^*T_D/T_C) \otimes_{\mathbf{k}} K$. As we never used the properness, the same is true for the local or complete problem as f^*T_D/T_C is supported on the ramification locus.

As for the last part, the formula $t \mapsto x^n + \sum_{0 \le i < n-1} a_i x^i$ gives a deformation over $\mathbf{k}[[a_0, \dots, a_{n-2}]]$ and hence a map to the miniversal deformation. As $\mathbf{k}[[a_0, \dots, a_{n-2}]]$ is (formally) smooth, to show that this map is an isomorphism it is enough to show that it induces an isomorphism on tangent spaces, and for that we can use the description of deformations over $\mathbf{k}[\delta]$, $\delta^2 = 0$, just given to show that. Indeed, the action of the sections of T_C on such deformations is by interpreting a derivation of $\mathbf{k}[[x]]$ as an automorphism of the scalar extension to $\mathbf{k}[\delta]$, $\mathbf{k}[[x]][\delta]$, that is the identity modulo δ, and then composing the given map $\mathbf{k}[[t]][\delta] \to \mathbf{k}[[x]][\delta]$ with that automorphism. If the vector field is $h(x)d/dx$ and the map has the form $f(x) + g(x)\delta$ with $f, g, h \in \mathbf{k}[[x]]$, then this composite is $f(x) + (h(x)f'(x) + g(x))\delta$. This shows that the tangent vector of the map is given by the residue of g modulo $f'(x)$, which makes it clear that the tangent map is an isomorphism. □

When we make an infinitesimal study of the moduli stack we shall not just deal with the stack as such but also with the natural stratification of it given by the ramification exponents of a map between curves. We recall its definition and first properties given in [3, App.]. We begin by noting that for technical reasons we shall need to assume that we deal with schemes over Spec $\mathbb{Q}$ for the rest of this section.

Remark 1. Note that this restriction is not just due to the fact that one would need some slight modifications to get similar results in positive characteristic. In fact there are some truly new phenomena in positive characteristic. Consider for instance the case of Proposition 3. Condensed it says that for a finite flat map there is a stratification of the base such that on each stratum there is a closed subscheme of the total space which is étale over the base and whose defining ideal is nilpotent.

A similar result is not possible in positive characteristic. Consider for instance an inseparable field extension $k \subset K$ of degree p say. For the corresponding map of schemes $\operatorname{Spec} K \to \operatorname{Spec} k$ if it had a similar stratification then there could only be one stratum but $\operatorname{Spec} K$ does not have a closed subscheme which is étale over $\operatorname{Spec} k$ and whose ideal is nilpotent.

Recall that if $f: Y \to X$ is a finite flat map then we define its *trace form* to be the symmetric bilinear form $(r, s) \mapsto \operatorname{Tr}(rs)$. We then define, for each natural number n, the closed subscheme of X given by the condition that the corank of the map $f_*\mathcal{O}_Y \to \operatorname{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$ induced by the trace form is $\geq n$. We shall call the stratification thus obtained the *trace stratification* wrt f. In an open stratum we get the following primary decomposition result.

Proposition 3. *Let $f: Y \to X$ be a finite flat map for which X equals a single open trace stratum. Then the radical of the trace form (i.e., the kernel of the map $f_*\mathcal{O}_Y \to \operatorname{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$) is a subbundle and an ideal. The closed subscheme Y' defined by it is an étale covering of X. Furthermore, there is a unique X-retraction $Y \to Y'$ which makes Y a flat Y'-scheme.*

Finally, if $Y \to Y'$ has rank n then the n'th power of the radical of the trace map is zero.

Proof. The fact that the radical is a subbundle follows directly from the fact that by assumption $f_*\mathcal{O}_Y \to \operatorname{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$ has constant rank (in the schematic sense defined by the vanishing and non-vanishing of subdeterminants when X is not reduced) and then its kernel is a subbundle. That it is an ideal follows directly from the definition of the trace form. Then $Y' \to X$ is flat so to prove that it is étale it is enough to do it when X is the spectrum of a field in which case it is well known and easy to see that Y' is étale. Assume that we know that the existence and unicity of a retraction locally. Then the unicity forces it to exist globally by (étale) descent. By descent again, the flatness of the retraction needs to be checked only in the case when Y' is the disjoint union of copies of X in which case it is clear.

For the last statement, replacing X by Y' we may assume that f has rank n. If $\mathcal{I}$ is the radical and $x \in \mathcal{I}$ (i.e., is a local section of that sheaf) then we have that $x^i \in \mathcal{I}$ for all $i > 0$ and hence $\operatorname{Tr}(x^i) = 0$. As we are in characteristic zero this implies that the characteristic polynomial of multiplication by x is t^n and by the Cayley–Hamilton theorem $x^n = 0$. Again, as we are in characteristic zero, we get by polarisation that any product of n local sections of $\mathcal{I}$ is zero.

It remains to prove local existence and unicity of a retraction. Locally we may assume that Y' is a disjoint union of copies of X which makes the existence of a retraction obvious and the unicity clear. □

When the corank of the trace form is constant, the proposition shows that the function on the points of Y defined by the rank at a point of the radical is a locally constant function and hence the function which to a point of X associates the multisets of those ranks is locally constant. The stratification obtained in this way will be referred to as the *stratification by multiplicity*. Thus while the trace stratification is a decreasing sequence of closed subschemes, the stratification by multiplicity is a further decomposition of the open strata. We shall also in this case denote the subscheme defined by the radical by X^{fred} and call it the *fibrewise reduced subscheme*.

Using the primary decomposition we get an extension of the pointwise result that expresses an effective Cartier divisor on a smooth curve as the sum of points.

Proposition 4. *Let $f\colon X \to S$ be a smooth map of* $\operatorname{Spec}\mathbb{Q}$*-schemes of relative dimension* 1 *and suppose $D \subset X$ is a relative effective Cartier divisor and assume that the corank of the trace form of $D \to S$ is of locally constant rank.*

i) The fibrewise reduced subscheme D^{fred} of $\operatorname{Spec}\mathcal{O}_X/\mathcal{I}_D$ *is a relative effective Cartier divisor. It can be written as the disjoint of subschemes D_i having the property that at a point $d \in D_i$ the defining ideal of D_i in D has rank e_i as $\mathcal{O}_{D_i}$-module. If that is done then we have that $D = \sum_i e_i D_i$ as divisors.*

ii) Conversely if D can be written as $D = \sum_i e_i D_i$ with D_i étale disjoint Cartier divisors then the corank of the trace form of $D \to S$ is locally constant and the union of the D_i is the fibrewise reduced subscheme of D.

Proof. That D^{fred} is a Cartier divisor is clear and we get from Proposition 3 that its defining ideal $\mathcal{I}$ in D is a locally free $\mathcal{O}_{D^{\mathrm{fred}}}$-module and hence its rank is locally constant. This gives the components D_i and to prove the equality of Cartier divisors we may work locally around one of the D_i, i.e., assume that the rank of $\mathcal{I}$ as $\mathcal{O}_{D^{\mathrm{fred}}}$-module is everywhere equal to some n. By Proposition 3 the n'th power of $\mathcal{I}_{C^{\mathrm{fred}}}$ is contained in $\mathcal{I}_D$. To check that it is an equality it is enough to check on fibres over closed points of S and then it is true as they have the same degree at all the points of D^{fred}.

Finally, if $D = \sum_i e_i D_i$ and D' is the union of the D_i then $\mathcal{I}_{D'}/\mathcal{I}_D$ is an S-flat nilpotent ideal of $\mathcal{O}_X/\mathcal{I}_D$ such that the quotient by it is étale. This shows that $\mathcal{I}_{D'}/\mathcal{I}_D$ is the radical of the trace form and thus that D' is the fibrewise reduced subscheme of D. □

The multiplicities e_i can be considered as locally constant functions on D^{fred} and we may choose the D_i such that the e_i are all distinct. Having done that the decomposition is unique and we shall call it the *primitive decomposition* of D. We shall also consider the locally constant function on the base S which to a point s associates the multiset of the multiplicities of the points of D in the fibre over s. This will be called the *multiplicity multiset* associated to D.

If S is a scheme and $f\colon Y \to X$ is a finite S-map between smooth (possibly formal) S-schemes of relative dimension 1, then we get two finite S-schemes, the ramification locus which is a relative Cartier divisor of Y and the branch locus which is a relative Cartier divisor of X (by definition the branch locus is the norm wrt to f of the ramification locus considered as a Cartier divisor which is defined as f is finite flat). We shall call the stratifications by multiplicity on S induced by them the *ramification stratification* and *branch stratification* respectively.

2 The Original Problem

We shall begin by formulating in modern terms Abel's question and the answer he gave to it. The initial setup is that of a square free monic polynomial $R(x)$ of even degree over the complex numbers and the rational differential form $\omega := \rho dx/\sqrt{R}$, $\rho \in \mathbb{C}(x)$, on the compact Riemann surface C with field of rational functions $\mathbb{C}(x, \sqrt{R}) := \mathbb{C}(x)[y]/(y^2 - R(x))$. The general question Abel poses is when this form is the logarithmic differential $\operatorname{dlog} f := df/f$ for a non-zero rational function f. If ι is the hyperelliptic involution of C which takes x to x and $\sqrt{R}$ to $-\sqrt{R}$, then $\iota^*\omega = -\omega$ and as f is determined up to a constant we get $f \circ \iota \cdot f = \lambda \in \mathbb{C}^*$. Modifying f by multiplying by a square root of λ^{-1} allows us to assume that $f \circ \iota \cdot f = 1$. By a somewhat anachronistic appeal to Hilbert's Theorem 90 we get that f has the form $g/g \circ \iota$ and by clearing denominators we may assume that $g = P + \sqrt{R}Q$ with $P, Q \in \mathbb{C}(x)$, which is indeed the form that Abel assumes the solution to have. We note that g is uniquely determined up to a rational function in x. Abel then almost immediately restricts himself to the case where ρ is a polynomial. This implies that ω is regular over $C^\circ := \operatorname{Spec} \mathbb{C}[x, \sqrt{R}]$ and hence in particular that f does not have poles or zeroes in C°, or otherwise put, if ∞_1 and ∞_2 are the two points of C in the complement of C° in C, then $(f) = m\infty_1 - m\infty_2$ for some integer m. For a divisor D on C we denote by D° the part of D that has support on C° and then we have $0 = (f)^\circ = ((g) - \iota(g))^\circ$. Now by assumption $(g)^\circ \geq 0$ and hence for $r \in C^\circ$, r and ιr appear with the same multiplicity in g. Now, for any $r \in C^\circ$ $r + \iota r - (\infty_1 + \infty_2)$ is the divisor of a rational function in x so that if r is a non-Weierstrass point (i.e., $r \neq \iota r$) we may modify g by a rational function in x so that r does not appear in (g). Similarly, if r is a Weierstrass point we may assume that it appears with at most multiplicity 1 in (g). In particular, $(g)^\circ$ has support at the Weierstrass points of C.

Even though Abel treats the general case, we shall only be interested in the case when $(g)^\circ = 0$. The reason for this is that we shall be mainly interested in the existence of a g as a condition on the curve C and we have that $2\omega = \operatorname{dlog} f^2 = \operatorname{dlog} g^2/g^2 \circ \iota$ and all the Weierstrass points appear with even multiplicity in (g^2), and they can therefore be removed completely. Hence at the price of possibly replacing ρ with 2ρ we see that Abel's problem has been reduced to the problem of finding $g \in \mathbb{C}(x, \sqrt{R})$ with $(g)^\circ = 0$, i.e., $(g) = n(\infty_1 - \infty_2)$ for some integer n and excluding the trivial case of $\rho = 0$ we may assume that n is non-zero. In any case $(g)^\circ = 0$ and hence g is a unit in $\mathbb{C}[x, \sqrt{R}]$. This implies that $g = P + \sqrt{R}Q$, with $P, Q \in \mathbb{C}[x]$ and furthermore that the norm $N(g) = P^2 - RQ^2$ of g with respect to the finite flat extension $\mathbb{C}[x, \sqrt{R}]/\mathbb{C}[x]$ is a unit in $\mathbb{C}[x]$, i.e., a non-zero constant. After changing g by a constant we may assume that $P^2 - RQ^2 = 1$. Abel then notes that this is analogous to Pell's equation and proceeds to use continued fractions in analogy with the case of Pell's equation. There is a difference however in that the number theoretic case gives a method for solving Pell's equation while the geometric case gives a criterion for the existence of a solution (as well as a method for constructing it when it does exist).

Remark 2. Abel's approach gives a condition on R for a solution to the problem with n arbitrary to exist. This is not appropriate for our purposes as when n varies we get a countable union of closed subvarities in the space of R's which is unnatural from a geometric point of view.

3 Abel Curves

We shall now give the formal definition of an Abel curve. To simplify the presentation (and make it closer to Abel's original results) we shall *from now on assume that* 2 *is invertible in all our schemes*. If we want to consider the moduli space of Abel curves we want to make as few choices as possible as any choice leads to a larger space which is the reason for the somewhat lengthy definition. To avoid ambiguities in the case of genus 1 to us a *hyperelliptic curve* will be a smooth proper curve C together with a *choice* of an involution ι such that the quotient C/ι is of genus zero.[3]

Definition 1. *A* (smooth) Abel curve of genus g and order n *over a scheme* S *consists of*

- *a smooth and proper* S*-curve* $\pi\colon C \to S$,
- *an* S*-involution* ι *of* C *making each fibre a hyperelliptic curve of genus* g,
- *two disjoint sections* ∞_1 *and* ∞_2 *of* π *such that* $\infty_2 = \iota\infty_1$,
- *a line bundle* $\mathcal{L}$ *on* S *together with a trivialisation* $\varphi\colon \mathcal{O}_S \xrightarrow{\sim} \mathcal{L}^2$, *and*
- *a finite flat* S*-map* $f\colon C \to P$, *where* (P, σ) *is the involutive bundle associated to* $\mathcal{L}$ *and the trivialisation* φ, *of degree* n *such that the sections* ∞_1 *and* ∞_2 *map to the sections* $\mathbf{0}$ *and* ∞ *of the involutive bundle* $\mathbb{P}$ *and* $\sigma \circ f = f \circ \iota$.

A split Abel curve *is an Abel curve together with a splitting of the involutive bundle.*

An isomorphism between Abel curves consists of isomorphisms between the C *and* P *parts of the curves transporting all the structures of the first curve to those of the second.*

Associating to each S *the groupoid of Abel curves and isomorphisms between them gives a stack (in say the flat topology) that we shall denote* $\mathcal{H}_g^n$ *and similarly we get the stack of split Abel curves* $\mathcal{H}_g^{s,n}$.

When the base is an algebraically closed field we get exactly the description that came out of Abel's problem. Note that in that case it follows from the equation $P^2 - RQ^2 = 1$ that $2\deg P \geq \deg R$, i.e., $2n \geq 2g + 2$ which means $n \geq g + 1$.

It is not immediately clear that this is the right definition for families as one could worry that we have made an unnecessary choice in choosing two sections ∞_1 and ∞_2 instead of a divisor of degree 2 that only after a base change splits up into two disjoint sections. The following definition expresses that concern.

Definition 2. *A* twisted (smooth) Abel curve of order n *over a scheme* S *consists of*

- *a smooth and proper* S*-curve* $\pi\colon C \to S$,
- *an* S*-involution* ι *of* C *making each fibre a hyperelliptic curve,*
- *a* ι*-invariant relative effective divisor* D *of degree* 2 *of* C *which is étale over* S *and on which* ι *acts freely,*
- *a smooth and proper* S*-curve* $\rho\colon P \to D$ *all of whose fibres have genus zero,*
- *an* S*-involution* σ *of* P,

[3] With this definition we can have hyperelliptic curves of genus zero which for our purposes is quite acceptable though rather trivial.

- *a σ-invariant relative effective divisor D' of P which is étale over S and on which σ acts freely,*
- *a finite flat S-map $f: C \to P$ of degree n such that the inverse image (as effective divisors or equivalently as subschemes) of D' is nD and for which $f \circ \iota = \sigma \circ f$.*

Isomorphisms between Abel curves consist of an isomorphism g between the C-parts preserving the ι's, ∞_1's, and ∞_2's and an automorphism h of $\mathbb{P}^1 \times S$ such that $f \circ g = h \circ f$.

Associating to each S the groupoid of twisted Abel curves and isomorphisms between them gives a stack (in say the flat topology) that we shall denote $\mathcal{H}_g^{t,n}$.

The relation between these definitions is expressed in the following result.

Proposition 5. *i) The stack of twisted Abel curves of genus g and order n, $\mathcal{H}_g^{t,n}$, is equivalent to $B\Sigma_2 \times \mathcal{H}_g^n$, where $B\Sigma_2$ is the stack of Σ_2-torsors, i.e., the stack of étale double covers. The projection on the first factor associates to a twisted Abel curve, using the notation of Definition 2, the étale double cover $D \to S$.*

ii) The forgetful map $\mathcal{H}_g^{s,n} \to \mathcal{H}_g^n$ is an étale double cover.

Proof. Using the notation of Definition 2 we get from a twisted Abel curve over S an étale double cover $D \to S$ which gives a map from the stack of twisted Abel curves to $B\Sigma_2$. On the other hand, (ι, σ) gives an involution of the Abel curve and we may use it and the double cover $D \to S$ to twist the Abel curve, in particular the twist, $\tilde{C}$, of C is obtained by taking the quotient of $D \times_S C$ by the action of (ι, ι). The section given by the graph of the inclusion of D in C is invariant under this map and hence descends to a section of $\tilde{C} \to S$ and the same is true of the group of the map $D \to C$ composed with ι. In other words, the divisor D twists to give a divisor that is the disjoint union of two sections. Now, the map $f: C \to P$ maps D isomorphically to D' so that also D' is the disjoint union of two sections. The existence of these two disjoint sections makes $P \to S$ isomorphic to $\mathbb{P}(\mathcal{L} \oplus \mathcal{M})$ for some line bundles $\mathcal{L}$ and $\mathcal{M}$ on S, where the two sections correspond to the two summands. Now, σ permutes the two sections, which forces $\mathcal{L}$ and $\mathcal{M}$ to be isomorphic, so that $P \to S$ is isomorphic to $\mathbb{P}^1 \times S \to S$ with the two sections given by $0 \times S$ and $\infty \times S$. As the inverse images of $0 \times S$ and $\infty \times S$ are n times the two sections of D, we get an Abel curve, and consequently a map $\mathcal{H}_g^{t,n} \to \mathcal{H}_g^n$ and combining the two constructed maps we get a map $\mathcal{H}_g^{t,n} \to B\Sigma_2 \times \mathcal{H}_g^n$. Conversely, given an Abel curve over S we can consider the map f as an isomorphism $\mathcal{O}_C(n\infty_1) \xrightarrow{\sim} \mathcal{O}_C(n\infty_2)$. Letting ι act on that isomorphism gives another isomorphism $\mathcal{O}_C(n\infty_2) \xrightarrow{\sim} \mathcal{O}_C(n\infty_1)$. Their composites are then multiplication by an invertible function λ on S. That means that if we define σ on $\mathbb{P}^1 \times S$ by $(x : y) \mapsto (\lambda y : x)$ then $f \circ \iota = \sigma \circ f$ so that we have a twisted Abel curve over S. Now, (ι, σ) is an involution of that object and so that if we have an étale double cover $\tilde{D} \to S$ we can use it to twist our twisted Abel curve and we obtain thus a map $B\Sigma_2 \times \mathcal{H}_g^n \to \mathcal{H}_g^{t,n}$ which is clearly an inverse to the map just constructed.

As for the second part it is clear. □

The proposition shows that it is no real loss in generality to restrict ourselves to Abel curves which we shall do from now on with the exception of the following result which confirms the representability of the two stacks.

Proposition 6. *The stacks* $\mathcal{H}_g^n$, $\mathcal{H}_g^{s,n}$, *and* $\mathcal{H}_g^{t,n}$ *are Deligne–Mumford stacks of finite type over* $\operatorname{Spec}\mathbb{Z}[1/2]$.

Proof. This is quite standard as soon as we have verified that the automorphism group scheme of an (twisted) Abel curve over an algebraically closed field is finite étale. For $g \geq 2$ this is clear as it is true for all curves of genus g. For $g = 1$ we have to use the fact that the hyperelliptic involution is part of the structure so that an automorphism has to commute with it. For a hyperelliptic involution ι we may choose a fixed point as origin and in the thus obtained group structure on the curve, the involution is multiplication by -1; then it is clear that the automorphism group scheme centralizer of the curve is finite étale. Finally, for genus 0 we have to look at the automorphism group scheme of automorphisms of $\mathbb{P}^1$ fixing two points and commuting with an involution that permutes the two points. It is clear that the points and the involution is conjugate to 0, ∞, and $x \mapsto 1/x$ and then the automorphism group scheme that fixes these is clearly finite étale. □

Our definition of an Abel curve is chosen to be closely modeled on Abel's original condition. On the other hand – at least punctually – the relevant condition is that the divisor class $\infty_1 - \infty_2$ is killed by n as there is then a map to $\mathbb{P}^1$ whose zero and pole divisor is $n(\infty_1 - \infty_2)$. This turns out to be true for families.

Proposition 7. *Let* $\mathcal{H}_{g,2}$ *be the stack of hyperelliptic curves with two distinct points* (C, ι, a, b) *of genus* g *and let* $s\colon \mathcal{H}_{g,2} \to J_g$ *be the section of the Jacobian of the universal curve given by* $a - b$. *Let* $\mathcal{H}$ *be the closed substack of* $\mathcal{H}_{g,2}$ *defined by the conditions* $\iota a = b$ *and* $ns = 0$. *Let* ρ *be the involution of* $\mathcal{H}_g^n$ *which takes an object* $(C \to S, \iota, \infty_1, \infty_2, f)$ *to* $(C \to S, \iota, \infty_1, \infty_2, -f)$. *Then the map given by*

$$\begin{array}{ccc} \mathcal{H}_g^n & \to & \mathcal{H} \\ (C, \infty_1, \infty_2, f) & \mapsto & (C, \infty_1, \infty_2) \end{array}$$

is an isomorphism of stacks.

Proof. As has been noted above, f may be thought of as an isomorphism $\phi\colon \mathcal{O}_C(n\infty_1) \xrightarrow{\sim} \mathcal{O}_C(n\infty_2)$ and then ρ takes it to $-\phi$. On the other hand, an S-object of $\mathcal{H}$ has the property that $\mathcal{O}(n\infty_1 - n\infty_2)$ is a pullback of a (unique) line bundle $\mathcal{L}$ on S. Now, applying ι to $\mathcal{O}(n\infty_1 - n\infty_2)$ gives its inverse which translates into an isomorphism $\mathcal{L} \xrightarrow{\sim} \mathcal{L}^{-1}$, i.e., a trivialisation of $\mathcal{L}^2$. This gives an object of $\mathcal{H}_g^n$ over S. □

As the zero-section in an abelian scheme is a local complete intersection subscheme we get one immediate consequence.

Corollary 1. $\mathcal{H}_g^n \to \operatorname{Spec}\mathbb{Z}[1/2]$ *is of relative dimension at least* g *at each of its points and at a point where the relative dimension is* g *it is a local complete intersection.*

Proof. The substack $\mathcal{H}$ of $\mathcal{H}_{g,2}$ fulfilling $\iota a = b$ is an open substack of the stack $\mathcal{H}_{g,1}$ of hyperelliptic curves with one chosen point, namely the complement of the locus of fixed points of the hyperelliptic involution, where the isomorphism maps (C, ι, a) to $(C, \iota, a, \iota a)$. Hence that substack is smooth of relative dimension $2g - 1 + 1 = 2g$. Now, by the proposition $\mathcal{H}_g^n$ is the inverse image in $\mathcal{H}$ of the zero section of $J_g \to \mathcal{H}_{g,2}$ under the map ns and $J_g \to \mathcal{H}_{g,2}$ being smooth, the zero section is a local complete intersection map of codimension g. □

Remark 3. In characteristic 0 we shall show that the codimension is in fact g and that $\mathcal{H}_g^n$ is in fact smooth.

Fix n and g with $n \geq g + 1$ and consider $\mathbf{A} := \mathbf{A}^{2n+g+3}_{\mathbb{Z}[1/2]}$ that we shall regard as the parameter space for triples (P, Q, R) of polynomials of degrees n, $n - g - 1$, and $2g + 2$ respectively with R monic. We let $\mathcal{V}_g^n$ be the subscheme of triples that fulfill $P^2 - RQ^2 = 1$ and for which R is square free (i.e., its discriminant is invertible) and P and Q have invertible top coefficients. We let $\mathcal{U}_g^n$ be the subscheme of $\mathcal{V}_g^n$ defined by the condition that R is normalised and P and Q are monic. Over $\mathcal{V}_g^n$ we have an Abel curve given by

$$C := \operatorname{Proj} \mathcal{O}_{\mathcal{U}_g^n}[s, t, y]/\left(y^2 - t^{2g+2}R(s/t)\right) ,$$

where $\deg s = \deg t = 1$ and $\deg y = g + 1$, ι is given by $(s\colon t\colon y) \mapsto (s\colon t\colon -y)$, ∞_1 and ∞_2 are given by $(0\colon 1\colon 1)$ resp. $(0\colon 1\colon -1)$, and f is given by $(s\colon t\colon y) \mapsto (t^n(P(s/t) + yQ(s/t))\colon t^n)$. This therefore gives a map $\mathcal{V}_g^n \to \mathcal{H}_g^n$. We shall call any Abel curve that is a pullback of this family by a map to $\mathcal{V}_g^n$ a *Pell family* and if it is given as a pullback by a map to the closed subscheme $\mathcal{U}_g^n$ we shall call it a *normalised Pell family*.

Theorem 3.1. *i)* $\mathcal{V}_g^n \to \mathcal{H}_g^n$ *factors through the map* $\mathcal{H}_g^{s,n} \to \mathcal{H}_g^n$.

ii) Over Spec $\mathbb{Q}$ *the map* $\mathcal{V}_g^n \to \mathcal{H}_g^{s,n}$ *is a torsor under the subgroup of* $\mathbf{G}_m \times \mathrm{Aff}$, *where* Aff *is the group of affine transformations of the affine line, of pairs* $(\lambda, z \mapsto az + b)$ *for which* $\lambda^2 = a^{2g+2}$.

iii) Over Spec $\mathbb{Q}$ *the map* $\mathcal{U}_g^n \to \mathcal{H}_g^{s,n}$ *is a torsor under the subgroup of* $\mathbf{G}_m \times \mathrm{Aff}$ *of pairs* $(\lambda, z \mapsto az+b)$ *for which* $\lambda = a^{g+1}$, $a^n = 1$ *and* $b = 0$, *a group isomorphic to the group* μ_n *of n'th roots of unity.*

iv) In particular the map $\mathcal{V}_g^n \to \mathcal{H}_g^n$ *is a chart. i.e., smooth and surjective, and* $\mathcal{U}_g^n \to \mathcal{H}_g^n$ *is even an étale chart.*

Proof. To prove the first part we note that for a Pell family the involution on $\mathbb{P}^1 \times S$ compatible with f and ι is $x \mapsto 1/x$ whose fixed point scheme is ± 1 and by ordering it as $\{1, -1\}$ we get a family in $\mathcal{H}_g^{s,n}$.

Assume now that $(C \to S, \iota, \infty_1, \infty_2, f)$ is a family in $\mathcal{H}_g^{s,n}$. By assumption, using the notation of Definition 1, P is isomorphic to $\mathbb{P}^1 \times S$ in a way such that $\mathbf{0}$ on P is $0 \times S$ and ∞ is $\infty \times S$ and the involution σ is $x \mapsto 1/x$. Consider now the quotient D of C by ι. As 2 is invertible, taking the quotient by ι commutes with base change so that in particular $\pi\colon D \to S$ is a smooth proper map with genus 0 fibres. Furthermore, either of the sections ∞_1 or ∞_2 give a section ∞ of π. Now, again as 2 is invertible, the double cover $C \to D$ is given by a line bundle $\mathcal{M}$ on D and a section of $\mathcal{M}^2$. As $\mathcal{M}$ has degree $g+1$ on each fibre $\mathcal{M}(-(g+1)\infty)$ is the pullback from S of a line bundle $\mathcal{L}$.

We shall now show that giving an isomorphism of D with $\mathbb{P}^1 \times S$ taking ∞ to $\infty \times S$ and trivialising $\mathcal{L}$ is the same thing as giving a Pell family over S and an isomorphism with it and our split Abel curve. This will prove the second part and the third follows as the group of affine transformations is smooth.

In one direction it is clear as a Pell family gives by construction a trivialisation of D as well as $\mathcal{L}$.

For the converse we shall need to use (cf. Corollary 2 which assumes that we are over $\mathbb{Q}$) that $\mathcal{U}_g^n$ is smooth so that we may assume that S is smooth.[4]

Assume now that an isomorphism $D \xrightarrow{\sim} \mathbb{P}^1 \times S$ and a trivialisation of $\mathcal{L}$ has been given. This means that $\mathcal{M}$ is isomorphic to $\mathcal{O}(g+1)$ so that the section of $\mathcal{M}^2$ is a homogeneous form $R(s,t)$ of degree $2g+2$ with coefficients in $\Gamma(S, \mathcal{O}_S)$. The existence of the sections ∞_1 and ∞_2 show that $R(1,0)$ is a non-zero square and hence after scaling R we can assume that $R(s,1)$ is monic. Now, as the Abel curve is split we may regard f as an isomorphism $f\colon \mathcal{O}_C(\infty_1) \to \mathcal{O}_C(\infty_2)$ and then, again by the fact that the curve is split, $f \circ \iota^*(f)$ is scalar multiplication by a square and hence by scaling f we may assume that $f \circ \iota^*(f) = 1$. On $C^\circ := C \setminus \{\infty_1\} \cup \{\infty_2\}$ f maps into $\mathbf{G}_{m\,S}$ so that f is a unit in $\Gamma(C^\circ, \mathcal{O})$. This ring is equal to $\Gamma(S, \mathcal{O}_S)[s,y]/(y^2 - R(s,1))$ so that f has the form $P(s) + yQ(s)$ and the condition $f \circ \iota^*(f) = 1$ translates into $P^2 - RQ^2 = 1$. Now, if the base is a field it is easy to see that the degree of P is equal to the degree n and hence, as S is reduced P is of degree n and its top coefficient is a unit. The equation $P^2 - RQ^2 = 1$ and the fact that R is monic shows that Q has degree $n-g-1$ with invertible top coefficient, i.e., we have a map to $\mathcal{V}_g^n$. The possible changes in choices is given by a scaling factor λ, which is a unit in $\mathcal{O}_S$, in the choice of trivialisation of $\mathcal{L}$ and an affine transformation $s \mapsto as + b$ where $s \in \Gamma(S, \mathcal{O}_S^\times)$ and $b \in \Gamma(S, \mathcal{O}_S)$. This change takes y to λy and then $(P(s), Q(s), R(s))$ to $(P(as+b), \lambda^{-2}R(as+b), \lambda Q(as+b))$ so that if we want to keep R monic we need $\lambda^2 = a^{2g+2}$ which shows ii).

Turning to iii) we may after an étale extension which extracts an n'th root of the top coefficient compose with a change of trivialisation and affine transformation such that P is monic. As R is also monic this forces the top coefficient of Q to be ± 1 and if -1 we may change the trivialisation by -1 to get that Q is also monic. We may then by an appropriate affine transformation of the form $s \mapsto a + s$ assume that R is normalised, i.e., we have obtained an S-point of $\mathcal{U}_g^n$. The ambiguities in our choices are then reduced to a pair $(\lambda, s \mapsto as)$ with $a^n = 1$, $\lambda^2 = a^{2g+2}$ and $1 = \lambda a^{n-g-1}$ conditions which are equivalent to $a^n = 1$ and $\lambda = a^{g+1}$.

The last statement is now clear. □

Remark 4. Despite the very explicit form of these charts it seems difficult to use them. I have for instance not been able to show the smoothness of the moduli space

[4] In fact we only use that it is reduced.

using the Pell equation directly (in the generic case when R and Q have no common zeros it can be done).

We may use this result to show that $\mathcal{H}_g^{s,n}$ is almost a scheme by computing the fixed point sets for the action of subgroups of μ_n on $\mathcal{U}_g^n$. For this we introduce $\mathcal{W}_g^n$ as the closed subscheme of $\mathcal{V}_g^n$ consisting of tuples (P, R, Q) if $\mathcal{V}_g^n$ for which P and Q are monic.

Proposition 8. *Let $m > 1$ be an integer that divides n so that $\mu_m \subseteq \mu_n$. Then the fixed point locus for μ_m acting on $\mathcal{U}_g^n$ is empty unless $2g+2 \equiv 0, 1 \bmod m$.*

i) If $m|g+1$ then the fixed point scheme is of the form $(p(s^m), r(s^m), q(s^m))$, where $(p(t), r(t), q(t))$ is the universal family of $\mathcal{W}^{n/m}_{(g+1)/m-1}$.

ii) If $m|2g+2$ but $m\nmid g+1$ then the fixed point scheme is of the form $(p(s^m), r(s^m), s^{m/2}q(s^m))$, with $(p(t), r(t)t, q(t))$ the universal family (P, R, Q) of $\mathcal{W}^{n/m}_{(g+1)/m-1/2}$ restricted to the closed subscheme given by $R(0) = 0$.

iii) Assuming that $m|2g+1$ then the fixed point scheme is of the form $(p(s^m), sr(s^m), s^{(m-1)/2}q(s^m))$, where $(p(t), r(t)t, q(t))$ is the universal family (P, R, Q) of $\mathcal{W}^{n/m}_{(2g+1)/(2m)-1/2}$ restricted to the closed subscheme given by $R(0) = 0$.

Proof. If a tuple $(P(s), R(s), Q(s))$ is a point of $\mathcal{U}_g^n$ and ζ an m'th root of unity, then ζ takes the tuple to $(P(\zeta s), \zeta^{-2g-2}R(\zeta s), \zeta^{g+1}Q(\zeta s))$. Hence, that the tuple is fixed under μ_m, is equivalent to P, R, resp. Q being homogeneous of degrees $0, 2g+2$, resp. $-g-1$, where the grading takes values in $\mathbb{Z}/m\mathbb{Z}$ and s has degree 1. This means that for a tuple that is a fixed point, $R(s)$ is of the form $r(s^m)$. Furthermore, if k is the residue modulo m of $2g+2$, then s^k will be the lowest order non-zero monomial of R and as R does not have any multiple roots this implies that k is 0 or 1. Assume that $m|g+1$. Then $R(s)$ has the form $r(s^m)$ and $Q(s)$ has the form $q(s^m)$. Clearly, p, q, and r are all monic and as $p^2(s^m) - r(s^m)q^2(s^m) = 1$ we get $p^2(t) - r(t)q^2(t) = 1$ so that (p, r, q) gives a family in $\mathcal{W}^{n/m}_{(g+1)/m-1}$ and conversely such a family gives a fixed point $(p(s^m), r(s^m), q(s^m))$ (note that as $m > 1$ $p(s^m)$ is automatically normalised and that $r(t)$ is multiplicity free precisely when $r(s^m)$ is). Assume that $m|2g+2$ but $m\nmid g+1$. Then we still have $P(s) = p(s^m)$ and $R(s) = r(s^m)$, but $Q(s) = s^{m/2}q(s)$ and $P^2(s) - R(s)Q^2(s) = 1$ gives $p^2(t) - r(t)tq^2(t) = 1$ so that $(p(t), r(t)t, q(t))$ gives a family in $\mathcal{W}^{n/m}_{(g+1)/m-1/2}$ for which the R-component is 0 at 0. Finally if $2g+2 \equiv 1 \bmod m$ we get $P(s) = p(s^m)$, $R(s) = r(s^m)s$, and $Q(s) = s^{(m-1)/2}q(s)$, which gives $p^2(t) - r(t)tq^2(t) = 1$. □

Remark 5. By the arguments of the proof of Theorem 3.1 (and assuming we are in characteristic zero) $\mathcal{W}_g^n$ is isomorphic to $\mathbf{G}_a \times \mathcal{U}_g^n$ through affine translations $s \mapsto s + a$ in the polynomial variable. The subscheme defined by $R(0) = 0$ is by the same argument isomorphic to the finite étale cover of $\mathcal{U}_g^n$ whose S-object are (P, Q, R), an S-object of $\mathcal{U}_g^n$, together with a choice of zero of R.

4 Low Genera

It should come as no surprise that the cases of Abel curves of genus 0 and 1 are special and we start by treating them.

Proposition 9. *i)* $\mathcal{H}_0^n$ *is isomorphic to* $B\Sigma_2$ *with universal family having* $\mathbb{P}^1$ *as curve with hyperelliptic involution* $x \mapsto 1/x$, *function* $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ *given by* $x \mapsto x^n$ *and involution* $\sigma(x) = 1/x$. *The mapping to* $B\Sigma_2$ *giving the isomorphism is given by associating to an Abel curve the fixed point locus of its hyperelliptic involution.*

ii) Let $\mathcal{A}_1 \to \mathcal{M}_1$ *be the universal elliptic curve. Let* $\mathcal{U}$ *be the open substack of the fibre square of* $\mathcal{A}_1 \to \mathcal{M}_1$ *which is the complement of the diagonal and let* $\varphi\colon \mathcal{U} \to \mathcal{A}_1$ *be the map* $(x, y) \mapsto x - y$. *Then* $\mathcal{H}_1^n$ *is isomorphic to the inverse image of the kernel of multiplication by n by* φ

Proof. Starting with the genus zero case suppose we have a family of Abel curves of genus zero and degree n $(C \to S, f, \infty_1, \infty_2, \mathcal{L}, \varphi)$. Then $\mathcal{O}(\infty_1 - \infty_2)$ is the pullback of a (unique) line bundle $\mathcal{M}$ on S and the involution ι induces an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^{-1}$ (which identifies C and ι with the involutive bundle and involution associated to the obtained trivialisation of $\mathcal{M}^2$). Now, f corresponds to an isomorphism $\mathcal{O}(n\infty_1) \xrightarrow{\sim} \mathcal{O}(n\infty_2) \otimes \mathcal{L}$, i.e., an isomorphism $\mathcal{M}^{\otimes n} \xrightarrow{\sim} \mathcal{L}$ and the fact that $f \circ \iota = \sigma \circ f$, where σ is involutive involution, implies that φ equals the n'th power of the given trivialisation $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^{-1}$. This shows the whole Abel curve is determined by the involutive line bundle $\mathcal{M}$.

As for the genus 1 case we start by identifying the closed substack of $\mathcal{H}_{1,2}$ of triples (ι, a, b) with $\iota a = b$. In fact for any two disjoint sections a and b of a family of genus 1 curves there is a unique hyperelliptic involution that takes a to b, namely $x \mapsto -x + a + b$. This implies is isomorphic to $\mathcal{U}$ and the rest follows from Proposition 7. □

5 Hurwitz Type Description

If $f\colon C \to \mathbb{P}^1$ is a split Abel curve with hyperelliptic involution ι then $f \circ \iota = f^{-1}$. The map $\tau\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $\tau(x) = 1/2(x + x^{-1})$ is a quotient map for the action of the involution $x \mapsto x^{-1}$. We therefore get a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & D := C/\iota \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ \mathbb{P}^1 & \xrightarrow[\tau]{} & \mathbb{P}^1 \end{array}$$

and we see that we may recover C from the map g by taking the normalisation of its pullback along τ. The map f is then also determined. This gives the possibility of describing Abel curves in terms of maps of the form g. This is precisely what we are going to do in this section.

Definition 3. *An* Abel map *of genus g and degree n over a scheme S consists of*

- *a smooth proper map $\pi\colon P \to S$, the fibres of which are genus 0 curves,*
- *a section ∞ of π and an effective Cartier divisor C of P that is étale over S,*
- *an involutive line bundle $(\mathcal{L}, \varphi)$ over S with $\pi\colon Q \to S$ the associated projective bundle,*
- *an S-morphism $g\colon P \to Q$ fibrewise of degree n such that, $g^*\infty = n\infty$ as Cartier divisors, and*
- *a relative effective Cartier divisor $D \subset P$ such that $g^*\mathcal{F} = C + 2D$, where $\mathcal{F}$ is the fixed point scheme of φ which is an effective Cartier divisor.*

A split Abel map *is an Abel map together with a splitting of the involutive bundle.*

Remark 6. Note that g is flat so that g^* of Cartier divisors is well defined.

Given an Abel family $(f\colon X \to P, (\mathcal{L}, \varphi), \iota, \sigma)$ over a scheme S, where P is involutive bundle associated to the involutive line bundle $(\mathcal{L}, \varphi)$, we may consider the induced map $g\colon C/\iota \to P/\sigma$. As 2 is invertible, taking the quotient by an involution commutes with base change, so that $X/\iota \to S$ is a smooth genus 0 fibration, whereas P/σ is the involutive quotient and hence is isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{L})$. Consider now the induced map $X \to P \times_{P/\sigma} X/\iota$. The composite with it and the projection $P \times_{P/\sigma} C/\iota \to X/\iota$ is the quotient map and both $X \to X/\iota$ and $P \times_{P/\sigma} X/\iota \to X/\iota$ are double covers. As such they are specified by line bundles $\mathcal{M}$ and $\mathcal{N}$ and sections s and t of $\mathcal{M}^{-2}$ resp. $\mathcal{N}^{-2}$. The map $C \to P \times_{P/\sigma} C/\iota$ corresponds to a map $\mathcal{N} \to \mathcal{M}$ compatible with the sections of $\mathcal{M}^{-2}$ and $\mathcal{N}^{-2}$. The map $\mathcal{N} \to \mathcal{M}$ defines a relative Cartier divisor D as it defines a Cartier divisor on each fibre (over S). Let C be the divisor of s and note that the divisor of t is the pullback by g of the divisor of the involutive quotient map, i.e., $\mathcal{F}$, where $\mathcal{F}$ is the involutive locus. The compatibility between the coverings then gives that $g^*\mathcal{F} = C + 2D$, and as $g^*\infty = n\infty$ as $f^*\infty = n\infty_1$, we have an Abel map. Finally, as X is smooth, C is étale over S. This construction can be reversed.

Proposition 10. *The stack of Abel curves is isomorphic to the stack of Abel maps.*

Proof. We have just defined a map in one direction. Conversely, assume given an Abel curve and using the notations of Definition 3 we recall that the involutive quotient map is given by $\mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{L})}(-1)$ and the section $\otimes 1 \oplus \varphi(1)$ of $\mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{L})}(2)$ whose Cartier divisor is the involutive locus $\mathcal{F}$. The pullback of it by g is then given by $\mathcal{N} := g^*\mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{L})}(-1)$ and the Cartier divisor $g^*\mathcal{F}$. If we put $\mathcal{M} := \mathcal{N}(D)$ then by assumption the section of $\mathcal{N}^{-2}$ comes from one of $\mathcal{M}^{-2}$ and hence gives a double covering $X \to P$ that maps to the g-pullback of the involutive double cover and X is smooth as C is étale and 2 is invertible. This gives an inverse map. □

In the future we shall pass freely back and forth between Abel maps and Abel curves.

6 Infinitesimal Calculations

In this section we shall study the deformation theory of Abel maps (and hence of Abel curves). To avoid problems with wild ramification (and worse still, inseparable maps) we shall from now on assume that all our schemes and stacks are over Spec $\mathbb{Q}$.

If $(p_1(x), p_2(x), \dots, p_n(x))$ is a sequence of monic polynomials over S (i.e., with coefficients in $\Gamma(X, \mathcal{O}_X)$) then we may put X and Y equal to Spec $(\oplus_i \mathcal{O}_S[[x]])$ and let f be given by $x \mapsto p_i$ on the i'th component and we shall refer to the ramification and branch stratifications associated to f as the ramification resp. branch stratifications of the sequence $(p_1(x), p_2(x), \dots, p_n(x))$.

Definition 4. *Let $T = (S, S_1, S_2)$ be a sequence of disjoint finite sets and r a function from S', the disjoint union of the components of T to the positive integers; for $i = 1, 2$ let S_i^e and S_i^o be the subsets of S_i where r takes even resp. odd values and set*

$$n := \sum_{s \in S}(r(s) - 1) + \sum_{s \in S_1^e \dot{\cup} S_2^e} (r(s)/2 - 1) + \sum_{s \in S_1^o \dot{\cup} S_2^o} (r(s) - 1)/2.$$

We define $\mathcal{P}(T)$ to be the affine space $\mathbf{A}^n$ seen as the parameter space of tuples $(p_s)_{s \in S'}$ where p_s is a normalised polynomial of degree $r(s)$ if $s \in S$, a normalised polynomial of degree $r(s)/2$ if $s \in S_1^e \dot{\cup} S_2^e$ and a monic polynomial of degree $(r(s) - 1)/2$ if $s \in S_1^o \dot{\cup} S_2^o$. Despite this interpretation we shall continue to refer to the origin as the origin.

To a point (p_s) of $\mathcal{P}(T)$ we associate the tuple $(q_s)_{s \in S'}$, where $q_s = p_s$ if $s \in S$, $q_s = p_s^2$ if $s \in S_1^e \dot{\cup} S_2^e$ and $q_s(x) = (x - a_s)p_s^2(x)$ with a_s being twice the next to highest coefficient of p_s. (Thus q_s is always a normalised polynomial of degree $r(s)$.) The ramification and branch stratifications of the sequence (q_s) associated to the tautological sequence will be referred to as simply the ramification resp. branch stratification of $\mathcal{P}(T)$.

We have the following characterisation of the points of a stratum.

Proposition 11. *Let $f : Y \to X$ be a finite map of (possibly formal) smooth 1-dimensional schemes over a field $\mathbf{k}$. Then the corank of the trace map of a closed point s of the ramification locus of f is equal to the ramification index at s minus 1 and the corank of the trace map of a closed point s of the branch locus of f is equal to the sum of the ramification indices of points of the fibres over s of f minus the number of points of the fibre.*

Proof. This is clear for the ramification locus. For the branch locus it follows from the fact that locally at a closed point branch divisor is the sum of the norms of the ramification divisors at the points of the fibres and that the norm of a closed point considered as a divisor equals to the image point which is seen by looking at valuations of a defining element. □

We are now ready to give a description of the deformation theory of Abel maps (and equivalently Abel curves). To simplify descriptions, for an Abel map $(f : P \to Q, C, D)$ over a base S by its *assigned branch points* we shall mean the divisor of Q which is the sum of the involutive locus and the ∞-divisor.

Theorem 6.2. *Let* $(f: P \to Q, C, D)$ *be a split Abel map over an algebraically closed field* $\mathbf{k}$. *Let* $S \subset P(\mathbf{k})$ *be the ramification points that do not map to the assigned branch points, let* S_1^e *and* S_2^e *resp.* S_1^o *and* S_2^o *be the ramification points over* $(1: 1)$ *and* $(-1: 1)$ *with even resp. odd ramification index (wrt to* f*) and let* $T := (S, S_1^o, S_1^e, S_2^o, S_2^e)$. *Finally, let* r *associate to a point its ramification index wrt the map* f. *Then the completion of the local ring of the stack of Abel maps at the Abel map is isomorphic to the completion of the local ring of* $\mathcal{P}(T)$ *at the origin and the isomorphism may be assumed to be stratification preserving.*

Proof. We shall give an isomorphism of deformation functors so we consider a deformation of the given Abel map over a local Artinian ring R with residue field $\mathbf{k}$. Note that as 2 is invertible, the involutive bundle has just the trivial deformation so we may restrict ourselves to split Abel maps. If we just consider deformations of the map f, then Proposition 2 shows that such deformations are in bijection with tuples $(q_s)_{s\in S'}$, S' being as in Definition 4, where q_s is a normalised polynomial over R of degree $r(s)$. It remains to understand the influence the choice of relative Cartier divisors has. Now, the Weierstrass preparation theorem is equivalent to saying that the ideal of a relative Cartier divisor of $\operatorname{Spec} R[[x]] \to \operatorname{Spec} R$ is generated by a unique Weierstrass polynomial (i.e., of the form $x^n + a_1x^{n-1} + \cdots + a_0$ with $a_i \in \mathfrak{m}_R$) and it is clear from the uniqueness that inclusion of divisors corresponds to divisibility of polynomials and addition of divisors corresponds to product of divisors.

Hence for $s \in S_1^o \dot\cup S_1^e \dot\cup S_2^o \dot\cup S_2^e$ the inverse image of the assigned branch points at s is defined by q_s, C_s being étale is defined by a polynomial of degree 0 or 1 depending on whether the degree of q_s is odd or even (as the difference is even). In the even case, if p_s is the polynomial of D_s we have $q_s = p_s^2$ and p_s is normalised as q_s is. In the odd case, if C_s is given by $x - a_s$ and D_s by p_s we have that $q_s = (x - a_s)p_s^2$ and as q_s is normalised we have that a_s is twice the next to highest coefficient of p_s. This shows that the q_s for $s \in S$ and the p_s for s in the complement gives an R-point of $\mathcal{P}(T)$ and the converse is also clear. The definition of the stratification of $\mathcal{P}(T)$ has been set up so that the constructed isomorphism preserves the strata. □

Remark 7. The assumption of an algebraically closed field as base is just for notational convenience as is the existence of a splitting.

We put the most important consequences of this theorem in the following corollary. Note that we have identified the stack of Abel maps with that of Abel curves.

Corollary 2. *i)* $\mathcal{H}_g^n$ *is a smooth stack everywhere of dimension* g.

ii) The open substack of $\mathcal{H}_g^n$ *consisting of Abel maps with only simple ramification (i.e., all ramification indices are* ≤ 2*) and for which for all branch points outside of the assigned branch points there is only one ramification point above it, is dense.*

Proof. The map $\mathcal{H}_g^{s,n} \to \mathcal{H}_g^{s,n}$ is an étale cover so we may deal with the split case instead. The smoothness follows immediately from the theorem and we postpone the calculation of the dimension. For the second part we may complete the local ring at a point and then transfer the problem to the complete local ring at the origin of $\mathcal{P}(T)$. We shall now show that the set of $\mathcal{P}(T)$ where the corank of the trace form is 0 is non-empty and by Proposition 11 it is enough to show that generically on $\mathcal{P}(T)$ the derivative of each q_s has no multiple roots. If $s \in S$ this is clear as then q_s is a generic monic polynomial and then so is $1/nq_s'$, where n is the degree of q_s. If $s \in S_i^e$, $i = 1, 2$, then $q_s = p_s^2$ where p_s is a generic monic polynomial and thus $q_s' = 2p_s p_s'$. Generically p_s and p_s' have no roots in common, p_s has no double roots and neither has p_s' by the argument just given. If $s \in S_i^e$ then $q_s = (x - a_s)p_s^2$ where p_s is a generic monic polynomial with next to highest coefficient a_s. Then we have $q_s' = p_s(p_s + 2(x - a_s)p_s')$. Again, generically p_s has no double roots. Roots that are common to p_s and $p_s + 2(x - a_s)p_s'$ are also roots of either $x - a_s$ but generically a_s is not a root of p_s or of p_s' which again is not the case generically. We are left with showing that generically $p_s + 2(x - a_s)p_s'$ has no double roots. Now, $p_s + 2(x - a_s)p_s'$ divided by $2n + 1$, n being the degree of p_s, is a generic monic polynomial. Indeed, it is easily seen that the coefficients of p_s can be expressed as polynomials in those of $p_s + 2(x - a_s)p_s'$.

We have thus shown that generically all ramification points are simple and it remains to show that away from the assigned branch points there is generically only one ramification point above one branch point. For this we note that for a given $s \in S$ the contribution from that ramification point to the branch locus is defined by the norm of q_s' and hence what needs to be shown is that for two $s, s' \in S$ that map to the same point under f, the two norms do not have a common component. Now q_s and $q_{s'}$ are generic polynomials with independent coefficients. Hence the locus defined by the common components would have to be independent of both the coefficients of q_s and $q_{s'}$ (and of course only depend on their union) and would hence have to be constant. At the origin the full ramification loci consist just of 0 and so the common locus would have to be 0 everywhere. However, q_s has generically no factor in common with q_s'.

Finally, to compute the dimension we may by what has just been proved, look only at the case where all the ramification is simple and outside of the assigned branch points there is only one ramification point over a given branch point. We may also assume that the Abel curve is split. Now, if $s \in S$ is a ramification point, then the local deformation at that point depends on one parameter as q_s is a normalised second degree polynomial, whereas for a ramification point over the assigned branch points the local deformation at that point depends on zero parameters, as p_s is a normalised first degree polynomial. Hence the dimension is equal to the cardinality of S. Let now e_i, $i = 1, 2$, be the number of ramification points over ± 1, let $e' := e_1 + e_2$, and let e be the number of ramification points not above ± 1. By the Hurwitz formula applied to f we have

$$-2 = -2n + n - 1 + e' + e$$

and by the Hurwitz formula applied to double covering ramified at the non-ramification points over ± 1 we have

$$2g - 2 = -4 + (2n - 2e')$$

and elimination gives $e = g$. □

7 The Lyashko–Looijenga Map

By the Lyashko–Looijenga map is generally meant the map that to a family of finite maps between smooth curves associates the branch locus of each member. Sometimes one restricts oneself to families where the trace corank of the branch locus is constant and then it is natural to consider the fibrewise reduced subscheme of the branch locus. Furthermore, sometimes some of the branch points are by assumption fixed and then of course it is natural to exclude them from consideration. Our situation is of this type as the involutive fixed points are essentially fixed (i.e., they can not move non-trivially in a continuous fashion) and actually fixed in the split case.

We shall see that the situation is not completely straightforward; our strata on which the LL-map is defined will generally turn out to be non-reduced which certainly kills all hope of the LL-map being étale. All is not lost, however, as a stratum is locally the product of a smooth stack and a zero-dimensional one and the LL-map turns out to be étale on the reduced substack. The most obvious reason for the stratum being non-reduced is our definition of the branch locus. This definition is, however, more or less forced upon us if one wants the branch divisor to vary continuously (i.e., be a relative Cartier divisor) as generically the branch divisor is étale and hence determined by the condition that its support be the branch locus.

Definition 5. *i) A* ramification specification of degree *n consists of a finite multiset S of multisets of (strictly) positive integers such that for each multiset s in S $\sum_{e\in s} e = n$. The* multiplicity multiset *associated to S is the multiset $\{\rho(s) \mid s \in S\}$, where $\rho(s) = \{e-1 \mid e \in S, e > 1\}$. The* total ramification *of S is*

$$\sum_{e\in s\in S} (e-1).$$

ii) An Abel ramification specification of order *n is a ramification specification S of order n whose total ramification equals $n-1$ together with the choice of a sub-multiset T of S of cardinality 2. If t is the number of odd integers, counted with multiplicity, of the members of T, then the* genus *of S is equal to $(t-4)/2$.*

Remark 8. A ramification specification is determined by its associated multiplicity multiset and the degree n. A multiplicity multiset is the same as a *passport* of [4].

We shall now consider stratifications of $\mathcal{H}_g^n$. First we consider the ramification stratification of the universal map of $\mathcal{H}_g^n$ giving a multiset of multiplicities associated to each stratum. Then we consider its intersection with the trace stratification associated to the branch locus which gives a further division of the multiplicities according to which branch point they are mapped to. This gives exactly an Abel ramification specification S of order n associated to each such stratum. Conversely, for each Abel ramification specification S of order n we denote by $\mathcal{H}_{g,S}^n$ the corresponding stratum.

If $X \to S$ is a map of algebraic stacks and m a positive integer then $\mathrm{Conf}^m(X/S)$ (or just $\mathrm{Conf}^m(X)$ if S is understood) is the *m-point configuration space*, i.e., the stack quotient by the permutation action of the symmetric group Σ_m on the open substack of the m'th fibre power of $X \to S$ consisting of distinct points.

Our main use of this construction is to the universal involutive projective bundle; namely the projective bundle $\mathbb{P} \to B\Sigma_2$ that to an involutive line bundle over S (i.e., a map $S \to B\Sigma_2$) associates the involutive projective bundle. We then let $\mathbb{P}' \to B\Sigma_2$ be the open substack of $\mathbb{P}$ obtained by removing the section of infinity and the involutive fixed point set. Note that the universal involutive line bundle of $\mathcal{H}_g^n$ gives a map $\mathcal{H}_g^n \to B\Sigma_2$ and the base of the universal Abel map is just the pullback of $\mathbb{P}$ under this map.

Definition 6. *Let S be an Abel ramification specification (S, T) of order n and let m be the cardinality of S minus 2. We define the Lyashko–Looijenga map $LL\colon \mathcal{H}_{g,S}^n \to$ $\mathrm{Conf}^m(\mathbb{P}'/B\Sigma_2)$ by associating to an Abel map over S its reduced branch locus minus assigned base points.*

We can now prove the major result on the LL-map after we have proven the following lemma.

Lemma 1. *Let R be a commutative ring which contains $\mathbb{Q}$ and $a \in R$. Then for a strictly positive integer n the polynomial $(t^{2n+1} - a^{2n+1})/(t-a)$ is a square of a polynomial precisely when $a^{n+1} = 0$.*

Proof. In the ring of Laurent power series in t^{-1}, $R((t^{-1}))$ the polynomial has the unique square root

$$t^n\sqrt{1 + at^{-1} + \cdots + a^{2n}t^{-2n}}$$

and hence the polynomial has a polynomial square root precisely when all powers beyond t^{-n} have zero coefficients in $\sqrt{1 + at^{-1} + \cdots + a^n t^{-2n}}$. This series is obtained by substituting $s \mapsto at^{-1}$ in $\sqrt{(1-s^{2n+1})/(1-s)}$ which makes it clear that if $a^{n+1} = 0$ then the square root is a polynomial. It is equally clear that the converse is true if the coefficient of s^{n+1} in $\sqrt{(1-s^{2n+1})/(1-s)}$ is non-zero. However, as $n > 0$, $2n+1 \geq n+2$, and thus modulo s^{n+2} $\sqrt{(1-s^{2n+1})/(1-s)}$ is congruent to $(1-t)^{-1/2}$ which clearly has all of its coefficients non-zero. □

Theorem 7.3. *Let S be an Abel ramification specification (S, T) of order n and let m be the cardinality of S minus 2.*

i) The completion of $\mathcal{H}_{g,S}^n$ at any geometric point $s = \mathrm{Spec}\,\mathbf{k}$ is isomorphic to

$$\prod_{\substack{2n+1 \in t \in T \\ n>0}} \mathrm{Specf}\,\mathbf{k}[[a]]/\left(a^{n+1}\right) \times \prod_{s \in S\setminus T} \mathrm{Specf}\,\mathbf{k}[[\sigma, a_1, \ldots, a_{m(s)}]]/(\sigma_1, \ldots, \sigma_e)$$

where $m(s) := |\{e \mid e \in s; e \geq 2\}|$, $e(s) := \sum_{e \in s}(e-1)$ and

$$\prod_i (s - a_i)^{e_i - 1} = s^e + \sum_{1 \leq j \leq e} (-1)^j \sigma_j s^{e-j}$$

as polynomials in s. In particular, $\mathcal{H}^n_{g,S}$ is smooth precisely when there is exactly one ramification point over each unassigned branch point and no ramification point of odd ramification index above involutive fixed points. It is always the case that the reduced substack $(\mathcal{H}^n_{g,S})^{\mathrm{red}}$ is smooth.

ii) The Lyashko–Looijenga map $LL\colon (\mathcal{H}^n_{g,S})^{\mathrm{red}} \to \mathrm{Conf}^m(\mathbb{P}'/B\Sigma_2)$ restricted to the reduced subscheme is an étale covering map.

Proof. We start by making a local calculation. It is clear from Theorem 6.2 that we get a product over the elements of S. Let us first consider an unassigned branch point. Let $\{e_1, \ldots, e_k\}$ be an element of $S \setminus T$ with the members equal to 1 removed. Hence a deformation over a local Artinian ring R is given by a collection $(p_i)_{1\le i\le k}$ of normalised polynomials with deg $p_i = e_i$. Now the condition, that the deformation stay inside the stratum given by $\{e_1, \ldots, e_k\}$ means, according to Proposition 4 and the identification of Cartier divisors with Weierstrass polynomials, that each p'_i has the form $e_i(x - \alpha_i)^{e_i-1}$ and as p_i is normalised we get that $\alpha_i = 0$ and hence that $p_i(x) = x^{e_i} + b_i$. Furthermore the ramification divisor is defined by x^{e-1}. To compute the branch divisor we have to compute the norm of x^{e-1}, and using the multiplicativity of the norm it is enough to compute the norm of x. Now, it is clear that under the map $R[[t]] \to R[[x]]$ given by p_i we have that $R[[x]]$ is isomorphic to $R[[t, x]]/(x^{e_i} + b_i - t)$ which gives that the norm of x is $\pm(t - b_i)$. Hence the branch divisor is given by $\prod_i (t - b_i)^{e_i-1}$. Now, we are working in the stratum where the fibrewise reduced branch divisor exists, which means that there is a $\sigma \in \mathfrak{m}_R$ such that $\prod_i (t - b_i)^{e_i-1} = (t - \sigma)^e$, where $e = \sum_i (e_i - 1)$. Comparing next to highest coefficients gives $\sigma = \sum_i (e_i - 1)b_i$ and changing variable $s = t - \sigma$ and putting $a_i = b_i - \sigma$ gives us $\prod_i (s - b_i)^{e_i-1} = s^e$. This shows that the universal R is $\mathbf{k}[[\sigma, a_1, \ldots, a_k]]/(\sigma_1, \ldots, \sigma_e)$. Now, for degree reasons, as soon as $k > 1$, $\mathbf{k}[[\sigma, a_1, \ldots, a_k]]/(\sigma_1, \ldots, \sigma_e)$ is strictly larger than $\mathbf{k}[[\sigma]]$. On the other hand putting s equal to a_i gives $a_i^e = 0$, which shows that when dividing out by the nilradical of $\mathbf{k}[[\sigma, a_1, \ldots, a_k]]/(\sigma_1, \ldots, \sigma_e)$, this ring equals $\mathbf{k}[[\sigma]]$.

Considering now instead one of the involutive fixed points again as we are in a fixed ramification stratum, we get the form $p_i = x^{e_i} + b_i$. This time, however, we have that when e_i is even, p_i is a square and when it is odd, p_i is a square times a linear polynomial. In the first case it is easy to see that if $x^{e_i} + b_i$ is a square, then $a_i = 0$. In the second case, if $x^{e_i} + b_i = (x - a_i)q^2(x)$, then setting $x = a_i$ we get $b_i = -a_i^{e_i}$, so that $(x^{e_i} - a_i^{e_i})/(a - a_i) = q^2(x)$ and we conclude from Lemma 1 that this is possible precisely when $a_i^{(e_i+1)/2} = 0$. This concludes the proof of i).

Turning to ii), that the map is étale is clear from the local calculation, as the fibrewise reduced branch divisor is defined by $t - \sigma$, using the notations of the first part. It remains to prove that it is proper and for that we shall use the valuative criterion, and as everything is of finite type over Spec $\mathbb{Q}$ we may restrict ourselves to discrete valuations, which we may assume to be strictly Henselian. Hence we may assume that the map is split and by Theorem 3.1 we may assume that it is given by a Pell family (P, Q, R) such that $P^2 - RQ^2 = 1$, and P is then the Abel map in question. By for instance [4, Lemma 3.1] (and the fact also noted in [loc. cit.] that the inverse image of the origin under the LL-map is the origin) P has coefficients in R.

By Gauss' lemma so does R and Q. The next step is to show that the discriminant of R is a unit. For this one may reduce modulo the maximal ideal of R and apply the Hurwitz formula to the map given by P. Indeed, by assumption the number of branch points of P is fixed and hence by Hurwitz' formula the number of ramification points is also fixed. This makes it impossible for zeros of R to come together. □

Remark 9. i) The local description of the stratum contradicts [3, Prop. A.3] which claims that the LL-map always is étale. In view of the theorem (very slightly modified to fit into the context of [loc. cit.]) this is now seen to be false when there is more than one ramification point over a branch point. It thus has to be modified to saying that the restriction to the reduced subscheme of an equisingular stratum is étale. Luckily, this is what is used in the main text and it is also given a topological proof in [3, Thm. 4.2].

ii) It is possible to get a natural interpretation of the reduced structure on the strata. This will be treated elsewhere.

8 Topological Construction

In this section we shall study the covering given by the LL-map. Even though we can easily get a description for all strata, using the same methods, we shall only deal with the open stratum as that gives a combinatorial algorithm for computing the number of connected components of the stack of Abel curves. Note however that other strata are also interesting. For instance the lowest stratum where all the branch points are assigned has been considered in connection with Grothendieck's "dessin d'enfants" (cf., [5]).

As usual the fibres of the LL-map are in bijection with conjugacy classes of certain sequences of the symmetric group. In order to give a procedure for computing the number of components of a stratum we need generators for the fundamental group of the appropriate configuration space. The following result can most certainly be extracted from the literature but for the convenience of the reader as well as the author we give a proof. We start by giving some notation. If C is a simple non-closed oriented curve in $\mathbb{C}$ and $S \subset C$ is a finite set then the orientation of C induces a total order on S. If i is a positive integer strictly smaller than the number s of elements of S then we define, as usual, the elements σ_i of the braid group on s strands given by letting the i'th point move along C to the position of the $i+1$'st point to the right of C and letting the $i+1$'st point move along C to the position of the i'th point to the left of C ("right" and "left" being from the point of view of the orientation of C). If $s \geq 3$ and we define τ_1 resp. τ_2 to be the braids that takes the second resp. $s-1$'st point and moves along C on the left resp. right hand side till just before the first resp. last point, then circles that point once counter-clockwise, and returns back to its original position along the right resp. left hand side of C. (They are equal to σ_1^2 resp. σ_{s-1}^{-2}.)

Proposition 12. *Let $C := [-1, 1]$ oriented in any direction and $S \subset C$ a finite subset with s elements containing ± 1, and let $A := \mathbb{C} \setminus \{\pm 1\}$. Then the map induced by the inclusion* $\{x_1, \ldots, x_{s-2}\} \mapsto \{-1, 1, x_1, \ldots, x_{s-2}\}$

$$\pi_1\left(\mathrm{Conf}^{s-2}(A), S \setminus \{\pm 1\}\right) \to \pi_1(\mathrm{Conf}^s(\mathbb{C}), S)$$

is an injection whose image is generated by σ_i, $2 \leq i \leq s-2$, τ_1 *and* τ_2.

Proof. By possibly applying $z \to -z$ we may assume that the orientation of $[-1, 1]$ is such that -1 becomes its first element.

To begin with it is clear the σ_i, $2 \leq i \leq s-2$, τ_1 and τ_2 lie in the image. Recall that we have a surjection $\pi_1(\mathrm{Conf}^s(\mathbb{C}), S) \to \Sigma_s$ taking σ_i to the transposition $(i-1, i)$. The image of $\pi_1(\mathrm{Conf}^A(s-2), S \setminus \{\pm 1\})$ maps into Σ_{s-2}, considered as the subgroup that fixes the first and last elements, and as σ_i maps to $(i-1, i)$, the subgroup generated by them maps surjectively onto Σ_{s-2}. Hence both for injectivity and generation it suffices to consider $\pi_1(\widetilde{\mathrm{Conf}}^{s-2}(A), S\setminus\{\pm 1\}) \to \pi_1(\widetilde{\mathrm{Conf}}^{s}(\mathbb{C}), S)$, where $\widetilde{\mathrm{Conf}}^{t}(X)$ is the space of ordered t-subsets of X, and to show that the map is injective and the image is generated by the conjugates of σ_i^2, $1 \leq s \leq s-1$ in the group generated by $2 \leq i \leq s-2$, τ_1 and τ_2.

Now, by conjugating by the σ_i, $2 \leq i \leq s-1$, we can get from τ_1 and τ_2 all braids A_i^1 and A_i^2 defined like τ_1 resp. τ_2 only starting at the i'th point for $2 \leq i \leq s-1$, as well as the A_{ij}, $2 \leq i < j \leq s-2$, defined like τ_1 only starting at the j'th point and encircling the i'th point. (The A_{ij} are the A_{ij} of [2, 1-11], A_i^1 is A_{1i} and A_i^2 is a mirror image of $A_{s-i,s}$.) Our aim is to show the injectivity and that these elements generate the image. In this we shall follow the proof of [2, Lemma 1.8.2] and we start following [2] in using the notation $F_{m,n}(X)$ for $\widetilde{\mathrm{Conf}}^{n}(X \setminus Q_m)$ where Q_m is a fixed subset of X of cardinality m and will use of the theorem of Fadell and Neuwirth (cf. [2, Thm. 1.2]) which says that when X is a manifold, then the projection on the first r factors $F_{m,n} \to F_{m,r}$ is a fibration with fibre $F_{m+r,n-r}$. Applied to $r = n-1$ and $X = \mathbb{C}$ and $X = A$ this will allow us to prove the statement by induction. As the involved spaces are acyclic, the fibrations give short exact sequences, and by induction we are reduced to showing that for $1 \leq i < s$

$$\pi_1\big(A \setminus S_i, x_{i+1}\big) \to \pi_1\big(\mathbb{C} \setminus S_i, x_{i+1}\big),$$

where S_i consists of the i first elements of S and x_{i+1} is the $i+1$'st element, is an injection and that the image is contained in the subgroup generated by $A_{k,i+1}$ and A_{i+1}^1 (and when $i = s-1$ also the A_k^2) for $1 \leq i \leq k$. This however is clear. □

This result, combined with Theorem 7.3 and Corollary 2 ii), allows us to give a combinatorial description of the number of components of $\mathcal{H}_g^n$ and $\mathcal{H}_g^{s,n}$. For this we first introduce the following definition.

Definition 7. *Let* $N_{g,n}$ *be the set of tuples* $(\sigma, \sigma_1, \ldots, \sigma_g, \tau) \in (\Sigma_n)^{g+2}$ *fulfilling the conditions*

- $\sigma\sigma_1 \cdots \sigma_g \tau$ *is an* n*-cycle and*
- *the* σ_i *are transpositions and* σ *and* τ *are products of disjoint transpositions and the sum of the number of fixed points of* σ *and of* τ *equals* $2g+2$.

Let $M_{g,n}$ be the set of orbits of the action of Σ_n on $N_{g,n}$ given by

$$(\rho, (\sigma, \sigma_1, \ldots, \sigma_g, \tau)) \mapsto \left(\rho\sigma\rho^{-1}, \rho\sigma_1\rho^{-1}, \ldots, \rho\sigma_g\rho^{-1}, \rho\tau\rho^{-1}\right).$$

Thus armed we can give a combinatorial description of the set of connected components of the stacks of (split) Abel curves.

Theorem 8.4. *The set of connected components of $\mathcal{H}_g^{s,n}$ is in bijection with equivalence classes of $M_{g,n}$ under the equivalence relation generated by the relations.*

- $(\sigma, \sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_g, \tau) \sim (\sigma, \sigma_1, \ldots, \sigma_i\sigma_{i+1}\sigma_i^{-1}, \sigma_i, \ldots, \sigma_g, \tau)$ *for all* $1 \le i < g$.
- $(\sigma, \sigma_1, \ldots, \sigma_g, \tau) \sim (\sigma[\sigma_1, \sigma], \sigma\sigma_1\sigma^{-1}, \ldots, \sigma_g, \tau)$, *with* $[\sigma_1, \sigma] = \sigma_1\sigma\sigma_1^{-1}\sigma^{-1}$.
- $(\sigma, \sigma_1, \ldots, \sigma_g, \tau) \sim (\sigma, \sigma_1, \ldots, \tau^{-1}\sigma_g\tau, [\tau^{-1}, \sigma_g^{-1}]\tau)$.

The set of connected components of $\mathcal{H}_g^n$ is in bijection with equivalence classes of $M_{g,n}$ under the equivalence relation generated by the above relations together with the relation

$$(a_1, a_2, \ldots, a_{g+2}) \sim (b_{g+2}a_{g+2}b_{g+2}^{-1}, \ldots, b_2a_2b_2^{-1}, a_1),$$

where $b_i = a_1 \ldots a_{i-1}$ for $i \ge 2$.

Proof. The part on $\mathcal{H}_g^{s,n}$ follows directly from the fact that the LL-map is an étale covering (Theorem 7.3), that the fibres of the LL-mapping are in bijection with $M_{g,n}$, the description of the generators for the fundamental group for the target of the LL-map (Proposition 12) and the formula for the action of the σ_i on $M_{g,n}$.

As for the $\mathcal{H}_g^n$-part the Lyashko–Looijenga map has as target the quotient of $\mathrm{Conf}^g(\mathbf{A}^1 \setminus \{\pm 1\})$ divided by the map induced by $z \mapsto -z$. Hence we have to add the relation that identifies an equivalence class of maps from the fundamental group to Σ_n with the one obtained by composing with the action of the (outer) automorphism induced by $z \mapsto -z$. For that we choose as basepoint of $\mathrm{Conf}^g(\mathbf{A}^1 \setminus \{\pm 1\})$ the set $\{-1/2, -1/3, \ldots, 1/3, 1/2\}$ (with 0 included if g is odd) and as basepoint for $\mathbf{A}^1 \setminus \{-1, -1/2, -1/3, \ldots, 1/3, 1/2, 1\}$ i. Acting by $z \mapsto -z$ gives us $-i$ as new basepoint and we identify fundamental groups by choosing a curve from $-i$ to i going to the left of $\{-1, -1/2, -1/3, \ldots, 1/3, 1/2, 1\}$. □

References

[1] N. H. Abel, Sur l'intégration de la formule différentielle $\frac{\varrho\, dx}{\sqrt{r}}$, r et ϱ étant des fonctions entiéres. *J. Reine Angew. Math.* 1:105–144, 1826.

[2] J. S. Birman, *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Sudies, No. 82.

[3] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves. *Invent. Math.* 146(2):297–327, 2001.

[4] S. K. Lando and D. Zvonkin, On multiplicities of the Lyashko–Looijenga mapping on strata of the discriminant. *Funktsional. Anal. i Prilozhen.* 33(3):21–34, 1999.

[5] G. Shabat and A. Zvonkin, Plane trees and algebraic numbers. In *Jerusalem combinatorics '93*, volume 178 of *Contemp. Math.*, pages 233–275. Amer. Math. Soc., Providence, RI, 1994.