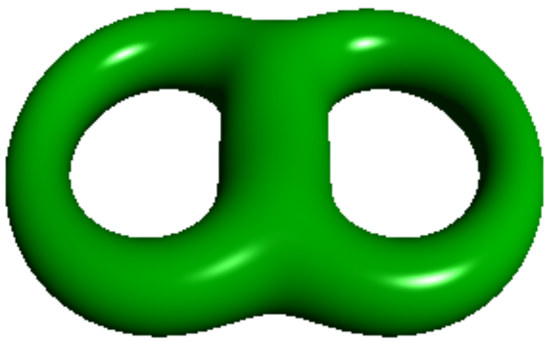
Hyperelliptic Curves and Cusps

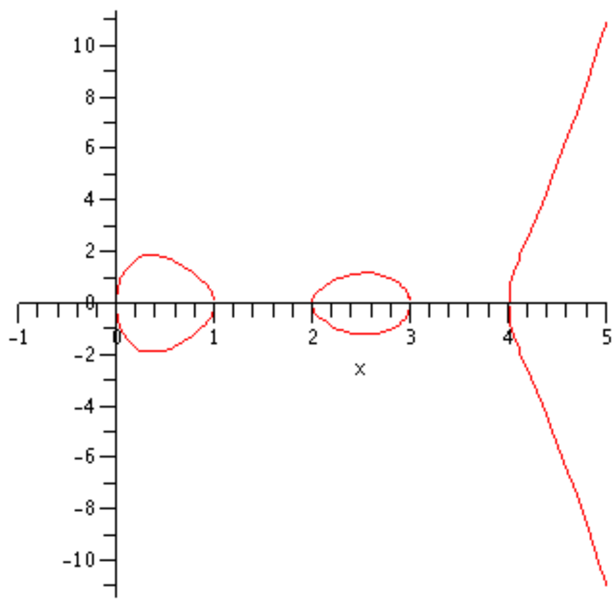
A *hyperelliptic curve* is defined by an equation $y^2 = p(x)$ where $p(x)$ is a polynomial of degree at least 5 with distinct roots. The last condition guarantees it's nonsingular. General theory tells us that this would have large genus (i.e. many holes). For example,

$$y^2 = x(x-1)(x-2)(x-3)(x-4)$$

should be a genus two surface or "two-holed doughnut".

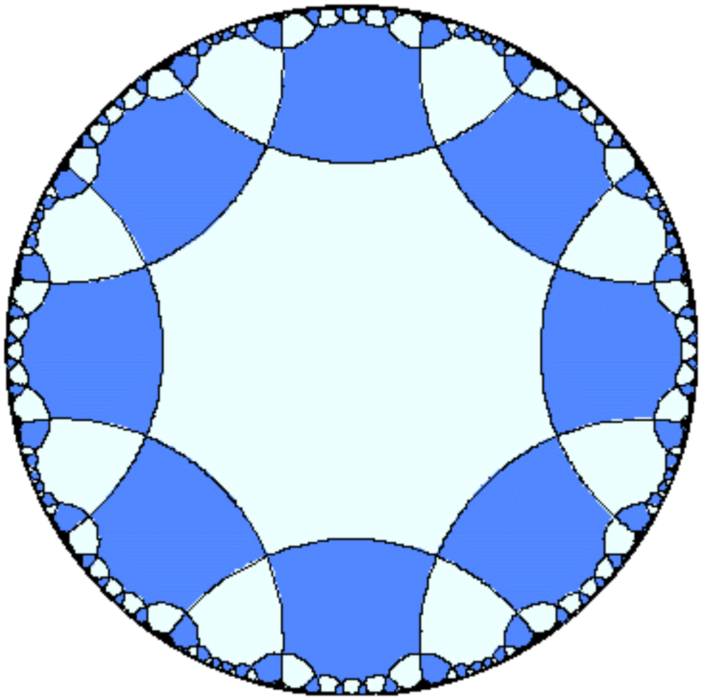


This is partly evident from the real graph:



where the holes can be thought of as the gaps between the components.

We were able to parameterize an elliptic curve by using elliptic functions on the complex plane. These were periodic with respect to a tiling of the plane by parallelograms. For hyperelliptic curves, we replace the plane by the disk, and tile these by polygons which are regular in the sense of hyperbolic geometry (in which the "straight lines" are circles meeting the boundary circle at right angles).



We can then parameterize such curves by using functions which are almost periodic or *automorphic* with respect to this tiling. This is usually formulated in terms of the group of symmetries of the tiling, rather than the tiling itself. Some discussion of these matters can be found for example in *Siegel, Topics in complex function theory vol 2.*

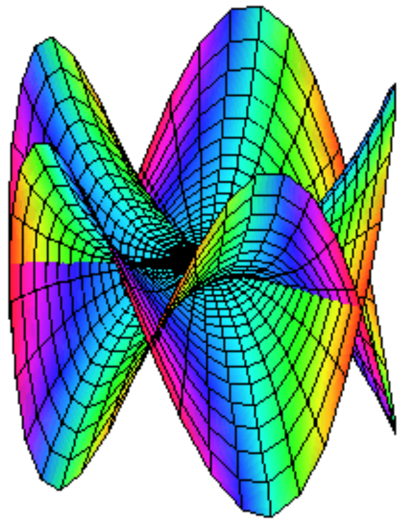
Making this explicit enough to draw a graph would be a daunting task. So we look at the simpler case where all the roots of $p(x)$ come together. Namely, $y^2 = x^5$. This is a singular curve called a cusp. It is easy to see that $x = t^2$, $y = t^5$ gives a rational parameterization. This can arrived at by carrying out a resolution proceedure as above, but this time with two blow ups. Switching to polar coordinates (for purely aesthetic reasons) and taking real and imaginary parts as above, leads to

$$x_1 = r^2 \cos 2\theta$$

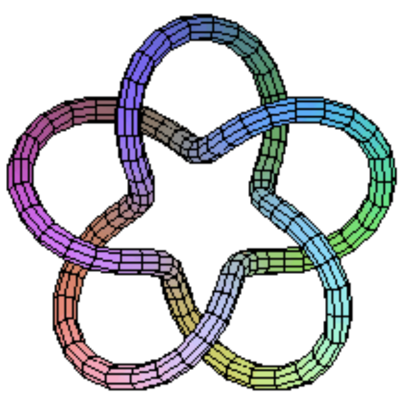
$$x_2 = r^2 \sin 2\theta$$

$$x_3 = r^5 \cos 5\theta$$

This yields



Now take a look at the outer edge of this picture. The way it is drawn, it repeatedly crosses itself. But we know this really represents a defect of our depiction more than anything else. To get a better feeling for what this ought to look like, form the intersection of our cusp with a small sphere $|x|^2 + |y|^2 = \varepsilon^2$. This is called the *link* of the singularity. The sphere can be identified with R^3 plus a point at infinity, and the link is a knot in this space. In this case, the link is a so called (2,5)-torus knot which looks as follows



If we did this for a nonsingular point, we would have gotten an unknotted circle. So the amount of "knottedness" is really telling us how bad the singularity is.

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