What does the Jacobian of a curve tell us about the curve?

[+18] [2] Lalit Jain

[2013-08-10 23:08:31]

[ag.algebraic-geometry algebraic-curves motives]

[https://mathoverflow.net/questions/139118/what-does-the-jacobian-of-a-curve-tell-us-about-the-curve]

A natural object in the study of curves is the Jacobian of a curve. What are some natural geometric properties of the curve that the Jacobian encapsulates? In other words, what can the Jacobian tell us about the curve that we didn't know already?

Note, I am asking for concrete examples, statements like "The Jacobian having property blah implies the curve has property blah."

Ideally these will be statements that are easier to prove using the Jacobian (whose construction is not so easy!) rather than directly from the curve.

(Also, if this question is more appropriate for math.se, I'd be happy to delete it.)

(3) By Torelli's theorem, everything. - **Felipe Voloch**
(8) @FelipeVoloch : not necessarily. You need the Jacobian plus the theta divisor (or polarization) to tell you everything - there could be two non-isomorphic curves with isomorphic Jacobians. - **Abhinav Kumar**
(1) You could also look at the question mathoverflow.net/questions/128593/... - **Dan Petersen**
(1) Abhinav: I believe "the Jacobian" usually denotes the polarized abelian variety, so I agree with Felipe. - **roy smith**
thus perhaps it is more useful to describe how the information is differently encoded - e.g a curve is hyperelliptic iff the theta divisor has singular locus of codimension two in theta. - **roy smith**

[+21] [2013-08-11 01:57:59] Donu Arapura

The question seems fine to me. Off the top of my head:

1) The Jacobian is a group, and in fact an abelian variety, whereas the curve usually isn't. This gives you a lot of structure to play with that you didn't have initially. For example, to show that a general curve doesn't map onto a curve of smaller positive genus, you can use the fact that the Jacobian of such a curve is simple.

2) The Jacobian is the motive of the curve, loosely speaking. In particular, all cohomological information about the curve can be read off from its Jacobian. Eg. Etale cohomology $H^1(X,\mathbb{Z}/n)$ is just the group of n-torsion points (up to twist if you're a stickler). I believe that Weil first constructed the Jacobian in the abstract setting precisely for this reason.

3) It has not just one but two universal properties. It's the universal abelian variety the curve maps to, and it's also universal parameter space for divisor classes of degree 0 (i.e. it's both Alb and Pic^0).

...

(1) Could it be possible to formulate a universal property in terms of cohomology or even motives? (i.e. it's the universal thing the curve maps into inducing an isomorphism on first cohomology in all cohomology theories). - **David Corwin**

[+12] [2013-08-11 08:34:18] Darius Math

I think at least historically the Jacobian is related to the function theory over a curve which was one the main areas of research back in the 19th century. In that time given a compact Riemann surface X over $\mathbb{C}$, the question was to understand the behavior of holomorphis and mermorphic functions on this curve. If we have two effective divisors D and E on X, when is $D-E$ the divisors of zeros and poles of a meromorphic function f on X? Let the genus of X be g. Then there are g basis elements of the vector space of differential forms on X. The clever solution that Abel proposed for this question was this: let ω_1,...,ω_g be the generators of $\Omega(X)$, the space of holomorphic differentials of X. Given a path γ in X, the set $L=\{(\int_\gamma \omega_1,\dots,\int_\gamma \omega_g)\}$ is additive in $\mathbb{C}^g\cong\Omega(X)$ because of additivity property of integrals and in fact is a lattice. Therefore we can quotient out and get a group $\mathbb{C}^g/L$. We also get a map

$A : X \to J(X)$ by choosing a base point and p_0 and sending each point $p \in X$ to $(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g)$ mod L. Abel realized that two divisors D and E (viewed as a collection of points on X) are linearly equivalent if and only if they have the same image under the map A. Note that the map $A : X \to J(X)$ in itself is a very interesting map: we have constructed an almost natural holomorphic map form X to a variety that has a structure of a group. In the first glance it is not at all clear that we can have such a map. The second funny property is that this map is not injective if and only if $X \cong \mathbb{P}^1$.

(5) to add to this nice answer, note that the abel map induces one on every symmetric product X^(d)-->J, whose fibers are linear systems ≈ P^r, and hence whenever d > dimJ= g(X), there must be divisors of degree d that are linearly equivalent. In particular dim.h^0(D) ≥ g-d. - **roy smith**