

NATIONAL INSTITUTE OF SCIENCE EDUCATION AND RESEARCH
SCHOOL OF MATHEMATICAL SCIENCES

M598: DISSERTATION

Sheaf-theoretic de Rham isomorphism

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November 20, 2018

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Abstract

A sheaf-theoretic proof of de Rham cohomology being a topological invariant has been presented. The de Rham cohomology of a smooth manifold is shown to be isomorphic to the Čech cohomology of that manifold with real coefficients.

Acknowledgements

Foremost, I would like to express my sincere gratitude to my supervisor *Dr. Ritwik Mukherjee* for his patience, motivation, and immense knowledge.

I would also like to thank the people who created the typesetting tools used for preparing this report.

- Donald Knuth for `TEX`
- Michael Spivak for `AMS-TEX`
- Sebastian Rahtz for `TEX Live`
- Leslie Lamport for `LATEX`
- American Mathematical Society for `AMS-LATEX`
- Hàn Thé Thành for `pdfTEX`
 - Alan Jeffrey & Frank Mittelbach for the `inputenc` package
 - Heiko Oberdiek for the `hyperref` package
 - Steven B. Segletes for the `stackengine` package
 - David Carlisle for the `graphicx` package
 - Javier Bezos for the `enumitem` package
 - Hideo Umeki for the `geometry` package
 - Peter R. Wilson & Will Robertson for the `epigraph` package
 - Sebastian Rahtz for the `textcomp` package
 - Walter Schmidt for the `gensymb` package
 - Patrick W. Daly for the `natbib` package
 - Jörg Knappen for the `mathrsfs` package
 - Jeremy Gibbons, Taco Hoekwater and Alan Jeffrey for the `stmaryrd` package
 - Olaf Kummer for the `dsfont` package
 - Marco Daniel for the `mdframed` package
 - Florêncio Neves for the `tikz-cd` package
- Philipp Kühn & Daniel Kirsch for Detexify (a tool for searching `LATEX` symbols)
- TeX.StackExchange community for helping me out with `LATEX` related problems

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Introduction

“The major virtue of sheaf theory is information-theoretic in nature. Most problems could be phrased and perhaps solved without sheaf theory, but the notation would be enormously more complicated and difficult to comprehend.”

— Raymond O. Wells, *Differential Analysis on Complex Manifolds*, p. 36

A fundamental problem of topology is that of determining, for two spaces, whether or not they are homeomorphic. Algebraic topology originated in the attempts by mathematicians to construct suitable topological invariants. In 1895, Henri Poincaré² introduced a certain group, called the *fundamental group* of a topological space; which is by definition a topological invariant. Enrico Betti, on the other hand, associated with each space certain sequence of abelian groups called its *homology groups* [14, p. 1]. It was eventually proved that homeomorphic spaces had isomorphic homology groups. It was not until 1935 that another sequence of abelian groups, called *cohomology groups*, was associated with each space. The origins of cohomology groups lie in algebra rather than geometry; in a certain algebraic sense they are dual to the homology groups [14, p. 245]. There are several different ways of defining (co)homology groups, most common ones being *simplicial* and *singular* groups³. A third way of defining homology groups for arbitrary spaces, using the notion of open cover, is due to Eduard Čech (1932). The Čech homology theory is still not completely satisfactory [14, p. 2]. Apparently, Čech himself did not introduce *Čech cohomology*. Clifford Hugh Dowker, Samuel Eilenberg, and Norman Steenrod introduced Čech cohomology in the early 1950’s [2, p. 24].

In 1920s, Élie Cartan’s extensive research lead to the global study of general *differential forms* of higher degrees. É. Cartan, speculating the connections between topology and differential geometry, conjectured the *de Rham theorem* in a 1928 paper [10, p. 95]. In 1931, in his doctoral thesis, Georges de Rham⁴ showed that differential forms satisfy the same axioms as *cycles* and *boundaries*, in effect proving a duality between what are now called *de Rham cohomology* and singular cohomology with real coefficients⁵. De Rham cohomology is considered to be one of the most important diffeomorphism invariant of a smooth manifold [18, p. 274].

Jean Leray, as a prisoner of war from 1940 to 1945, set himself the goal of discovering methods which could be applied to a very general class of topological space, while avoiding the use of simplicial approximation. The de Rham theorem and É. Cartan’s theory of differential forms were central to Leray’s thinking [8, §2]. After the war he published his results in 1945, which marked the birth of *sheaves* and *sheaf cohomology*⁶. His remarkable but rather obscure results were clarified by Émile Borel, Henri Cartan, Jean-Louis Koszul, Jean-Pierre Serre and

²Poincaré, Henri. “Analysis situs.” *Journal de l’École Polytechnique*. 2 (1895): 1-123. <https://gallica.bnf.fr/ark:/12148/bpt6k4337198>

³Singular homology emerged around 1925 in the work of Oswald Veblen, James Alexander and Solomon Lefschetz, and was defined rigorously and in complete generality by Samuel Eilenberg in 1944 [2, p. 10].

⁴De Rham, Georges. “Sur l’analyse situs des variétés à n dimensions.” 1931. <http://eudml.org/doc/192808>

⁵This can also be achieved directly via simplicial methods, see John Lee’s *Introduction to Smooth Manifolds*, Chapter 18. In fact, this theorem has several dozens of different proofs.

⁶The word *faisceau* was introduced in the first of the announcements made by Leray in meeting of the Académie des Sciences on May 27, 1946. In 1951, John Moore fixed on “sheaf” as the English equivalent of “faisceau”.

André Weil in the late 1940's and early 1950's⁷. In 1952, Weil⁸ found the modern proof of the de Rham theorem, this proof was a vindication of the local methods advocated by Leray [1, p. 5]. Weil's discovery provided the light which led H. Cartan to the modern formulation of *sheaf theory* [8, §2].

One can use Weil's approach, involving generalized Mayer-Vietoris principle, to study the relation between the de Rham theory to the Čech theory [1, p. 6]. However, we will follow the approach due to H. Cartan, written in the early 1950's, to give a sheaf theoretic proof of the isomorphism between de Rham and Čech cohomology with coefficients in \mathbb{R} [19, p. 163]. An outline of this approach for proving de Rham cohomology to be a topological invariant can be found in the the books by Griffiths and Harris [3, p. 44] and Hirzebruch [4, §2.9–2.12].

This report consists of three chapters. In [chapter 1](#) we will discuss various concepts related to differential forms and smooth manifolds needed to define de Rham cohomology. We will also develop the tools like Poincaré lemma, which will be used later to establish important sheaf theoretic results about the differential forms. In [chapter 2](#) we will first discuss the sheaf theory necessary for defining Čech cohomology, and then prove the key results about Čech cohomology of paracompact Hausdorff spaces, like “short exact sequence of sheaves induces a long exact sequence of Čech cohomology”, and “Čech cohomology vanishes on fine sheaves”. Finally, in [chapter 3](#) we will present the proof of de Rham-Čech isomorphism.

Apart from the three chapters, we have also included two appendices. In [Appendix A](#), to supplement the discussions in the first two chapters, we have stated few facts about paracompact spaces. In [Appendix B](#) we have discussed the theory of direct limits needed for understanding various definitions and proofs in the second chapter.

⁷Georges Elencwajg (<https://math.stackexchange.com/users/3217/georges-elencwajg>), Why was Sheaf cohomology invented?, URL (version: 2016-05-24): <https://math.stackexchange.com/q/1798796>

⁸Weil, André. “Sur les théorèmes de de Rham.” *Commentarii mathematici Helvetici* 26 (1952): 119-145. <http://eudml.org/doc/139040>.

Chapter 1

de Rham cohomology

“The differential equation $P(x, y)dx + Q(x, y)dy = 0$ is said to be exact if there is a function f such that $P = \partial f / \partial x$ and $Q = \partial f / \partial y$. In our terminology, this means simply that the 1-form $Pdx + Qdy$ is the differential of the 0-form f , so that it is exact.”

— James R. Munkres, *Analysis on Manifolds*, p. 260

1.1 Differential forms on \mathbb{R}^n

In this section some basic definitions and facts from [13, Chapter 6] and [18, Chapter 1] will be stated. All the vector spaces are over the field \mathbb{R} of real numbers.

1.1.1 Tangent space

Definition 1.1 (Tangent vector). Given $p \in \mathbb{R}^n$, a *tangent vector* to \mathbb{R}^n at p is a pair $(p; v)$, where $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$.

Definition 1.2 (Tangent space). The set of all tangent vectors to \mathbb{R}^n at p forms a vector space called *tangent space* of \mathbb{R}^n at p , defined by

$$(p; v) + (p; w) = (p; v + w) \quad \text{and} \quad c(p; v) = (p; cv)$$

It is denoted by $T_p(\mathbb{R}^n)$.

Definition 1.3 (Germ of smooth functions). Consider the set of all pairs (f, U) , where U is a neighborhood of $p \in \mathbb{R}^n$ and $f : U \rightarrow \mathbb{R}$ is a smooth function. (f, U) is said to be equivalent to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W . This equivalence class of (f, U) is called *germ of f at p* .

Remark 1.1. The set of all germs of smooth functions on \mathbb{R}^n at p is written as $C_p^\infty(\mathbb{R}^n)$. The addition and multiplication of functions induce corresponding operations of $C_p^\infty(\mathbb{R}^n)$, making it into a ring; with scalar multiplication by real numbers $C_p^\infty(\mathbb{R}^n)$ becomes an algebra over \mathbb{R} .

Definition 1.4 (Derivation at a point). A linear map $X_p : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfying the Leibniz rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

is called a *derivation at $p \in \mathbb{R}^n$* or a *point-derivation of $C_p^\infty(\mathbb{R}^n)$* .

Remark 1.2. The set of all derivations at p is denoted by $\mathcal{D}_p(\mathbb{R}^n)$. This set is a vector space.

Theorem I. *The linear map*

$$\begin{aligned}\phi : T_p(\mathbb{R}^n) &\rightarrow \mathcal{D}_p(\mathbb{R}^n) \\ (p; v) &\mapsto D_v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_p\end{aligned}$$

where $(p; v) = (p; v_1, \dots, v_n)$ and D_v is the directional derivative in the direction of v , is an isomorphism.

Remark 1.3. Under this vector space isomorphism, the standard basis $\{e_1, \dots, e_n\}$ of $T_p(\mathbb{R}^n)$ corresponds to the set $\{\partial/\partial x_1|_p, \dots, \partial/\partial x_n|_p\}$ of partial derivatives.

Definition 1.5 (Pushforward of a vector). Let U be an open set in \mathbb{R}^m , $\alpha : U \rightarrow \mathbb{R}^n$ be a smooth function. The function f induces the linear transformation

$$\begin{aligned}\alpha_* : T_p(\mathbb{R}^m) &\rightarrow T_{\alpha(p)}(\mathbb{R}^n) \\ (p; v) &\mapsto (\alpha(p); D\alpha(p) \cdot v)\end{aligned}$$

where $D\alpha(p)$ is the total derivative of α at p . In other words, $\alpha_*(D_v)f = D_v(f \circ \alpha)$ for $f \in C_{\alpha(p)}^\infty(\mathbb{R}^n)$. Then $\alpha_*(p; v)$ is called the *pushforward of the vector v at p* ,

Theorem II. *Let U be open in \mathbb{R}^m , and $\alpha : U \rightarrow \mathbb{R}^n$ be a smooth map. Let V be an open set of \mathbb{R}^m containing $\alpha(U)$, let $\beta : V \rightarrow \mathbb{R}^k$ be a smooth map. Then $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.*

1.1.2 Multilinear algebra

Unlike the preceding and succeeding (sub)sections, here V and W denote real vector spaces instead of open sets.

Definition 1.6 (k -tensor). Let V be a vector space over \mathbb{R} . Let $V^k = V \times \dots \times V$ denote the set of all k -tuples (v_1, \dots, v_k) of vectors of V . A function $f : V^k \rightarrow \mathbb{R}$ is said to be a k -tensor if f is linear in the i^{th} variable for each i .

Remark 1.4. The set of all k -tensors on V is denoted by the symbol $\mathcal{L}^k(V)$. If $k = 1$ then $\mathcal{L}^1(V) = V^*$, the dual space of V .

Theorem III. *Let V be a vector space of dimension n , then $\mathcal{L}^k(V)$ is a vector space of dimension n^k .*

Definition 1.7 (Tensor product). Let $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^\ell(V)$, then the *tensor product* $f \otimes g \in \mathcal{L}^{k+\ell}(V)$ is defined by the equation

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) \cdot g(v_{k+1}, \dots, v_{k+\ell})$$

Definition 1.8 (Pullback of tensors). Let $T : V \rightarrow W$ be a linear transformation and

$$T^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$$

be the dual transformation defined for each $f \in \mathcal{L}^k(W)$ and $v_1, \dots, v_k \in V$ as

$$(T^*f)(v_1, \dots, v_k) = f(T(v_1), \dots, T(v_k))$$

Then T^*f is called the *pullback of tensor $f \in \mathcal{L}^k(W)$* .

Theorem IV. *T^* is a linear transformation such that:*

1. $T^*(f \otimes g) = T^*f \otimes T^*g$

2. If $S : W \rightarrow W'$ is a linear transformation, then $(S \circ T)^* f = T^*(S^* f)$.

Definition 1.9 (Alternating k -tensor). Let f be a k -tensor on V and σ be a permutation of $\{1, \dots, k\}$. The k tensor f^σ on V is defined by the equation

$$f^\sigma(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The tensor f is said to be alternating if $f^\sigma = (\text{sgn } \sigma)f$ for all permutations σ of $\{1, \dots, k\}$.

Remark 1.5. The set of all alternating k -tensors on V is denoted by the symbol $\mathcal{A}^k(V)$. If $k = 1$ then $\mathcal{A}^1(V) = \mathcal{L}^1(V) = V^*$, the dual space of V .

Theorem V. Let $T : V \rightarrow W$ be a linear transformation and $T^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$ be the dual transformation. If f is an alternating tensor on W , then $T^* f$ is an alternating tensor on V .

Definition 1.10 (Alternating operator). The linear transformation $A : \mathcal{L}^k(V) \rightarrow \mathcal{A}^k(V)$ defined as

$$Af = \sum_{\sigma} (\text{sgn } \sigma) f^\sigma$$

is called the *alternating operator*.

Remark 1.6. One can easily verify that this is a well defined linear transformation. Let τ be any permutation and $f \in \mathcal{L}^k(V)$ then

$$(Af)^\tau = \sum_{\sigma} (\text{sgn } \sigma) (f^\sigma)^\tau = \sum_{\sigma} (\text{sgn } \sigma) f^{\tau \circ \sigma} = (\text{sgn } \tau) \sum_{\sigma} (\text{sgn } \tau \circ \sigma) f^{\tau \circ \sigma} = (\text{sgn } \tau) Af$$

hence $Af \in \mathcal{A}^k(V)$ for all $f \in \mathcal{L}^k(V)$.

Definition 1.11 (Wedge product). Let $f \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^\ell(V)$, then the *wedge product* $f \wedge g \in \mathcal{A}^{k+\ell}(V)$ is defined as

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g)$$

where A is the alternating operator.

Remark 1.7. The reason for the coefficient $1/k!\ell!$ follows from the fact that $Af = k!f$ if $f \in \mathcal{A}^k(V)$.

Theorem VI. Let f, g, h be alternating tensors on V . Then the following properties hold:

1. (Associative) $f \wedge (g \wedge h) = (f \wedge g) \wedge h$
2. (Homogeneous) $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$ for all $c \in \mathbb{R}$
3. (Distributive) If f and g have the same order, then $(f + g) \wedge h = f \wedge h + g \wedge h$ and $h \wedge (f + g) = h \wedge f + h \wedge g$
4. (Anti-commutative) If f and g have orders k and ℓ , respectively, then $g \wedge f = (-1)^{k\ell} f \wedge g$
5. Let $T : V \rightarrow W$ be a linear transformation and $T^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$ be the dual transformation. If f and g are alternating tensors on W , then $T^*(f \wedge g) = T^* f \wedge T^* g$

Theorem VII. Let V be a vector space of dimension n , with basis $\{e_1, \dots, e_n\}$, and $\{f_1, \dots, f_n\}$ be the dual basis for $V^* = \mathcal{A}^1(V)$. Then $\mathcal{A}^k(V)$ is a vector space of dimension $\binom{n}{k}$ with the set $\{f_I = f_{i_1} \wedge \dots \wedge f_{i_k} : I = (i_1, \dots, i_k)\}$ as basis.

Remark 1.8. If $k > \dim V$, then $\mathcal{A}^k(V) = 0$. This is because the anti-commutativity of wedge product implies that if $f \in V^*$ then $f \wedge f = 0$.

1.1.3 Differential forms

Definition 1.12 (Tensor field). Let U be an open set in \mathbb{R}^n . A k -tensor field in U is a function ω assigning each $p \in U$, a k -tensor ω_p defined on the tangent space $T_p(\mathbb{R}^n)$. That is, $\omega_p \in \mathcal{L}^k(T_p(\mathbb{R}^n))$ for each $p \in U$.

Remark 1.9. Thus ω_p is a function mapping k -tuples of tangent vectors to \mathbb{R}^n at p into \mathbb{R} . The tensor field ω is said to be of class C^r if it is of class C^r as a function of (p, v_1, \dots, v_k) for all $p \in U$ and $v_i \in T_p(\mathbb{R}^n)$.

Definition 1.13 (Differential k -form). A *differential form of order k* , or *differential k -form* on an open subset U of \mathbb{R}^n is a k -tensor field with the additional property that $\omega_p \in \mathcal{A}^k(T_p(\mathbb{R}^n))$ for all $p \in U$.

Definition 1.14 (Differential 0-form). If U is open in \mathbb{R}^n , and if $f : U \rightarrow \mathbb{R}$ is a map of class C^r , then f is called a *differential 0-form* in U .

Definition 1.15 (Wedge product of 0-form and k -form). The *wedge product* of a 0-form f and k -form ω on the open set U of \mathbb{R}^n is defined by the rule

$$(\omega \wedge f)_p = (f \wedge \omega)_p = f(p) \cdot \omega_p$$

for all $p \in U$.

Remark 1.10. Henceforth, we restrict ourselves to differential forms of class C^∞ . If U is an open set in \mathbb{R}^n , let $\Omega^k(U)$ denote the set of all smooth k -forms on U . The sum of two such k -forms is another k -form, and so is the product of a k -form by a scalar. Hence $\Omega^k(U)$ is the vector space of k -forms on U . Also, $\Omega^0(U) = C^\infty(U)$.

1.1.4 Exterior derivative

Definition 1.16 (Differential of a function). Let U be open in \mathbb{R}^n and $f : U \rightarrow \mathbb{R}$ be a smooth real-valued function. Then the *differential of f* is defined to be the smooth 1-form df on U such that for any $p \in U$ and $(p; v) \in T_p(\mathbb{R}^n)$

$$(df)_p(p; v) = Df(p) \cdot v$$

where $Df(p)$ is the total derivative of f at p . In other words, $(df)_p(X_p) = X_p f$ for all derivations $X_p \in T_p(\mathbb{R}^n)$.

Remark 1.11. If x denotes the general point of \mathbb{R}^n , the i^{th} projection function mapping \mathbb{R}^n to \mathbb{R} is denoted by the symbol x_i . Then dx_i equals the elementary 1-form in \mathbb{R}^n , i.e. the set $\{dx_1, \dots, dx_n\}$ is a basis of $\Omega^1(\mathbb{R}^n)$. If $I = (i_1, \dots, i_k)$ is an ascending k -tuple from the set $\{1, \dots, n\}$, then

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

denotes the elementary k -forms in \mathbb{R}^n , i.e. the set $\{dx_I : I \text{ is an ascending set of } k \text{ elements}\}$ is a basis of $\Omega^k(\mathbb{R}^n)$. The general k -form $\omega \in \Omega^k(U)$ can be written uniquely in the form

$$\omega = \sum_{[I]} a_I dx_I$$

for some $a_I \in C^\infty(U)$.

Theorem VIII. Let U be open in \mathbb{R}^n and $f \in C^\infty(U)$. Then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

In particular, $df = 0$ if f is a constant function.

Definition 1.17 (Differential of a k -form). Let U be an open set in \mathbb{R}^n and $\omega \in \Omega^k(U)$ such that $\omega = \sum_{[I]} f_I dx_I$. Then for $k \geq 0$, the *differential of a k -form* ω is defined by the linear transformation

$$\begin{aligned} d : \Omega^k(U) &\rightarrow \Omega^{k+1}(U) \\ \omega &\mapsto \sum_{[I]} df_I \wedge dx_I \end{aligned}$$

where df_I is the differential of function.

Theorem IX. Let U be an open set in \mathbb{R}^n . If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^\ell(U)$ then

1. (Antiderivation) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
2. $d \circ d = 0$

Definition 1.18 (Pullback of a k -form). Let U be open in \mathbb{R}^m and $\alpha : U \rightarrow \mathbb{R}^n$ be a smooth map. Let V be an open set in \mathbb{R}^n containing $\alpha(U)$. For $k \geq 1$

$$\alpha^* : \Omega^k(V) \rightarrow \Omega^k(U)$$

is the dual transformation defined for each $\omega \in \Omega^k(V)$ and $(p; v_1), \dots, (p; v_k) \in T_p(\mathbb{R}^m)$ as

$$(\alpha^*\omega)_p((p; v_1), \dots, (p; v_k)) = \omega_{\alpha(p)}(\alpha_*(p; v_1), \dots, \alpha_*(p; v_k))$$

Then the k -form $\alpha^*\omega \in \Omega^k(U)$ is called the *pullback of $\omega \in \Omega^k(V)$* .

Definition 1.19 (Pullback of a 0-form). Let U be open in \mathbb{R}^m and $\alpha : U \rightarrow \mathbb{R}^n$ be a smooth map. Let V be an open set in \mathbb{R}^n containing $\alpha(U)$. If $f : V \rightarrow \mathbb{R}$ be a smooth map, then the pullback of $f \in \Omega^0(V)$ is the the 0-form $\alpha^*f = f \circ \alpha \in \Omega^0(U)$, i.e. $(\alpha^*f)(p) = f(\alpha(p))$ for all $p \in U$.

Theorem X. Let U be open in \mathbb{R}^ℓ and $\alpha : U \rightarrow \mathbb{R}^m$ be a smooth map. Let V be open in \mathbb{R}^m which contains $\alpha(U)$ and $\beta : V \rightarrow \mathbb{R}^n$ be a smooth map. Then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$, i.e. $(\beta \circ \alpha)^*\omega = \alpha^*(\beta^*\omega)$ for all $\omega \in \Omega^k(W)$ where W is an open set in \mathbb{R}^n containing $\beta(V)$.

Theorem XI. Let U be open in \mathbb{R}^m and $\alpha : U \rightarrow \mathbb{R}^n$ be a smooth map. If ω, η and θ are differential forms defined in an open set V of \mathbb{R}^n containing $\alpha(U)$, such that ω and η have same order, then

1. (preservation of the vector space structure) $\alpha^*(a\omega + b\eta) = a(\alpha^*\omega) + b(\alpha^*\eta)$ for all $a, b \in \mathbb{R}$.
2. (preservation of the wedge product) $\alpha^*(\omega \wedge \theta) = \alpha^*\omega \wedge \alpha^*\theta$.
3. (commutation with the differential) $\alpha^*(d\omega) = d(\alpha^*\omega)$, i.e. the following diagram commutes

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{d} & \Omega^{k+1}(V) \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ \Omega^k(U) & \xrightarrow{d} & \Omega^{k+1}(U) \end{array}$$

1.2 Closed and exact forms on \mathbb{R}^n

In this section the proof of Poincaré lemma following [13, Chapter 8] will be discussed.

Definition 1.20 (Closed forms). Let U be an open set in \mathbb{R}^n and $\omega \in \Omega^k(U)$ for $k \geq 0$. Then ω is said to be *closed* if $d\omega = 0$.

Remark 1.12. If U is an open set in \mathbb{R}^n , let $\mathcal{Z}^k(U)$ denote the set of all closed k -forms on U . The sum of two such k -forms is another closed k -form, and so is the product of a closed k -form by a scalar. Hence $\mathcal{Z}^k(U)$ is the vector sub-space of $\Omega^k(U)$. Also, $\mathcal{Z}^0(U)$ is the set of all locally constant¹ functions on U .

Definition 1.21 (Exact k -forms). Let U be an open set in \mathbb{R}^n and $\omega \in \Omega^k(U)$ for $k \geq 1$. Then ω is said to be *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$.

Remark 1.13. If U is an open set in \mathbb{R}^n , let $\mathcal{B}^k(U)$ denote the set of all exact k -forms on U . The sum of two such k -forms is another exact k -form, and so is the product of a exact k -form by a scalar. Hence $\mathcal{B}^k(U)$ is the vector sub-space of $\Omega^k(U)$. Also, $\mathcal{B}^0(U)$ is defined to be the set consisting only zero.

Theorem 1.1. *Every exact form is closed.*

Proof. Let U be an open set in \mathbb{R}^n and $\omega \in \mathcal{B}^k(U)$ such that $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$. From **Theorem IX** we know that $d\omega = d(d\eta) = 0$ hence $\omega \in \mathcal{Z}^k(U)$ for all $k \geq 1$. For $k = 0$, the statement is trivially true. \square

Remark 1.14. This theorem implies that $\mathcal{B}^k(U) \subseteq \mathcal{Z}^k(U)$ for all $k \geq 0$. However, the converse doesn't always hold for $k \geq 1$. For example, if $U = \mathbb{R}^2 - 0$ then the 1-form

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

is closed but not exact [13, Exercise 30.5, p. 261].

1.2.1 Differentiable homotopy

Definition 1.22 (Differentiable homotopy). Let U and V be open sets in \mathbb{R}^m and \mathbb{R}^n , respectively; let $g, h : U \rightarrow V$ be smooth maps. Then g and h are said to be *differentially homotopic* if there is a smooth map² $H : U \times [0, 1] \rightarrow V$ such that

$$H(x, 0) = g(x) \quad \text{and} \quad H(x, 1) = h(x)$$

for all $x \in U$. The map H is called *differentiable homotopy* between g and h .

Lemma 1.1. *Let U be an open set in \mathbb{R}^n and W be an open set in \mathbb{R}^{n+1} such that $U \times [0, 1] \subset W$. Let $\alpha, \beta : U \rightarrow W$ be smooth maps such that $\alpha(x) = (x, 0)$ and $\beta(x) = (x, 1)$. Then there is a linear transformation*

$$L : \Omega^{k+1}(W) \rightarrow \Omega^k(U)$$

defined for all $k \geq 0$, such that

$$\begin{cases} dL\eta + Ld\eta = \beta^*\eta - \alpha^*\eta & \text{if } \eta \in \Omega^{k+1}(W), k \geq 0 \\ Ld\gamma = \beta^*\gamma - \alpha^*\gamma & \text{if } \gamma \in C^\infty(W) = \Omega^0(W) \end{cases}$$

where $\alpha^, \beta^* : \Omega^k(W) \rightarrow \Omega^k(U)$ are the pullback maps defined for all $k \geq 0$.*

¹Locally constant functions are constant on any connected component of domain.

²This means that H is smooth in some open neighborhood of $U \times [0, 1]$, like $U \times (-\epsilon, 1 + \epsilon)$.

Proof. Let $x = (x_1, \dots, x_n)$ denote the general point of \mathbb{R}^n , and let t denote the general point of \mathbb{R} . Then, as in [Remark 1.11](#), dx_1, \dots, dx_n, dt are the elementary 1-forms in \mathbb{R}^{n+1} . Also, for any continuous function $b : U \times [0, 1] \rightarrow \mathbb{R}$ a scalar function Γb is defined on U by the formula

$$(\Gamma b)(x) = \int_{t=0}^{t=1} b(x, t)$$

Then for any $\eta \in \Omega^{k+1}(W)$

$$\eta = \sum_{[I]} a_I dx_I + \sum_{[J]} b_J dx_J \wedge dt$$

where I is an ascending $(k+1)$ -tuple and J is an ascending k -tuple from the set $\{1, \dots, n\}$, we define

$$\begin{aligned} L : \Omega^{k+1}(W) &\rightarrow \Omega^k(U) \\ \eta &\mapsto \sum_{[I]} L(a_I dx_I) + \sum_{[J]} L(b_J dx_J \wedge dt) \end{aligned}$$

where $L(a_I dx_I) = 0$ and $L(b_J dx_J \wedge dt) = (-1)^k (\Gamma b_J) dx_J$.

Step 1. L is a well defined linear transformation.

We need to show that $L\eta \in \Omega^k(U)$. Clearly, $L\eta$ is a k -form on the subset U of \mathbb{R}^n . To prove that $L\eta$ is smooth, it's sufficient to show that the function Γb_J is smooth; and this result follows from Leibniz's rule [[13](#), Theorem 39.1], since b_J is smooth.

Linearity of L follows from the uniqueness of the representation of η and linearity of the integral operator Γ .

Step 2. $L(a dx_I) = 0$ and $L(b dx_J \wedge dt) = (-1)^k (\Gamma b) dx_J$ for any arbitrary $(k+1)$ -tuple I and k -tuple J from the set $\{1, \dots, n\}$.

If the indices are not distinct, then these formulae hold trivially, since $dx_I = 0$ and $dx_J = 0$ in that case. If the indices are distinct and in ascending order then these formulas hold by definition. Since rearranging the indices changes the values of dx_I and dx_J only by a sign, the formulae hold even in that case (the signs will cancel out due to linearity).

Step 3. $L d\gamma = \beta^* \gamma - \alpha^* \gamma$ if $\gamma \in C^\infty(W)$

$$\begin{aligned} L d\gamma &= L \left(\sum_{i=1}^n \frac{\partial \gamma}{\partial x_i} dx_i \right) + L \left(\frac{\partial \gamma}{\partial t} dt \right) \\ &= 0 + (-1)^0 \left(\Gamma \frac{d\gamma}{dt} \right) \\ &= \int_{t=0}^{t=1} \frac{\partial \gamma}{\partial t}(x, t) \\ &= \gamma(x, 1) - \gamma(x, 0) \\ &= \gamma \circ \beta - \gamma \circ \alpha \\ &= \beta^* \gamma - \alpha^* \gamma \end{aligned}$$

Step 4. $dL\eta + L d\eta = \beta^* \eta - \alpha^* \eta$ if $\eta \in \Omega^{k+1}(W), k \geq 0$

Since d, L, α^* and β^* are all linear transformations, it suffices to verify the formula for the $(k+1)$ -forms $\eta = a dx_I$ and $\eta = b dx_J \wedge dt$. We will use Step 2 and [Theorem XI](#) to simplify and compare *left hand side* (LHS) and *right hand side* (RHS) of the formula for both the cases.

Case 1. $\eta = a dx_I$ for any $(k+1)$ -tuple I from $\{1, \dots, n\}$

Simplify the LHS:

$$\begin{aligned}
dL\eta + L d\eta &= d0 + L(da \wedge dx_I) \\
&= L \left(\sum_{i=1}^n \frac{\partial a}{\partial x_i} dx_i \wedge dx_I + \frac{\partial a}{\partial t} dt \wedge dx_I \right) \\
&= L \left(\sum_{i=1}^n \frac{\partial a}{\partial x_i} dx_i \wedge dx_I \right) + L \left(\frac{\partial a}{\partial t} dt \wedge dx_I \right) \\
&= 0 + (-1)^{k+1} L \left(\frac{\partial a}{\partial t} dx_I \wedge dt \right) \\
&= (-1)^{k+1} \cdot (-1)^{k+1} \left(\Gamma \frac{\partial a}{\partial t} \right) dx_I \\
&= \left(\int_{t=0}^{t=1} \frac{\partial a}{\partial t}(x, t) \right) dx_I \\
&= (a(x, 1) - a(x, 0)) dx_I \\
&= (a \circ \beta - a \circ \alpha) dx_I
\end{aligned}$$

Simplify the RHS:

$$\begin{aligned}
\beta^* \eta - \alpha^* \eta &= \beta^*(a dx_I) - \alpha^*(a dx_I) \\
&= \beta^*(a) \beta^*(dx_I) - \alpha^*(a) \alpha^*(dx_I) \\
&= (a \circ \beta) \beta^*(dx_{i_1} \wedge \dots \wedge dx_{i_{k+1}}) - (a \circ \alpha) \alpha^*(dx_{i_1} \wedge \dots \wedge dx_{i_{k+1}}) \\
&= (a \circ \beta) (d(\beta^* x_{i_1}) \wedge \dots \wedge d(\beta^* x_{i_{k+1}})) - \\
&\quad (a \circ \alpha) (d(\alpha^* x_{i_1}) \wedge \dots \wedge d(\alpha^* x_{i_{k+1}})) \\
&= (a \circ \beta) (dx_{i_1} \wedge \dots \wedge dx_{i_{k+1}}) - (a \circ \alpha) (dx_{i_1} \wedge \dots \wedge dx_{i_{k+1}}) \\
&= (a \circ \beta - a \circ \alpha) dx_I
\end{aligned}$$

Case 2. $\eta = b dx_J \wedge dt$ for any k -tuple J from $\{1, \dots, n\}$

Simplify the LHS:

$$\begin{aligned}
dL\eta + L d\eta &= d \left((-1)^k (\Gamma b) dx_J \right) + L (db \wedge dx_J \wedge dt) \\
&= \left[(-1)^k d(\Gamma b) \wedge dx_J \right] + \\
&\quad \left[L \left(\sum_{j=1}^n \frac{\partial b}{\partial x_j} dx_j \wedge dx_J \wedge dt + \frac{\partial b}{\partial t} dt \wedge dx_J \wedge dt \right) \right] \\
&= \left[(-1)^k \sum_{j=1}^n \frac{\partial}{\partial x_j} (\Gamma b) dx_j \wedge dx_J \right] + \left[\sum_{j=1}^n L \left(\frac{\partial b}{\partial x_j} dx_j \wedge dx_J \wedge dt \right) \right] \\
&= \left[(-1)^k \sum_{j=1}^n \frac{\partial}{\partial x_j} (\Gamma b) dx_j \wedge dx_J \right] + \left[\sum_{j=1}^n (-1)^{k+1} \left(\Gamma \frac{\partial b}{\partial x_j} \right) dx_j \wedge dx_J \right] \\
&= 0
\end{aligned}$$

since by Leibniz's rule [13, Theorem 39.1], $\frac{\partial}{\partial x_j} (\Gamma b) = \Gamma \frac{\partial b}{\partial x_j}$ for all j . Now we simplify the RHS:

$$\beta^* \eta - \alpha^* \eta = \beta^*(b dx_J \wedge dt) - \alpha^*(b dx_J \wedge dt)$$

$$\begin{aligned}
&= [(\beta^*b) d(\beta^*x_{j_1}) \wedge \cdots \wedge d(\beta^*x_{j_k}) \wedge d(\beta^*t)] - \\
&\quad [(\alpha^*b) d(\alpha^*x_{j_1}) \wedge \cdots \wedge d(\alpha^*x_{j_k}) \wedge d(\alpha^*t)] \\
&= [(b \circ \beta) dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dt] - [(b \circ \alpha) dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dt] \\
&= 0 - 0 = 0
\end{aligned}$$

This completes the proof of the lemma. \square

Remark 1.15. For the special case, when $k = 0$ we have $\eta = \sum_{i=1}^n a_i dx_i + b dt$. In this case, we have $L\eta = \Gamma b$ since J is empty. Hence, just as d is in some sense a “differentiation operator”, the operator L is in some sense an “integration operator”, one that integrates η in the direction of the last coordinate [13, Exercise 39.4].

Proposition 1.1. *Let U and V be open sets in \mathbb{R}^n and \mathbb{R}^m , respectively. Let $g, h : U \rightarrow V$ be smooth maps that are differentiably homotopic. Then there is a linear transformation*

$$T : \Omega^{k+1}(V) \rightarrow \Omega^k(U)$$

defined for all $k \geq 0$, such that

$$\begin{cases} dT\omega + T d\omega = h^*\omega - g^*\omega & \text{if } \omega \in \Omega^{k+1}(V), k \geq 0 \\ T df = h^*f - g^*f & \text{if } f \in C^\infty(V) = \Omega^0(V) \end{cases}$$

where $g^*, h^* : \Omega^k(V) \rightarrow \Omega^k(U)$ are the pullback maps defined for all $k \geq 0$.

Proof. The preceding lemma was a special case of this proposition since α and β were differentiably homotopic. We borrow notations from the preceding lemma.

Let $H : U \times [0, 1] \rightarrow V$ be the differentiable homotopy between g and h , i.e. $H(x, 0) = H(\alpha(x)) = g(x)$ and $H(x, 1) = H(\beta(x)) = h(x)$. Then we have the pullback map $H^* : \Omega^k(V) \rightarrow \Omega^k(W)$ defined on an open neighborhood W of $U \times [0, 1]$ and $k \geq 0$. Hence for $k \geq 0$ we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^{k+1}(V) & \xrightarrow{H^*} & \Omega^{k+1}(W) \\ & \searrow L \circ H^* & \downarrow L \\ & & \Omega^k(U) \end{array}$$

Claim: $T = L \circ H^*$

We will verify both the desired properties separately.

Step 1. $dT\omega + T d\omega = h^*\omega - g^*\omega$ if $\omega \in \Omega^{k+1}(V), k \geq 0$

Let $H^*\omega = \eta \in \Omega^{k+1}(W)$, then using **Theorem XI**, **Theorem X**, and the preceding lemma

$$\begin{aligned}
dT\omega + T d\omega &= d(L(H^*\omega)) + L(H^*(d\omega)) \\
&= dL\eta + L d\eta \\
&= \beta^*\eta - \alpha^*\eta \\
&= \beta^*(H^*\omega) - \alpha^*(H^*\omega) \\
&= (H \circ \beta)^*\omega - (H \circ \alpha)^*\omega \\
&= h^*\omega - g^*\omega
\end{aligned}$$

Step 2. $Tdf = h^*f - g^*f$ if $f \in C^\infty(V) = \Omega^0(V)$

Let $H^*f = \gamma \in \Omega^0(W)$, then using **Theorem XI**, **Theorem X**, and the preceding lemma

$$\begin{aligned}
Tdf &= L(H^*df) \\
&= Ld\gamma \\
&= \beta^*\gamma - \alpha^*\gamma \\
&= \beta^*(H^*f) - \alpha^*(H^*f) \\
&= (H \circ \beta)^*f - (H \circ \alpha)^*f \\
&= h^*f - g^*f
\end{aligned}$$

This completes the proof. □

1.2.2 Poincaré lemma

Definition 1.23 (Star-convex). Let U be an open set in \mathbb{R}^n . Then U is said to be *star-convex* with respect to the point $p \in U$ if for each $x \in U$, the line segment joining x and p lies in U .

Theorem 1.2 (Poincaré lemma). *Let U be a star-convex open set in \mathbb{R}^n . If $k \geq 1$, then every closed k -form on U is exact.*

Proof. Let $\omega \in \mathcal{Z}^k(U)$ for $k \geq 1$. We apply the preceding proposition. Let p be a point with respect to which U is star-convex. We define the maps g and h as follows:

$$\begin{array}{ccc}
g : U \rightarrow U & & h : U \rightarrow U \\
x \mapsto p & & x \mapsto x
\end{array}$$

Since U is star-convex with respect to p , there always exists a straight line in U joining any point $x \in U$ with p . Hence we have the differentiable homotopy between g and h given by this straight line

$$\begin{aligned}
H : U \times [0, 1] &\rightarrow U \\
(x, t) &\mapsto th(x) + (1-t)g(x)
\end{aligned}$$

Therefore the maps g and h are differentiably homotopic.

Now we use the previous proposition, i.e. there exists $T : \Omega^k(U) \rightarrow \Omega^{k-1}(U)$ such that $dT\omega + Td\omega = h^*\omega - g^*\omega$. Hence if $d\omega = 0$ then $dT\omega = \omega$ since pullback map corresponding to the identity map is the identity map i.e. $h^*\omega = \omega$ and pullback map corresponding to a constant map is the zero map i.e. $g^*\omega = 0$. Hence $\omega \in \mathcal{B}^k(U)$ for all $k \geq 1$. This completes the proof³. □

Remark 1.16. Being star-convex is not such a restrictive condition, since any open ball

$$B(p, \varepsilon) = \{x \in \mathbb{R}^n : \|x - p\| < \varepsilon\}$$

is star-convex with respect to p . Hence, Poincaré lemma holds for any open ball in \mathbb{R}^n .

³If we also use the second condition of the preceding proposition we get that if $df = 0$ then f is a constant map. This is Munkres' definition of exact 0-form [13, p. 259].

1.3 Differential forms on smooth manifolds

In this section some basic definitions and facts from [18, Chapter 2, 3 and 5] and [11, §1.1, 2.1, 3.2, 3.4 and 5.1] will be stated.

Definition 1.24 (Smooth manifold). A *smooth manifold* M of dimension n is a second countable Hausdorff space together with a smooth structure on it. A *smooth structure* \mathcal{U} is the collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ where U_α is an open set of M and ϕ_α is a homeomorphism of U_α onto an open set of \mathbb{R}^n such that

1. the open sets $\{U_\alpha\}_{\alpha \in A}$ cover M .
2. for every pair of indices $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the homeomorphisms

$$\begin{aligned}\phi_\alpha \circ \phi_\beta^{-1} &: \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta), \\ \phi_\beta \circ \phi_\alpha^{-1} &: \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)\end{aligned}$$

are smooth maps between open subsets of \mathbb{R}^n .

3. the family \mathcal{U} is maximal in the sense that it contains all possible pairs (U_α, ϕ_α) satisfying the properties 1. and 2.

Example 1.1. Following two smooth manifolds will be used throughout this report:

1. The Euclidean space \mathbb{R}^n is a smooth manifold with single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n}$ is the identity map. In other words, $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n}) = (\mathbb{R}^n, x_1, \dots, x_n)$ where x_1, \dots, x_n are the standard coordinates on \mathbb{R}^n .
2. Any open subset V of a smooth manifold M is also a smooth manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M , then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})\}$ is an atlas for V , where $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \rightarrow \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$.

Theorem XII. Every smooth manifold M is paracompact⁴.

Definition 1.25 (Smooth function on a manifold). Let M be a smooth manifold of dimension n . A function $f : M \rightarrow \mathbb{R}$ is said to be a *smooth function at a point* p in M if there is a chart (U, ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of \mathbb{R}^n , is smooth at $\phi(p)$. The function f is said to be smooth on M if it is smooth at every point of M .

$$\begin{array}{ccc} (U, p) & \xrightarrow{\phi} & (\mathbb{R}^n, \phi(p)) \\ & \searrow f & \downarrow f \circ \phi^{-1} \\ & & (\mathbb{R}, f(p)) \end{array}$$

Definition 1.26 (Smooth partition of unity). Let M be a smooth manifold with an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$. Then a *smooth partition of unity* on M subordinate to \mathcal{U} is a family of smooth functions $\{\psi_\alpha : M \rightarrow \mathbb{R}\}_{\alpha \in A}$ satisfying the following conditions

1. $\text{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$.
2. $0 \leq \psi_\alpha(p) \leq 1$ for all $p \in M$ and $\alpha \in A$
3. the collection of supports $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$ is locally finite.

⁴For definition and general properties of paracompact spaces, see [Appendix A](#).

$$4. \sum_{\alpha \in A} \psi_{\alpha}(p) = 1 \text{ for all } p \in M$$

where $\text{supp}(\psi_{\alpha})$ is the closure of the set of those $p \in M$ for which $\psi_{\alpha}(p) \neq 0$.

Theorem XIII. Any smooth manifold M with an open covering $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in A}$ admits a smooth partition of unity subordinate to $\{U_{\alpha}\}$.

Remark 1.17. If $\{\psi_{\alpha}\}$ is a smooth partition of unity on M subordinate to $\{U_{\alpha}\}$, and $\{f_{\alpha} : U_{\alpha} \rightarrow \mathbb{R}\}$ is a family of smooth functions, then the function $f : M \rightarrow \mathbb{R}$ defined by $f(x) = \sum_{\alpha \in A} \psi_{\alpha} f_{\alpha}$ is smooth.

Definition 1.27 (Smooth map between smooth manifolds). Let M and N be smooth manifolds of dimension m and n , respectively. A continuous map $F : M \rightarrow N$ is *smooth at a point* p if M if there are charts (V, ψ) about $F(p)$ in N and (U, ϕ) about p in M such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of \mathbb{R}^m to \mathbb{R}^n , is smooth at $\phi(p)$.

$$\begin{array}{ccc} (U, p) & \xrightarrow{F} & (V, F(p)) \\ \downarrow \phi & & \downarrow \psi \\ (\mathbb{R}^m, \phi(p)) & \xrightarrow{\psi \circ F \circ \phi^{-1}} & (\mathbb{R}^n, \psi(F(p))) \end{array}$$

The continuous map $F : M \rightarrow N$ is said to be *smooth* if it is smooth at every point in M .

Remark 1.18. In the definition of smooth maps between manifolds it's assumed that $F : M \rightarrow N$ is continuous to ensure that $F^{-1}(V)$ is an open set in M . Thus, smooth maps between manifolds are by definition continuous.

Theorem XIV. Let M and N be smooth manifolds of dimension m and n , respectively, and $F : M \rightarrow N$ a continuous map. The following are equivalent

1. The map $F : M \rightarrow N$ is smooth
2. There are atlases \mathcal{U} for M and \mathcal{V} for N such that for every chart (U, ϕ) in \mathcal{U} and (V, ψ) in \mathcal{V} the map

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^n$$

is smooth.

3. For every chart (U, ϕ) on M and (V, ψ) on N , the map

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^n$$

is smooth.

Theorem XV. If (U, ϕ) is a chart on a smooth manifold M of dimension n , then the coordinate map $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Remark 1.19. One can generalize the notation for projection maps introduced in [Remark 1.11](#). If (U, ϕ) is a chart of a manifold, i.e. $\phi : U \rightarrow \mathbb{R}^n$, then let $r_i = x_i \circ \phi$ be the i^{th} component of ϕ and write $\phi = (r_1, \dots, r_n)$ and $(U, \phi) = (U, r_1, \dots, r_n)$. Thus, for $p \in U$, $(r_1(p), \dots, r_n(p))$ is a point in \mathbb{R}^n . The functions r_1, \dots, r_n are called *coordinates* or *local coordinates* on U .

Theorem XVI. Let M and N be smooth manifolds of dimension m and n , respectively, and $F : M \rightarrow N$ a continuous map. The following are equivalent

1. The map $F : M \rightarrow N$ is smooth

2. The manifold N has an atlas such that for every chart $(V, \psi) = (V, s_1, \dots, s_n)$ in the atlas⁵, the components $s_i \circ F : F^{-1}(V) \rightarrow \mathbb{R}$ of f relative to the chart are all smooth.
3. For every chart $(V, \psi) = (V, s_1, \dots, s_n)$ on N , the components $s_i \circ F : F^{-1}(V) \rightarrow \mathbb{R}$ of F relative to the chart are all smooth.

1.3.1 Tangent space

Definition 1.28 (Germ of smooth functions). Consider the set of all pairs (f, U) , where U is a neighborhood of $p \in M$ and $f : U \rightarrow \mathbb{R}$ is a smooth function. Then (f, U) is said to be equivalent to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W . This equivalence class of (f, U) is called *germ of f at p* .

Remark 1.20. The set of all germs of smooth functions on M at p is denoted by $C_p^\infty(M)$. The addition and multiplication of functions induce corresponding operations of $C_p^\infty(M)$, making it into a ring; with scalar multiplication by real numbers $C_p^\infty(M)$ becomes an algebra over \mathbb{R} .

Definition 1.29 (Derivation at a point). A linear map $X_p : C_p^\infty(M) \rightarrow \mathbb{R}$ satisfying the Leibniz rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

is called a *derivation at $p \in M$* or a *point-derivation of $C_p^\infty(M)$* .

Definition 1.30 (Tangent vector). A *tangent vector* at a point p in a manifold M is a derivation at p .

Definition 1.31 (Tangent space). The tangent vectors at p form a real vector space T_pM , called the *tangent space* of M at p .

Definition 1.32 (Partial derivative). Let M be a smooth manifold of dimension n , $(U, \phi) = (U, r_1, \dots, r_n)$ be a chart and $f : M \rightarrow \mathbb{R}$ be a smooth function. For $p \in U$, the *partial derivative* $\partial f / \partial r_i$ of f with respect to r_i at p is defined to be

$$\frac{\partial}{\partial r_i} \Big|_p f := \frac{\partial f}{\partial r_i}(p) := \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p)) := \frac{\partial}{\partial x_i} \Big|_{\phi(p)} (f \circ \phi^{-1})$$

where $r_i = x_i \circ \phi$ and $\{x_1, \dots, x_n\}$ are the standard coordinates on \mathbb{R}^n .

Definition 1.33 (Pushforward of a vector). Let $F : M \rightarrow N$ be a smooth map between two smooth manifolds. At each point $p \in M$, the map F induces a linear map of tangent spaces

$$F_* : T_pM \rightarrow T_{F(p)}N$$

such that given $X_p \in T_pM$ we have $(F_*(X_p))f = X_p(f \circ F) \in \mathbb{R}$ for $f \in C_{F(p)}^\infty(N)$.

Remark 1.21. The pushforward map induced by the identity map of manifolds is the identity map of vector spaces, i.e. $(\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM}$.

Theorem XVII. Let $F : M \rightarrow N$ and $G : N \rightarrow N'$ be smooth maps of manifolds, and $p \in M$, then $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$

$$\begin{array}{ccc} T_pM & \xrightarrow{F_{*,p}} & T_{F(p)}N \\ & \searrow & \downarrow G_{*,F(p)} \\ & & T_{G(F(p))}N' \end{array}$$

⁵Here $s_i = y_i \circ \psi$ if we consider the coordinates of \mathbb{R}^n to be (y_1, \dots, y_n) and coordinates of \mathbb{R}^m to be (x_1, \dots, x_m) .

Theorem XVIII. Let $(U, \phi) = (U, r_1, \dots, r_n)$ be a chart about a point p in a manifold M of dimension n . Then $\phi_* : T_p M \rightarrow T_{\phi(p)} \mathbb{R}^n$ is a vector space isomorphism and $T_p M$ has the basis

$$\left\{ \frac{\partial}{\partial r_i} \Big|_p, \dots, \frac{\partial}{\partial r_n} \Big|_p \right\}$$

where $r_i = x_i \circ \phi$ and $\{x_1, \dots, x_n\}$ the standard coordinates of \mathbb{R}^n .

Remark 1.22. Hence one observes that if M is n dimensional manifold then $T_p M$ is a vector space of dimension n over \mathbb{R} .

1.3.2 Cotangent bundle

Definition 1.34 (Cotangent space). Let M be a smooth manifold and p a point in M . The *cotangent space* of M at point p denoted by $T_p^* M$ is defined to be the dual space of the tangent space $T_p M$, i.e. the set of all linear maps from $T_p M$ to \mathbb{R} .

Remark 1.23. Hence, if M is n dimensional manifold then $T_p^* M$ is a vector space of dimension n over \mathbb{R} .

Definition 1.35 (Cotangent bundle). The *cotangent bundle* $T^* M$ of a manifold M is the union of the tangent spaces at all points of M

$$T^* M := \bigcup_{p \in M} T_p^* M$$

Remark 1.24. The union in the definition above is disjoint, i.e. $T^* M = \coprod_{p \in M} T_p^* M$, since for distinct points p and q in M , the cotangent spaces $T_p^* M$ and $T_q^* M$ are already disjoint.

Theorem XIX. Let M is a smooth manifold of dimension n , then its cotangent bundle $T^* M$ is a smooth manifold of dimension $2n$.

Definition 1.36 (Smooth vector bundle). A *smooth vector bundle* of rank n is a triple (E, M, π) consisting of a pair of smooth manifolds E and M , and a smooth surjective map $\pi : E \rightarrow M$ satisfying the following conditions

1. for each $p \in M$, the inverse image $E_p = \pi^{-1}(p)$ is an n -dimensional vector space over \mathbb{R} ,
2. for each $p \in M$, there is an open neighborhood U of p and a diffeomorphism $\phi : U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$ such that

(a) the following diagram commutes

$$\begin{array}{ccc} U \times \mathbb{R}^n & \xrightarrow{\phi} & \pi^{-1}(U) \\ & \searrow p_1 & \swarrow \pi \\ & & U \end{array}$$

where p_1 is the projection onto the first factor,

- (b) for each $q \in U$, the map $\phi_q : \mathbb{R}^n \rightarrow \pi^{-1}(q)$, defined by $\phi_q(x) = \phi(q, x)$, is a linear isomorphism.

Theorem XX. The cotangent bundle $T^* M$ with the projection map $\pi : T^* M \rightarrow M$ given by $\pi(\alpha) = p$ if $\alpha \in T_p^* M$, is a vector bundle of rank n over M .

Definition 1.37 (Exterior power of cotangent bundle). Let M be a smooth manifold. Then the k^{th} exterior power of the cotangent bundle $\Lambda^k(T^*M)$ is the disjoint union of all alternating k -tensors at all points of the manifold, i.e.

$$\Lambda^k(T^*M) = \bigcup_{p \in M} \mathcal{A}^k(T_p M)$$

Theorem XXI. If M is a manifold of dimension n , then the exterior power of the cotangent bundle $\Lambda^k(M)$ is a manifold of dimension $n + \binom{n}{k}$.

Theorem XXII. The exterior power of cotangent bundle $\Lambda^k(T^*M)$ with the projection map $\pi : \Lambda^k(T^*M) \rightarrow M$ given by $\pi(\alpha) = p$ if $\alpha \in \mathcal{A}^k(T_p M)$, is a vector bundle of rank $\binom{n}{k}$ over M .

1.3.3 Differential forms

Definition 1.38 (Smooth section). A smooth section of a vector bundle $\pi : E \rightarrow M$ is a smooth map $s : M \rightarrow E$ such that $\pi \circ s = \mathbb{1}_M$.

Remark 1.25. The condition $\pi \circ s = \mathbb{1}_M$ precisely means that for each p in M , s maps p into E_p .

Definition 1.39 (Differential k -form). A differential k -form on M is a smooth section of the vector bundle $\pi : \Lambda^k(T^*M) \rightarrow M$.

Remark 1.26. The vector space of all smooth k -forms on M is denoted by $\Omega^k(M)$. If $\omega \in \Omega^k(M)$ then $\omega : M \rightarrow \Lambda^k(T^*M)$ is a smooth map such that ω assigns each point $p \in M$ an alternating k -tensor, i.e. $\omega_p \in \mathcal{A}^k(T_p M)$ for all $p \in M$. In particular, if U is an open subset of M , then $\omega \in \Omega^k(U)$ if $\omega_p \in \mathcal{A}^k(T_p M)$ for all $p \in U$ (view U as open neighborhood of point p).

Definition 1.40 (Differential 0-form). A differential 0-form on M is a smooth real valued function on M , i.e. $\Omega^0(M) = C^\infty(M)$.

Definition 1.41 (Wedge product of 0-form and k -form). The wedge product of a 0-form $f \in C^\infty(M)$ and a k -form $\omega \in \Omega^k(M)$ is defined as the k -form $f\omega$ where

$$(\omega \wedge f)_p = (f \wedge \omega)_p = f(p) \cdot \omega_p$$

for all $p \in M$.

Definition 1.42. The wedge product extends pointwise to differential forms on a manifold, i.e. if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ then $\omega \wedge \eta \in \Omega^{k+\ell}(M)$ such that

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

at all $p \in M$.

1.3.4 Exterior derivative

Definition 1.43 (Differential of a function). Let $f : M \rightarrow \mathbb{R}$ be a smooth function, its differential is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_p M$

$$(df)_p(X_p) = X_p f$$

Remark 1.27. Let (U, r_1, \dots, r_n) be a coordinate chart on a smooth manifold M . Then the differentials $\{dr_1, \dots, dr_n\}$ are 1-forms on U . At each point $p \in U$, the 1-forms $\{(dr_1)_p, \dots, (dr_n)_p\}$

form a basis⁶ of $\mathcal{A}^1(T_p M) = T_p^* M$, dual to the basis $\{\partial/\partial r_1|_p, \dots, \partial/\partial r_n|_p\}$ for the tangent space $T_p M$. Hence, a 1-form on U is a linear combination $\omega = \sum_{i=1}^n a_i dr_i$ where a_i are smooth functions on U .

If $I = (i_1, \dots, i_k)$ is an ascending k -tuple from the set $\{1, \dots, n\}$, then

$$dr_I = dr_{i_1} \wedge \dots \wedge dr_{i_k}$$

denotes the elementary k -forms on $U \subset M$, i.e. the k -forms

$$\{(dr_I)_p : I \text{ is an ascending set } k\text{-tuple}\}$$

form a basis of $\mathcal{A}^k(T_p M)$ for all $p \in U$. The general k -form $\omega \in \Omega^k(U)$ can be written uniquely in the form

$$\omega = \sum_{[I]} a_I dr_I$$

for some $a_I \in C^\infty(U)$.

Theorem XXIII. *If f is a smooth function on M , then the restriction of the 1-form df to U can be expressed as*

$$df = \frac{\partial f}{\partial r_1} dr_1 + \dots + \frac{\partial f}{\partial r_n} dr_n$$

Theorem XXIV. *$\omega \in \Omega^k(M)$ if and only if on every chart $(U, \phi) = (U, r_1, \dots, r_n)$ on M , the coefficients a_I of $\omega = \sum_{[I]} a_I dr_I$ relative to the elementary k -forms $\{dr_I\}$ are all smooth.*

Theorem XXV. *Suppose ω is a smooth differential form defined on a neighborhood U of a point p in a manifold M , i.e. $\omega \in \Omega^k(U)$. Then there exists a smooth form $\tilde{\omega}$ on M , i.e. $\tilde{\omega} \in \Omega^k(M)$, that agrees with ω on a possible smaller neighborhood of p .*

Remark 1.28. The extension $\tilde{\omega}$ is not unique, it depends on p and on the choice of a bump function at p . All this can be generalized to a family of differential forms, as in [Remark 1.17](#), using smooth partitions of unity.

Definition 1.44 (Differential of a k -form). Let (U, r_1, \dots, r_n) be a coordinate chart on a smooth manifold M and $\omega \in \Omega^k(U)$ is written uniquely as a linear combination

$$\omega = \sum_{[I]} a_I dr_I, \quad a_I \in C^\infty(U)$$

The \mathbb{R} -linear map $d_U : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$ defined as

$$d_U \omega = \sum_{[I]} da_I \wedge dr_I$$

is called the exterior derivative of ω on U . Let $p \in U$, then $(d_U \omega)_p$ is independent of the chart containing p . Thus the *differential of a k -form* is defined by the linear operator

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

such that for $k \geq 0$ and $\omega \in \Omega^k(M)$ one has $(d\omega)_p = (d_U \omega)_p$ for all $p \in M$.

Theorem XXVI. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ then*

1. (*Antiderivation*) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

⁶In the case of $M = \mathbb{R}^n$ the expression was much more straightforward because $T_p M \cong \mathbb{R}^n$ (vector space isomorphism) for any n -dimensional manifold.

2. $d \circ d = 0$

Remark 1.29. Since the exterior derivative is an antiderivation, it is a local operator, i.e. for all $k \geq 0$, whenever a k -form $\omega \in \Omega^k(M)$ is such that $\omega_p = 0$ for all points p in an open set U of M , then $d\omega \equiv 0$ on U . Equivalently, for all $k \geq 0$, whenever two k -forms $\omega, \eta \in \Omega^k(M)$ agree on an open set U , then $d\omega \equiv d\eta$ on U [13, Proposition 19.3].

Definition 1.45 (Pullback of a k -form). Let $F : M \rightarrow N$ be a smooth map of manifolds. Then for $k \geq 1$

$$F^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

is the pullback map defined for each $\omega \in \Omega^k(N)$ at every point $p \in M$ as

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(F_{*,p}v_1, \dots, F_{*,p}v_k)$$

where $v_i \in T_pM$. Then the k -form $F^*\omega \in \Omega^k(M)$ is called the *pullback* of $\omega \in \Omega^k(N)$.

Definition 1.46 (Pullback of a 0-form). Let $F : M \rightarrow N$ be a smooth map and $f \in C^\infty(N) = \Omega^0(N)$, then the *pullback* of f is the 0-form $F^*f = f \circ F \in \Omega^0(M)$.

Remark 1.30. Pullback of the identity map is an identity map, i.e. $(\mathbb{1}_M)^* = \mathbb{1}_{\Omega^k(M)}$.

Theorem XXVII. If $F : M \rightarrow N$ and $G : N \rightarrow N'$ are smooth maps, then $(G \circ F)^* = F^* \circ G^*$.

$$\begin{array}{ccc} \Omega^k(N') & \xrightarrow{G^*} & \Omega^k(N) \\ & \searrow (G \circ F)^* & \downarrow F^* \\ & & \Omega^k(M) \end{array}$$

Theorem XXVIII. Let $F : M \rightarrow N$ be a smooth map. If ω, η and θ are differential forms on N , such that ω and η have same order, then

1. (preservation of the vector space structure) $F^*(a\omega + b\eta) = a(F^*\omega) + b(F^*\eta)$ for all $a, b \in \mathbb{R}$.
2. (preservation of the wedge product) $F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$.
3. (commutation with the differential) $F^*(d\omega) = d(F^*\omega)$, i.e. the following diagram commutes

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) \\ \downarrow F^* & & \downarrow F^* \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \end{array}$$

1.4 Closed and exact forms on smooth manifolds

In this section the de Rham cohomology will be defined and generalization of Poincaré lemma to smooth manifolds will be discussed following [18, §24].

Definition 1.47 (Closed forms). $\omega \in \Omega^k(U)$ for $k \geq 0$ is said to be *closed* if $d\omega = 0$.

Remark 1.31. We denote the set of all closed k -forms on M by $\mathcal{Z}^k(M)$. The sum of two such k -forms is another closed k -form, and so is the product of a closed k -form by a scalar. Hence $\mathcal{Z}^k(M)$ is the vector sub-space of $\Omega^k(M)$.

Definition 1.48 (Exact k -forms). $\omega \in \Omega^k(U)$ for $k \geq 1$ is said to be *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$.

Remark 1.32. We denote the set of all exact k -forms on M by $\mathcal{B}^k(U)$. The sum of two such k -forms is another exact k -form, and so is the product of a exact k -form by a scalar. Hence $\mathcal{B}^k(M)$ is the vector sub-space of $\Omega^k(M)$. Also, $\mathcal{B}^0(M)$ is defined to be the set consisting only zero.

Theorem 1.3. *On a smooth manifold M , every exact form is closed.*

Proof. Let $\omega \in \mathcal{B}^k(M)$ such that $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. From **Theorem XXVI** we know that $d\omega = d(d\eta) = 0$ hence $\omega \in \mathcal{Z}^k(M)$ for all $k \geq 1$. For $k = 0$, the statement is trivially true. \square

Lemma 1.2. *Let $F : M \rightarrow N$ be a smooth map of manifolds, then the pullback map F^* sends closed forms to closed forms, and sends exact forms to exact forms.*

Proof. Suppose ω is closed. From **Theorem XXVIII** we know that F^* commutes with d

$$dF^*\omega = F^*d\omega = 0$$

Hence, $F^*\omega$ is also closed. Next suppose $\theta = d\eta$ is exact. Then

$$F^*\theta = F^*d\eta = dF^*\eta$$

Hence, $F^*\theta$ is exact. \square

1.4.1 de Rham cohomology

Definition 1.49 (de Rham cohomology of a smooth manifold). The k^{th} *de Rham cohomology group*⁷ of M is the quotient group

$$H_{dR}^k(M) := \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)}$$

Remark 1.33. Hence, the de Rham cohomology of a smooth manifold measures the extent to which closed forms are not exact on that manifold.

Proposition 1.2. *If the smooth manifold M has ℓ connected components, then its de Rham cohomology in degree 0 is $H_{dR}^0(M) = \mathbb{R}^\ell$. An element of $H_{dR}^0(M)$ is specified by an ordered ℓ -tuple of real numbers, each real number representing a constant function on a connected component of M .*

Proof. Since there are no non-zero exact 0-forms

$$H_{dR}^0(M) = \mathcal{Z}^0(M)$$

Suppose f is a closed 0-form on M , i.e. $f \in C^\infty(M)$ such that $df = 0$. By **Theorem XXIII** we know that on any chart (U, r_1, \dots, r_n)

$$df = \sum_{i=1}^n \frac{\partial f}{\partial r_i} dr_i$$

Thus $df = 0$ on U if and only if all the partial derivatives $\partial f / \partial r_i$ vanish identically on U . This is equivalent to f being locally constant on U . Hence, $\mathcal{Z}^0(M)$ is the set of all locally constant⁸ functions on M . Such a function must be constant on each connected component of M . If M has ℓ connected components then a locally constant function of M can be specified by an ordered set of ℓ real numbers. Thus $\mathcal{Z}^0(M) = \mathbb{R}^\ell$. \square

⁷which is really a vector space over \mathbb{R}

⁸Locally constant functions are constant on any connected component of domain.

Proposition 1.3. *On a smooth manifold M of dimension n , the de Rham cohomology $H_{dR}^k(M)$ vanishes for $k > n$.*

Proof. At any point $p \in M$, the tangent space T_pM is a vector space of dimension n . If $\omega \in \Omega^k(M)$, then $\omega_p \in \mathcal{A}^k(T_pM)$, the space of alternating k -linear functions on T_pM . By [Remark 1.8](#), if $k > n$ then $\mathcal{A}^k(T_pM) = 0$. Hence for $k > n$, the only k -form on M is the zero form. \square

1.4.2 Poincaré lemma for smooth manifolds

Definition 1.50 (Pullback map in cohomology). Let $F : M \rightarrow N$ be a smooth map of manifolds, then its pullback F^* induces⁹ a linear map of quotient spaces, denoted by $F^\#$

$$F^\# : \frac{\mathcal{Z}^k(N)}{\mathcal{B}^k(N)} \rightarrow \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)}$$

$$[[\omega]] \mapsto [[F^*(\omega)]]$$

This is a map in cohomology $F^\# : H_{dR}^k(N) \rightarrow H_{dR}^k(M)$ called the *pullback map in cohomology*.

Remark 1.34. From [Remark 1.30](#) and [Theorem XXVII](#) it follows that:

1. If $\mathbb{1}_M : M \rightarrow M$ is the identity map, then $\mathbb{1}_M^\# : H_{dR}^k(M) \rightarrow H_{dR}^k(M)$ is also the identity map.
2. If $F : M \rightarrow N$ and $G : N \rightarrow N'$ are smooth maps, then $(G \circ F)^\# = F^\# \circ G^\#$.

Proposition 1.4 (Diffeomorphism invariance of de Rham cohomology). *Let $F : M \rightarrow N$ be a diffeomorphism of manifolds, then the pullback map in cohomology $F^\# : H_{dR}^k(N) \rightarrow H_{dR}^k(M)$ is an isomorphism.*

Proof. Since F is a diffeomorphism, $F^{-1} : N \rightarrow M$ is also a smooth map of manifolds. Therefore, using [Remark 1.34](#) we have

$$\mathbb{1}_{H_{dR}^k(M)} = \mathbb{1}_M^\# = (F^{-1} \circ F)^\# = F^\# \circ (F^{-1})^\#$$

This implies that $(F^{-1})^\#$ is the inverse of $F^\#$, i.e. $F^\#$ is an isomorphism. \square

Theorem 1.4 (Poincaré lemma for smooth manifold). *Let M be a smooth manifold, then for all $p \in M$ there exists an open neighborhood U such that every closed k -form on U is exact for $k \geq 1$.*

Proof. Let (U, ϕ) be a chart on a smooth manifold M of dimension n such that $p \in U$. By [Theorem XV](#) we know that the coordinate map $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism. We choose U such that $\phi(U)$ is an open ball in \mathbb{R}^n . Then by [Theorem 1.2](#) every closed k -form on $\phi(U)$ is exact for $k \geq 1$, i.e. $H_{dR}^k(\phi(U)) = 0$ for $k \geq 1$. Now we can use [Proposition 1.4](#) to conclude that $H_{dR}^k(U) = 0$ for $k \geq 1$, i.e. every closed k -form on U is exact for $k \geq 1$. \square

⁹Follows from [Lemma 1.2](#).

Chapter 2

Čech cohomology

“Cohomology is a way of attaching ordinary groups to sheaves of groups (or rings to sheaves of rings, etc.) which measure the global aspects of a sheaf. Sheaves are designed to make all local statements easy to formulate, and this is because in many instances local statements are easy to come by. However in geometry one is usually interested in global information.”

— Rick Miranda, *Algebraic Curves and Riemann Surfaces*, p. 290

2.1 Sheaf theory

Definition 2.1 (Presheaf). A *presheaf*¹ \mathcal{F} of abelian groups on a topological space X consists of an abelian group $\mathcal{F}(U)$ for every open subset $U \subset X$ and a group homomorphism $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for any two nested open subsets $V \subset U$ satisfying the following two conditions:

1. for any open subset U of X one has $\rho_{UU} = \mathbb{1}_{\mathcal{F}(U)}$
2. for open subsets $W \subset V \subset U$ one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho_{UW}} & \mathcal{F}(W) \\ & \searrow \rho_{UV} & \nearrow \rho_{VW} \\ & & \mathcal{F}(V) \end{array}$$

Example 2.1. Let G be a non-trivial abelian group and X be a topological space. Then the *constant presheaf* \mathcal{G}_X is defined to be the collection of abelian groups $\mathcal{G}_X(U) = G$ for all non-empty subsets U of X and $\mathcal{G}_X(\emptyset) = \{0\}$, along with the group homomorphisms $\rho_{UV} = \mathbb{1}_G$ for nested open subsets $V \subset U$. In particular, for $G = \mathbb{R}$ we get the constant presheaf \mathbb{R} which is the collection of constant real valued functions $\mathbb{R}(U)$ for every open subset U of X and restriction maps ρ_{UV} for nested open subsets $V \subset U$.

Example 2.2. Let X be a topological space. For each open subset U of X we define $\mathcal{F}(U)$ to be the set of (continuous/differentiable) real valued functions², and ρ_{UV} to be the natural restriction map for the nested open subsets $V \subset U$. Then \mathcal{F} is a presheaf of (continuous/differentiable) real valued functions.

Definition 2.2 (Sheaf). A presheaf \mathcal{F} on a topological space X is called a *sheaf* if for every collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X with $U = \cup_{\alpha \in A} U_\alpha$ the following conditions are satisfied

¹Presheaves and sheaves are typically denoted by calligraphic letters, \mathcal{F} being particularly common, presumably for the French word for sheaves, *faisceaux*.

²Note that there exists only one function from an empty set to any other set, hence $\mathcal{F}(\emptyset)$ is singleton.

1. (Uniqueness) If $f, g \in \mathcal{F}(U)$ and $\rho_{UU_\alpha}(f) = \rho_{UU_\alpha}(g)$ for all $\alpha \in A$, then $f = g$.
2. (Gluing) If for all $\alpha \in A$ we have $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $\rho_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$ for any $\alpha, \beta \in A$, then there exists a $f \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}(f) = f_\alpha$ for all $\alpha \in A$ (this f is unique by previous axiom).

Example 2.3. It is easy to observe that the gluing axiom doesn't hold for the constant presheaf \mathbb{R} on X if X is disconnected. We therefore define a *constant sheaf* $\underline{\mathbb{R}}$ on X to be the collection of *locally constant* real valued functions $\underline{\mathbb{R}}(U)$ corresponding to every open subset $U \subset X$ and restriction maps ρ_{UV} for nested open subsets $V \subset U$.

In general, given a non-trivial abelian group G , the *constant sheaf* \underline{G} on X is defined by endowing G with the discrete topology and assigning each open set U of X the set $\underline{G}(U)$ of all continuous functions $f : U \rightarrow G$ along with the restriction maps ϕ_{UV} for nested open sets $V \subset U$.

Example 2.4. If one has a presheaf of functions (or forms) on X which is defined by some property which is a local property³, then the presheaf is also a sheaf. This is because the agreement of functions (or forms) on the overlap intersections automatically gives a well defined unique function (or form) on the open set U , and one must only check that it satisfies the property [9, p. 272].

In particular, if X is a smooth manifold then Ω^q is the sheaf of smooth q -forms on X such that for every open subset U of X we have the abelian group $\Omega^q(U)$ of smooth q -forms on U (smooth sections of a exterior power of cotangent bundle, i.e. smooth maps of manifolds) along with the natural restriction maps as the group homomorphisms ρ_{UV} for nested open subsets $V \subset U$ [20, Example II.1.9].

Remark 2.1. When defining presheaf, many authors like Liu [7, §2.2.1] and Miranda [9, §IX.1], additionally require $\mathcal{F}(\emptyset) = 0$, i.e. the trivial group with one element. This is a necessary condition for the sheaf to be well defined, but this follows from our sheaf axioms. Namely, note that the empty set is covered by the empty open covering, and hence the collection $f_i \in \mathcal{F}(U_i)$ from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in⁴. In other words, we don't require this condition while defining presheaf (see [20, §II.1] or [1, §II.10]) since from the definition of sheaf one can deduce that that $\mathcal{F}(\emptyset)$ is equal to a final object, which in the case of a sheaf of sets is a singleton.

Remark 2.2. There is another equivalent way of defining sheaf \mathcal{F} (of abelian groups) over X as a triple (F, π, X) which satisfies certain axioms [4, §2.1]. For a discussion on the equivalence of both these definitions see [19, §5.7]. However, the definition that we have adopted is useful since for many important sheaves, particularly those that arise in algebraic geometry, the sheaf space F is obscure, and its topology complicated [6, Remark 2.6].

Remark 2.3. The definition of sheaf can be generalized to any category like groups, rings, modules, and algebras instead of abelian groups.

2.1.1 Stalks

Definition 2.3 (Stalk). Let \mathcal{F} be a sheaf on X , and let $x \in X$. Then the *stalk* of \mathcal{F} at x is

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

³A property \mathcal{P} is said to be *local* if whenever $\{U_\alpha\}_{\alpha \in A}$ is an open cover of an open set U , then the property holds on U if and only if it holds for each U_α . In other words, a *local property* \mathcal{P} of functions is the one which is initially defined at points, i.e. a function f defined in a neighborhood of a point $p \in X$ has property \mathcal{P} at p if and only if some condition holds at the point p . For example, the properties like continuity and differentiability.

⁴The Stacks project, Tag 006U: <https://stacks.math.columbia.edu/tag/006U>

where the direct limit⁵ is indexed over all the open subsets containing x , with order relation induced by reverse inclusion, i.e. $U < V$ if $V \subset U$. Also, the image of $f \in \mathcal{F}(U)$ in \mathcal{F}_x under the group homomorphism induced⁶ by the inclusion map $\mathcal{F}(U) \hookrightarrow \coprod_{U \ni x} \mathcal{F}(U)$ is denoted by f_x , i.e. $\llbracket f \rrbracket = f_x$.

Remark 2.4. This definition of stalks also holds for presheaves, which leads to the useful tool of *sheafification*, i.e. finding sheaf associated to a given presheaf. This technique of sheafification is very useful but we won't need it in this report. For more details, see the books by Hirzebruch [4, §2] and Liu [7, §2.2.1].

Lemma 2.1. *Let \mathcal{F} be a sheaf of abelian groups on X and $f, g \in \mathcal{F}(X)$ be such that $f_x = g_x$ for every $x \in X$. Then $f = g$.*

Proof. Without loss of generality, we may assume $g = 0$. Then $f_x = 0$ implies that f_x and 0 belong to same equivalence class, i.e. for every $x \in X$ there exists an open neighborhood U_x of x such that $\rho_{XU_x}(f) = 0$. As $\{U_x\}_{x \in X}$ covers X , we have $f = 0$ by the uniqueness condition of sheaf. \square

2.1.2 Sheaf maps

Definition 2.4 (Map of sheaves). Let \mathcal{F} and \mathcal{G} be sheaves of abelian groups on a topological space X . A maps of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ on X is given by a collection of group homomorphisms $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for any open subset U of X , which commute with the group homomorphisms ρ for the two sheaves, i.e. for $V \subset U$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow \rho_{UV}^{\mathcal{F}} & & \downarrow \rho_{UV}^{\mathcal{G}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

Example 2.5. The identity map $\mathbb{1}_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}$ is always a sheaf map, and the composition of sheaf maps is a sheaf map.

Example 2.6. As seen above, for the sheaf of functions (or forms) the natural restriction map is the group homomorphism ρ_{UV} for nested open subsets $V \subset U$. Also, from [Remark 1.29](#) we know that the exterior derivative is a local operator, hence it commutes with restriction. Therefore, $d : \Omega^q \rightarrow \Omega^{q+1}$ is a map of sheaves, where Ω^q and Ω^{q+1} are sheaves of smooth q -forms and $q+1$ -forms, respectively, defined on a smooth manifold X for $q \geq 0$. In other words, [Remark 1.29](#) implies that the following diagram commutes for $V \subset U$

$$\begin{array}{ccc} \Omega^q(U) & \xrightarrow{d_U} & \Omega^{q+1}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \Omega^q(V) & \xrightarrow{d_V} & \Omega^{q+1}(V) \end{array}$$

where by abuse of notation we use the same symbol for restriction maps of both sheaves.

Definition 2.5 (Associated presheaf). Given a sheaf map $\phi : \mathcal{F} \rightarrow \mathcal{G}$ between two sheaves of abelian groups on X , one constructs the *associated presheaves* $\ker(\phi)$, $\text{im}(\phi)$, and $\text{coker}(\phi)$

⁵For the definition of direct limit see [Appendix B](#). To get the direct system $\{\mathcal{F}(U), \rho_{UV}\}$, the “reverse inclusion” is defined to be the order relation for the directed set.

⁶As defined in the universal property of direct limit, see [Theorem B.1](#).

which are defined in the obvious way⁷, i.e. $\ker(\phi)(U) = \ker(\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U))$ with group homomorphism ρ inherited from \mathcal{F} .

Proposition 2.1. *Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a sheaf map between two sheaves of abelian groups on X , then $\ker(\phi)$ is a sheaf.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets of X , and $U = \cup_{\alpha \in A} U_\alpha$ be their union. It suffices to show that if for all $\alpha \in A$ we have $f_\alpha \in \ker(\phi_{U_\alpha})$ such that $\rho_{U_\alpha, U_\alpha \cap U_\beta}^{\mathcal{F}}(f_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}^{\mathcal{F}}(f_\beta)$ for any $\alpha, \beta \in A$, then there exists a unique $f \in \ker(\phi_U)$ such that $\rho_{U, U_\alpha}^{\mathcal{F}}(f) = f_\alpha$ for all $\alpha \in A$.

Since \mathcal{F} is a sheaf, there exists a unique element $f \in \mathcal{F}(U)$ such that $\rho_{U, U_\alpha}^{\mathcal{F}}(f) = f_\alpha$ for all $\alpha \in A$. We just need to show that $f \in \ker(\phi_U)$, i.e. $\phi_U(f) = 0$ in $\mathcal{G}(U)$.

Let $g_\alpha = \rho_{U, U_\alpha}^{\mathcal{G}}(\phi_U(f))$, then by the commutativity of ϕ with ρ , we have that

$$g_\alpha = \rho_{U, U_\alpha}^{\mathcal{G}}(\phi_U(f)) = \phi_{U_\alpha}(\rho_{U, U_\alpha}^{\mathcal{F}}(f)) = \phi_{U_\alpha}(f_\alpha) = 0$$

since $f_\alpha \in \ker(\phi_{U_\alpha})$. Now using the uniqueness axiom for the sheaf \mathcal{G} we conclude that $\phi_U(f) = 0$, since $\rho_{U, U_\alpha}^{\mathcal{G}}(\phi_U(f)) = 0$ for all $\alpha \in A$. \square

Example 2.7. Let X be a smooth manifold and $d : \Omega^q \rightarrow \Omega^{q+1}$ be the exterior derivative. Then $\ker(d) = \mathcal{Z}^q$ is the sheaf of closed q -forms on X .

Remark 2.5. There is an important subtlety here. The associated presheaves $\text{im}(\phi)$ and $\text{coker}(\phi)$ need not be sheaves in general. Also, in general, quotient of sheaves need not be a sheaf. In order to define the cokernel, image and quotient sheaf one need to use sheafification, see [5, Definition B.0.26] and [3, pp. 36-37].

Definition 2.6 (Injective map of sheaves). A map of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is called injective if for every open subset U of X , ϕ_U is an injective group homomorphism.

Definition 2.7 (Surjective map of sheaves). A map of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is called surjective if for every $x \in X$ the induced map of stalks⁸ $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is a surjective group homomorphism.

Remark 2.6. In other words, ϕ is surjective if for every point $x \in X$, every open set U containing x and every $g \in \mathcal{G}(U)$, there is an open subset $V \subset U$ containing x such that $\phi_V(f) = \rho_{U, V}^{\mathcal{G}}(g)$ for some $f \in \mathcal{F}(V)$.

Proposition 2.2. *The sheaf map $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is injective if and only if $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective for every $x \in X$.*

Proof. (\Rightarrow) This is trivial.

(\Leftarrow) Let U be any open subset of X , it suffices to show that $\ker(\phi_U) = \{0_{\mathcal{F}(U)}\}$. Let $f \in \mathcal{F}(U)$ such that $\phi_U(f) = 0_{\mathcal{G}(U)}$. Then for every $x \in U$, $\phi_x(f_x) = \llbracket \phi_U(f) \rrbracket = 0_{\mathcal{G}_x}$. Since ϕ_x is injective, we have $f_x = 0_{\mathcal{F}_x}$ for every $x \in U$. By **Lemma 2.1** we conclude that $f = 0_{\mathcal{F}(U)}$, hence completing the proof. \square

Remark 2.7. Analogous statement is not true for the surjective map of sheaves, see [7, Example 2.2.11]

Proposition 2.3. *Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be an injective map of sheaves. Then ϕ is surjective if and only if $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is surjective for all open subsets $U \subset X$.*

⁷Let U be an open subset and $f \in \ker(\phi_U)$, then for $V \subset U$ we have $\rho_{U, V}^{\mathcal{F}}(f) \in \ker(\phi_V)$ since $\phi_V \circ \rho_{U, V}^{\mathcal{F}} = \rho_{U, V}^{\mathcal{G}} \circ \phi_U$.

⁸The map of sheaves is a map of direct systems $\phi : \{(\mathcal{F}(U), \rho_{U, V}^{\mathcal{F}})\} \rightarrow \{(\mathcal{G}(U), \rho_{U, V}^{\mathcal{G}})\}$, and the map of stalks $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is the direct limit of the homomorphisms ϕ_U .

Proof. (\Rightarrow) Let U be any open subset of X , and $g \in \mathcal{G}(U)$. We need to show that there exists a $f \in \mathcal{F}(U)$ such that $\phi_U(f) = g$. Since ϕ_x is surjective for every $x \in X$, for every $g_x \in \mathcal{G}_x$ there exists an open neighborhood V of x and $f \in \mathcal{F}(V)$ such that $\phi_x(f_x) = \llbracket \phi_V(f) \rrbracket = g_x$. Therefore, we can find an open covering $\{U_\alpha\}_{\alpha \in A}$ of U such that each U_α is an open neighborhood of $x \in U$ such that $\phi_x(f_x) = \llbracket \phi_{U_\alpha}(f_\alpha) \rrbracket = g_x$ for some $f_\alpha \in \mathcal{F}(U_\alpha)$. In other words, there exist $f_\alpha \in \mathcal{F}(U_\alpha)$ such that

$$\phi_{U_\alpha}(f_\alpha) = \rho_{UU_\alpha}^{\mathcal{G}}(g) \quad (2.1)$$

for all $\alpha \in A$. In particular, for $f_\alpha \in \mathcal{F}(U_\alpha)$ and $f_\beta \in \mathcal{F}(U_\beta)$ we have

$$\phi_{U_\alpha \cap U_\beta}(\rho_{U_\alpha, U_\alpha \cap U_\beta}^{\mathcal{F}}(f_\alpha)) = \rho_{U, U_\alpha \cap U_\beta}^{\mathcal{G}}(g) \quad \text{and} \quad \phi_{U_\alpha \cap U_\beta}(\rho_{U_\beta, U_\alpha \cap U_\beta}^{\mathcal{F}}(f_\beta)) = \rho_{U, U_\alpha \cap U_\beta}^{\mathcal{G}}(g)$$

Since ϕ is injective, the map $\phi_{U_\alpha \cap U_\beta} : \mathcal{F}(U_\alpha \cap U_\beta) \rightarrow \mathcal{G}(U_\alpha \cap U_\beta)$ is injective for all $\alpha, \beta \in A$. Hence we have

$$\rho_{U_\alpha, U_\alpha \cap U_\beta}^{\mathcal{F}}(f_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}^{\mathcal{F}}(f_\beta)$$

for all $\alpha, \beta \in A$. Now by the gluing axiom of the sheaf \mathcal{F} , there exists a $f \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}^{\mathcal{F}}(f) = f_\alpha$ for all $\alpha \in A$. Using this in (2.1) we get

$$\rho_{UU_\alpha}^{\mathcal{G}}(g) = \phi_{U_\alpha}(\rho_{UU_\alpha}^{\mathcal{F}}(f)) = \rho_{UU_\alpha}^{\mathcal{G}}(\phi_U(f))$$

for all $\alpha \in A$. By the uniqueness axiom of the sheaf \mathcal{G} , we conclude that $g = \phi_U(f)$. Hence completing the proof.

(\Leftarrow) This is trivial. □

2.1.3 Exact sequence of sheaves

Definition 2.8 (Exact sequence of sheaves). A sequence of sheaves $\mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is said to be exact if $\mathcal{F}'_x \xrightarrow{\phi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{F}''_x$ is an exact sequence of abelian groups for every $x \in X$.

Example 2.8. By [Theorem 1.3](#), [Theorem 1.4](#) and [Proposition 1.2](#) we know that for every point x in a smooth manifold X there exists an open subset U containing x such that

$$0 \longrightarrow \underline{\mathbb{R}}(U) \hookrightarrow \Omega^0(U) \xrightarrow{d_U} \Omega^1(U) \xrightarrow{d_U} \Omega^2(U) \xrightarrow{d_U} \dots$$

is an exact sequence of abelian groups. In other words, for all $x \in X$ we have a long exact sequence at the level of stalks

$$0 \longrightarrow \underline{\mathbb{R}}_x \hookrightarrow \Omega_x^0 \xrightarrow{d_x} \Omega_x^1 \xrightarrow{d_x} \Omega_x^2 \xrightarrow{d_x} \dots$$

Therefore, by Poincaré lemma, the sequence of sheaves of differential forms on a smooth manifold

$$0 \longrightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

is exact.

Lemma 2.2. If $0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is an exact sequence of sheaves over X , then the induced sequence of abelian groups for any open set $U \subset X$

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\phi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U)$$

is also exact.

Proof. For all $x \in X$ we have an exact sequence of stalks

$$0 \longrightarrow \mathcal{F}'_x \xrightarrow{\phi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{F}''_x$$

Using [Proposition 2.2](#) we conclude that ϕ_U is injective. Hence we just need to show that $\text{im}(\phi_U) = \ker(\psi_U)$.

$\boxed{\ker(\psi_U) \subseteq \text{im}(\phi_U)}$ Let $f \in \ker(\psi_U)$, then for all $x \in U$ we have $f_x \in \ker(\psi_x)$ since $\psi_x(f_x) = \llbracket \psi_U(f) \rrbracket$. Since the sequence of stalks is exact at \mathcal{F}_x , $f_x = \phi_x(g_x)$ for some $g_x \in \mathcal{F}'_x$. Therefore, we can find an open covering $\{U_\alpha\}_{\alpha \in A}$ of U such that each U_α is an open neighborhood of $x \in U$ such that $\phi_x(g_x) = \llbracket \phi_{U_\alpha}(g_\alpha) \rrbracket = f_x$ for some $g_\alpha \in \mathcal{F}'(U_\alpha)$. In other words, there exist $g_\alpha \in \mathcal{F}'(U_\alpha)$ such that

$$\phi_{U_\alpha}(g_\alpha) = \rho_{UU_\alpha}^{\mathcal{F}}(f) \quad (2.2)$$

for all $\alpha \in A$. In particular, for $g_\alpha \in \mathcal{F}'(U_\alpha)$ and $g_\beta \in \mathcal{F}'(U_\beta)$ we have

$$\phi_{U_\alpha \cap U_\beta}(\rho_{U_\alpha, U_\alpha \cap U_\beta}^{\mathcal{F}'}(g_\alpha)) = \rho_{U, U_\alpha \cap U_\beta}^{\mathcal{F}}(f) \quad \text{and} \quad \phi_{U_\alpha \cap U_\beta}(\rho_{U_\beta, U_\alpha \cap U_\beta}^{\mathcal{F}'}(g_\beta)) = \rho_{U, U_\alpha \cap U_\beta}^{\mathcal{F}}(f)$$

Since ϕ is injective, the map $\phi_{U_\alpha \cap U_\beta} : \mathcal{F}(U_\alpha \cap U_\beta) \rightarrow \mathcal{G}(U_\alpha \cap U_\beta)$ is injective for all $\alpha, \beta \in A$. Hence we have

$$\rho_{U_\alpha, U_\alpha \cap U_\beta}^{\mathcal{F}'}(g_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}^{\mathcal{F}'}(g_\beta)$$

for all $\alpha, \beta \in A$. Now by the gluing axiom of the sheaf \mathcal{F}' , there exists a $g \in \mathcal{F}'(U)$ such that $\rho_{UU_\alpha}^{\mathcal{F}'}(g) = g_\alpha$ for all $\alpha \in A$. Using this in (2.2) we get

$$\rho_{UU_\alpha}^{\mathcal{F}}(f) = \phi_{U_\alpha}(\rho_{UU_\alpha}^{\mathcal{F}'}(g)) = \rho_{UU_\alpha}^{\mathcal{F}}(\phi_U(g))$$

for all $\alpha \in A$. By the uniqueness axiom of the sheaf \mathcal{F} , we conclude that $f = \phi_U(g)$.

$\boxed{\text{im}(\phi_U) \subseteq \ker(\psi_U)}$ Let $f \in \text{im}(\phi_U)$, i.e. there exists $g \in \mathcal{F}'(U)$ such that $\phi_U(g) = f$. Then for all $x \in U$ we have $f_x \in \text{im} \phi_x$ since $\phi_x(g_x) = \llbracket \psi_U(f) \rrbracket = f_x$. Since the sequence of stalks is exact at \mathcal{F}_x , $\psi_x(f_x) = 0_{\mathcal{F}''_x}$ for all $x \in X$. Since $\psi_x(f_x) = \llbracket \psi_U(f) \rrbracket$, by [Lemma 2.1](#) we conclude that $\psi_U(f) = 0$. \square

Remark 2.8. In general, given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

Then the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{F}'(X) \xrightarrow{\phi_X} \mathcal{F}(X) \xrightarrow{\psi_X} \mathcal{F}''(X) \longrightarrow 0$$

is always exact at $\mathcal{F}'(X)$ and $\mathcal{F}(X)$ but not necessarily at $\mathcal{F}''(X)$, see [20, §II.3] and [16, §4.1].

2.2 Čech cohomology of sheaves

Definition 2.9 (Čech cochain). Let \mathcal{F} be sheaf of abelian groups on a topological space X . Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X , and fix an integer $k \geq 0$. A Čech k -cochain for the sheaf \mathcal{F} over the open cover \mathcal{U} is an element of $\prod_{(i_0, i_1, \dots, i_k)} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k})$ where Cartesian product is take over all collections of $k + 1$ indices (i_0, \dots, i_k) from I .

Remark 2.9. To simplify the notation, we will write

$$U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k} := U_{i_0, i_1, \dots, i_k} \quad \text{and} \quad \mathcal{F}(U_{i_0, i_1, \dots, i_k}) = \{f_{i_0, i_1, \dots, i_k}\}$$

Hence a Čech k -cochain is a tuple of the form $(f_{i_0, i_1, \dots, i_k})$. The abelian group of Čech k -cochains for \mathcal{F} over \mathcal{U} is denoted by $\check{C}^k(\mathcal{U}, \mathcal{F})$; thus

$$\check{C}^k(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, i_1, \dots, i_k)} \mathcal{F}(U_{i_0, i_1, \dots, i_k})$$

Definition 2.10 (Coboundary operator). The coboundary operator is defined as

$$\begin{aligned}\delta : \check{C}^k(\mathcal{U}, \mathcal{F}) &\rightarrow \check{C}^{k+1}(\mathcal{U}, \mathcal{F}) \\ (f_{i_0, i_1, \dots, i_k}) &\mapsto (g_{i_0, i_1, \dots, i_{k+1}})\end{aligned}$$

where

$$g_{i_0, i_1, \dots, i_{k+1}} = \sum_{\ell=0}^{k+1} (-1)^\ell \rho(f_{i_0, i_1, \dots, \widehat{i}_\ell, \dots, i_{k+1}})$$

and $\rho : \mathcal{F}(U_{i_0, i_1, \dots, \widehat{i}_\ell, \dots, i_{k+1}}) \rightarrow \mathcal{F}(U_{i_0, i_1, \dots, i_{k+1}})$ is the group homomorphism for the sheaf \mathcal{F} corresponding to the nested open subsets $U_{i_0, i_1, \dots, i_{k+1}} \subset U_{i_0, i_1, \dots, \widehat{i}_\ell, \dots, i_{k+1}}$.

Remark 2.10. To simplify the notations above, we wrote

$$U_{i_0, i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_k} := U_{i_0, i_1, \dots, \widehat{i}_\ell, \dots, i_k} \quad \text{and} \quad \mathcal{F}(U_{i_0, i_1, \dots, \widehat{i}_\ell, \dots, i_k}) = \{f_{i_0, i_1, \dots, \widehat{i}_\ell, \dots, i_k}\}$$

Definition 2.11 (Čech cocycle). A Čech k -cochain $f = (f_{i_0, i_1, \dots, i_k})$ with $\delta(f) = 0$ is called Čech k -cocycle.

Remark 2.11. The abelian group of k -cocycles is denoted by $\check{Z}^k(\mathcal{U}, \mathcal{F})$, i.e. kernel of δ at the k^{th} level.

Proposition 2.4. Let $f = (f_{i_0, \dots, i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, then

1. $f_{i_0, \dots, i_n} = 0$ if any two indices are equal.
2. $f_{\sigma(i_0), \sigma(i_1), \dots, \sigma(i_k)} = \text{sgn}(\sigma) f_{i_0, i_1, \dots, i_k}$ if σ is a permutation of $\{i_0, \dots, i_k\}$

Proof. We will check just for the simplest case, $k = 1$. Let $f = (f_{i_0 i_1})$ and $\delta(f) = (g_{i_0 i_1 i_2}) = 0$. For any $i \in I$ we have

$$0 = g_{i, i, i} = \rho_{U_{i, i} U_{i, i, i}}(f_{i, i}) - \rho_{U_{i, i} U_{i, i, i}}(f_{i, i}) + \rho_{U_{i, i} U_{i, i, i}}(f_{i, i})$$

This implies that $f_{i, i} = 0$ by the uniqueness axiom of sheaf. On the other hand, applied to (i, j, i) instead, it says

$$0 = g_{i, j, i} = \rho_{U_{j, i} U_{i, j, i}}(f_{j, i}) - \rho_{U_{i, i} U_{i, j, i}}(f_{i, i}) + \rho_{U_{i, j} U_{i, j, i}}(f_{i, j})$$

This implies that

$$\rho_{U_{j, i} U_{i, j, i}}(f_{j, i}) + \rho_{U_{i, j} U_{i, j, i}}(f_{i, j}) = 0 \quad \text{for all } i \in I$$

But the $\{U_{i, j, i}\}_{i \in I}$ is an open cover of $U_{i, j}$, and hence indeed $f_{i, j} = -f_{j, i}$ due to the uniqueness axiom of the sheaf \mathcal{F} . \square

Definition 2.12 (Čech coboundary). A Čech k -cochain $f = (f_{i_0, i_1, \dots, i_k})$ which is the image of δ , i.e. there exists $(k-1)$ -cochain $g = (g_{i_0, i_1, \dots, i_{k-1}})$ such that $\delta(g) = f$, is called Čech k -coboundary.

Remark 2.12. The abelian group of k -coboundaries is denoted by $\check{B}^k(\mathcal{U}, \mathcal{F})$, i.e. image of δ at the $(k-1)^{\text{th}}$ level. Also, we define $\check{B}^0(\mathcal{U}, \mathcal{F}) = 0$ for any sheaf \mathcal{F} and open cover \mathcal{U} .

Lemma 2.3. $\delta \circ \delta = 0$

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be the open cover. We will check it just for the simplest case

$$\begin{aligned} \check{C}^0(\mathcal{U}, \mathcal{F}) &\xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{F}) \\ (f_\alpha) &\longmapsto (g_{\alpha\beta}) \longmapsto (h_{\alpha\beta\gamma}) \end{aligned}$$

By the definition of coboundary operator, for $i_0 = \alpha$ and $i_1 = \beta$, we have

$$\begin{aligned} g_{\alpha\beta} &= (-1)^0 \rho_{U_\beta U_{\alpha\beta}}(f_\beta) + (-1)^1 \rho_{U_\alpha U_{\alpha\beta}}(f_\alpha) \\ &= \rho_{U_\beta U_{\alpha\beta}}(f_\beta) - \rho_{U_\alpha U_{\alpha\beta}}(f_\alpha) \end{aligned} \quad (2.3)$$

Also for $i_0 = \alpha, i_1 = \beta$ and $i_2 = \gamma$, we have

$$\begin{aligned} h_{\alpha\beta\gamma} &= (-1)^0 \rho_{U_{\beta\gamma} U_{\alpha\beta\gamma}}(g_{\beta\gamma}) + (-1)^1 \rho_{U_{\alpha\gamma} U_{\alpha\beta\gamma}}(g_{\alpha\gamma}) + (-1)^2 \rho_{U_{\alpha\beta} U_{\alpha\beta\gamma}}(g_{\alpha\beta}) \\ &= \rho_{U_{\beta\gamma} U_{\alpha\beta\gamma}}(g_{\beta\gamma}) - \rho_{U_{\alpha\gamma} U_{\alpha\beta\gamma}}(g_{\alpha\gamma}) + \rho_{U_{\alpha\beta} U_{\alpha\beta\gamma}}(g_{\alpha\beta}) \end{aligned} \quad (2.4)$$

Using (2.3) in (2.4) we get

$$\begin{aligned} h_{\alpha\beta\gamma} &= \rho_{U_{\beta\gamma} U_{\alpha\beta\gamma}}(\rho_{U_\gamma U_{\beta\gamma}}(f_\gamma) - \rho_{U_\beta U_{\beta\gamma}}(f_\beta)) - \rho_{U_{\alpha\gamma} U_{\alpha\beta\gamma}}(\rho_{U_\gamma U_{\alpha\gamma}}(f_\gamma) - \rho_{U_\alpha U_{\alpha\gamma}}(f_\alpha)) \\ &\quad + \rho_{U_{\alpha\beta} U_{\alpha\beta\gamma}}(\rho_{U_\beta U_{\alpha\beta}}(f_\beta) - \rho_{U_\alpha U_{\alpha\beta}}(f_\alpha)) \\ &= \rho_{U_\gamma U_{\alpha\beta\gamma}}(f_\gamma) - \rho_{U_\beta U_{\alpha\beta\gamma}}(f_\beta) - \rho_{U_\gamma U_{\alpha\beta\gamma}}(f_\gamma) + \rho_{U_\alpha U_{\alpha\beta\gamma}}(f_\alpha) + \rho_{U_\beta U_{\alpha\beta\gamma}}(f_\beta) - \rho_{U_\alpha U_{\alpha\beta\gamma}}(f_\alpha) \\ &= 0 \end{aligned}$$

Hence completing the verification. \square

Proposition 2.5. *Every k -coboundary is a k -cocycle.*

Proof. Let $f = (f_{i_0, i_1, \dots, i_k}) \in \check{B}^k(\mathcal{U}, \mathcal{F})$ such that $f = \delta(g)$ for some $g = (g_{i_0, i_1, \dots, i_{k-1}}) \in \check{C}^{k-1}(\mathcal{U}, \mathcal{F})$. From Lemma 2.3 we know that $\delta(f) = \delta(\delta(g)) = 0$ hence $f \in \check{Z}^k(\mathcal{U}, \mathcal{F})$ for all $k \geq 1$. For $k = 0$, the statement is trivially true. \square

Definition 2.13 (Čech cohomology with respect to a cover). The k^{th} Čech cohomology group of \mathcal{F} with respect to the open cover \mathcal{U} is the quotient group

$$\check{H}^k(\mathcal{U}, \mathcal{F}) := \frac{\check{Z}^k(\mathcal{U}, \mathcal{F})}{\check{B}^k(\mathcal{U}, \mathcal{F})}$$

Remark 2.13. Hence, the Čech cohomology with respect to a cover measures the extent to which cocycles are not coboundaries for a given open cover.

Lemma 2.4. *For any sheaf \mathcal{F} and open covering \mathcal{U} of X , we have $\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$.*

Proof. Since $\check{B}^0(\mathcal{U}, \mathcal{F}) = 0$, we just need to show that $\check{Z}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$. Consider the following group homomorphism

$$\begin{aligned} \psi : \mathcal{F}(X) &\rightarrow \check{C}^0(\mathcal{U}, \mathcal{F}) \\ f &\mapsto (f_i) = (\rho_{X U_i}(f)) \end{aligned}$$

Then $\delta((f_i)) = (g_{ij})$, where $g_{ij} = \rho_{U_j U_{ij}}(f_j) - \rho_{U_i U_{ij}}(f_i)$; this is zero for every i and j since $\rho_{U_i U_{ij}}(\rho_{X U_i}(f)) = \rho_{U_j U_{ij}}(\rho_{X U_j}(f))$. Hence ψ maps $\mathcal{F}(X)$ to $\check{Z}^0(\mathcal{U}, \mathcal{F})$. This map is injective and surjective by the uniqueness and gluing axioms of the sheaf \mathcal{F} , respectively. \square

Definition 2.14 (Refining map). Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open coverings of X such that \mathcal{V} is a refinement⁹ of \mathcal{U} . Then the map $r : J \rightarrow I$ such that $V_j \subset U_{r(j)}$ for every $j \in J$, is called the *refining map* for the coverings.

⁹For its definition see Appendix A.

Remark 2.14. The refining map is not unique. Also, the set of all open covers is a *directed set*¹⁰ where the ordering is done via refinement, i.e. $\mathcal{U} < \mathcal{V}$ if \mathcal{V} is a refinement of \mathcal{U} . The upper bound of \mathcal{U} and \mathcal{V} is given by $\mathcal{W} = \{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$ [14, §73, Example 2].

Lemma 2.5. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open coverings of X such that \mathcal{V} is a refinement of \mathcal{U} along with the refining map $r : J \rightarrow I$. The induced map at the level of cochains is given by

$$\begin{aligned} \tilde{r} : \check{C}^k(\mathcal{U}, \mathcal{F}) &\rightarrow \check{C}^k(\mathcal{V}, \mathcal{F}) \\ (f_{i_0, \dots, i_k}) &\mapsto (g_{j_0, \dots, j_k}) \end{aligned}$$

where

$$g_{j_0, \dots, j_k} = \rho(f_{r(j_0), \dots, r(j_k)})$$

and $\rho : \mathcal{F}(U_{r(j_0), \dots, r(j_k)}) \rightarrow \mathcal{F}(V_{j_0, \dots, j_k})$ is the group homomorphism for the sheaf \mathcal{F} corresponding to the nested open subsets $V_{j_0, \dots, j_k} \subset U_{r(j_0), \dots, r(j_k)}$. This map sends cocycles to cocycles and coboundaries to coboundaries.

Proof. We will check it just for the simplest case. We have the map

$$\begin{aligned} \tilde{r} : \check{C}^0(\mathcal{U}, \mathcal{F}) &\rightarrow \check{C}^0(\mathcal{V}, \mathcal{F}) \\ (f_{i_0}) &\mapsto \left(\rho_{U_{r(j_0)} V_{j_0}}(f_{r(j_0)}) \right) \end{aligned}$$

Let $\delta((f_{i_0})) = 0$, then $\rho_{U_{i_1} U_{i_0 i_1}}(f_{i_1}) = \rho_{U_{i_0} U_{i_0 i_1}}(f_{i_0})$ for every pair of indices $i_0, i_1 \in I$. Next we compute $\delta\left(\left(\rho_{U_{r(j_0)} V_{j_0}}(f_{r(j_0)})\right)\right) = (g_{j_0, j_1})$

$$\begin{aligned} g_{j_0, j_1} &= \rho_{V_{j_1} V_{j_0 j_1}}\left(\rho_{U_{r(j_1)} V_{j_1}}(f_{r(j_1)})\right) - \rho_{V_{j_0} V_{j_0 j_1}}\left(\rho_{U_{r(j_0)} V_{j_0}}(f_{r(j_0)})\right) \\ &= \rho_{U_{r(j_1)} V_{j_0 j_1}}(f_{r(j_1)}) - \rho_{U_{r(j_0)} V_{j_0 j_1}}(f_{r(j_0)}) \end{aligned}$$

But, we have

$$\rho_{U_{r(j_1)} U_{r(j_0) r(j_1)}}(f_{r(j_1)}) = \rho_{U_{r(j_0)} U_{r(j_0) r(j_1)}}(f_{r(j_0)})$$

and $V_{j_0, j_1} \subset U_{r(j_0) r(j_1)}$. Therefore $g_{j_0, j_1} = 0$, and \tilde{r} maps cocycle to cocycle. Since 0 is the only coboundary in this case, it also maps coboundary to coboundary. \square

Lemma 2.6. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open coverings of X such that \mathcal{V} is a refinement of \mathcal{U} along with the refining map $r : J \rightarrow I$. The induced map at the level of cohomology¹¹ is given by

$$\begin{aligned} H_r : \check{H}^k(\mathcal{U}, \mathcal{F}) &\rightarrow \check{H}^k(\mathcal{V}, \mathcal{F}) \\ \llbracket (f_{i_0, \dots, i_k}) \rrbracket &\mapsto \llbracket (g_{j_0, \dots, j_k}) \rrbracket \end{aligned}$$

for $(f_{i_0, \dots, i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, where

$$g_{j_0, \dots, j_k} = \rho(f_{r(j_0), \dots, r(j_k)})$$

and $\rho : \mathcal{F}(U_{r(j_0), \dots, r(j_k)}) \rightarrow \mathcal{F}(V_{j_0, \dots, j_k})$ is the group homomorphism for the sheaf \mathcal{F} corresponding to the nested open subsets $V_{j_0, \dots, j_k} \subset U_{r(j_0), \dots, r(j_k)}$. This map is independent of the refining map r and depends only on the two coverings \mathcal{U} and \mathcal{V} .

¹⁰For its definition see [Appendix B](#).

¹¹This map is well defined by the previous lemma.

Proof. Suppose the $r, r' : J \rightarrow I$ are two distinct refining maps for the refinement \mathcal{V} of \mathcal{U} .

Claim: $H_r = H_{r'}$

If $k = 0$, then $\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X) \cong \check{H}^0(\mathcal{V}, \mathcal{F})$. Therefore $H_r = \mathbb{1}_{\mathcal{F}(X)} = H_{r'}$. Let's assume that $k \geq 1$, and fix a cohomology class $f \in \check{H}^k(\mathcal{U}, \mathcal{F})$ represented by $(f_{i_0, i_1, \dots, i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, i.e. $f = \llbracket (f_{i_0, i_1, \dots, i_k}) \rrbracket$. Then we have

$$H_r(f) = \llbracket (g_{j_0, j_1, \dots, j_k}) \rrbracket \quad \text{and} \quad H_{r'}(f) = \llbracket (g'_{j_0, j_1, \dots, j_k}) \rrbracket$$

where

$$g_{j_0, j_1, \dots, j_k} = \rho_\alpha(f_{r(j_0), \dots, r(j_k)}) \quad \text{and} \quad g'_{j_0, j_1, \dots, j_k} = \rho_\beta(f_{r'(j_0), \dots, r'(j_k)})$$

where ρ_α and ρ_β are the appropriate group homomorphism for the sheaf \mathcal{F} . To prove our claim, it suffices to show that $(g_{j_0, j_1, \dots, j_k} - g'_{j_0, j_1, \dots, j_k}) \in \check{B}^k(\mathcal{V}, \mathcal{F})$.

Claim: $\delta(h) = (g'_{j_0, j_1, \dots, j_k} - g_{j_0, j_1, \dots, j_k})$ where $h = (h_{j_0, j_1, \dots, j_{k-1}}) \in \check{C}^{k-1}(\mathcal{V}, \mathcal{F})$ is such that¹²

$$h_{j_0, j_1, \dots, j_{k-1}} = \sum_{\ell=0}^{k-1} (-1)^\ell \rho(f_{r(j_0), \dots, r(j_\ell), r'(j_{\ell+1}), \dots, r'(j_{k-1})})$$

The claim follows from the fact that $(f_{i_0, i_1, \dots, i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$ for all indices (i_0, \dots, i_k) .

We will check the claim just for the simplest case, when $k = 1$. In this case we have $f = \llbracket (f_{i_0, i_1}) \rrbracket$, since $(f_{i_0, i_1}) \in \check{Z}^1(\mathcal{U}, \mathcal{F})$ we have $\delta((f_{i_0, i_1})) = 0$, that is

$$\rho_{U_{i_1, i_2} U_{i_0 i_1 i_2}}(f_{i_1, i_2}) - \rho_{U_{i_0, i_2} U_{i_0 i_1 i_2}}(f_{i_0, i_2}) + \rho_{U_{i_0, i_1} U_{i_0 i_1 i_2}}(f_{i_0, i_1}) = 0 \quad (2.5)$$

for any triplet of indices $i_0, i_1, i_2 \in I$. Also,

$$H_r(f) = \llbracket (g_{j_0, j_1}) \rrbracket \quad \text{and} \quad H_{r'}(f) = \llbracket (g'_{j_0, j_1}) \rrbracket$$

where

$$g_{j_0, j_1} = \rho_{U_{r(j_0), r(j_1)} V_{j_0, j_1}}(f_{r(j_0), r(j_1)}) \quad \text{and} \quad g'_{j_0, j_1} = \rho_{U_{r'(j_0), r'(j_1)} V_{j_0, j_1}}(f_{r'(j_0), r'(j_1)})$$

From this we get

$$g'_{j_0, j_1} - g_{j_0, j_1} = \rho_{U_{r'(j_0), r'(j_1)} V_{j_0, j_1}}(f_{r'(j_0), r'(j_1)}) - \rho_{U_{r(j_0), r(j_1)} V_{j_0, j_1}}(f_{r(j_0), r(j_1)}) \quad (2.6)$$

We have $h = (h_{j_0}) = (\rho_{U_{r(j_0), r'(j_0)} V_{j_0}}(f_{r(j_0), r'(j_0)}))$. Let $\delta(h) = (h'_{j_0 j_1})$, then

$$\begin{aligned} h'_{j_0 j_1} &= \rho_{V_{j_1} V_{j_0 j_1}}(h_{j_1}) - \rho_{V_{j_0} V_{j_0 j_1}}(h_{j_0}) \\ &= \rho_{V_{j_1} V_{j_0 j_1}}(\rho_{U_{r(j_1), r'(j_1)} V_{j_1}}(f_{r(j_1), r'(j_1)})) - \rho_{V_{j_0} V_{j_0 j_1}}(\rho_{U_{r(j_0), r'(j_0)} V_{j_0}}(f_{r(j_0), r'(j_0)})) \\ &= \rho_{U_{r(j_1), r'(j_1)} V_{j_0 j_1}}(f_{r(j_1), r'(j_1)}) - \rho_{U_{r(j_0), r'(j_0)} V_{j_0 j_1}}(f_{r(j_0), r'(j_0)}) \end{aligned} \quad (2.7)$$

To simplify the notations, we rename indices as $r(j_0) = i_0, r(j_1) = i_1, r'(j_0) = i_2$ and $r'(j_1) = i_3$. Since $V_{j_0 j_1} \subset U_{i_0 i_1 i_2}$ and $V_{j_0, j_1} \subset U_{i_1, i_2, i_3}$ from (2.5) we get

$$\begin{aligned} \rho_{U_{i_1 i_2} V_{j_0 j_1}}(f_{i_1, i_2}) - \rho_{U_{i_0 i_2} V_{j_0 j_1}}(f_{i_1, i_2}) + \rho_{U_{i_0 i_1} V_{j_0 j_1}}(f_{i_0, i_1}) &= 0 \\ \rho_{U_{i_2 i_3} V_{j_0 j_1}}(f_{i_2, i_3}) - \rho_{U_{i_1 i_3} V_{j_0 j_1}}(f_{i_1, i_3}) + \rho_{U_{i_1 i_2} V_{j_0 j_1}}(f_{i_1, i_2}) &= 0 \end{aligned} \quad (2.8)$$

¹²For a more general argument see [19, §5.33, equation (11)] and [4, Lemma 2.6.1].

We will use (2.8) to convert (2.7) to (2.6). Hence we have

$$\begin{aligned}
h'_{j_0, j_1} &= \rho_{U_{i_1, i_3} V_{j_0, j_1}}(f_{i_1, i_3}) - \rho_{U_{i_0, i_2} V_{j_0, j_1}}(f_{i_0, i_2}) \\
&= \left(\rho_{U_{i_1, i_2} V_{j_0, j_1}}(f_{i_1, i_2}) - \rho_{U_{i_0, i_2} V_{j_0, j_1}}(f_{i_0, i_2}) + \rho_{U_{i_0, i_1} V_{j_0, j_1}}(f_{i_0, i_1}) \right) \\
&\quad - \left(\rho_{U_{i_2, i_3} V_{j_0, j_1}}(f_{i_2, i_3}) - \rho_{U_{i_1, i_3} V_{j_0, j_1}}(f_{i_1, i_3}) + \rho_{U_{i_1, i_2} V_{j_0, j_1}}(f_{i_1, i_2}) \right) \\
&\quad + \rho_{U_{i_2, i_3} V_{j_0, j_1}}(f_{i_2, i_3}) - \rho_{U_{i_0, i_1} V_{j_0, j_1}}(f_{i_0, i_1}) \\
&= \rho_{U_{i_2, i_3} V_{j_0, j_1}}(f_{i_2, i_3}) - \rho_{U_{i_0, i_1} V_{j_0, j_1}}(f_{i_0, i_1}) \\
&= g'_{j_0, j_1} - g_{j_0, j_1}
\end{aligned}$$

Therefore these two cocycles differ by a coboundary. Hence completing the proof. \square

Remark 2.15. We will therefore denote this refining map on the cohomology level by $H_{\mathcal{U}\mathcal{V}}$ for $\mathcal{U} < \mathcal{V}$. Hence, $\{\check{H}^k(\mathcal{U}, \mathcal{F}), H_{\mathcal{U}\mathcal{V}}\}$ is a *direct system*¹³. We have $H_{\mathcal{U}\mathcal{U}} = \mathbb{1}_{\check{H}^k(\mathcal{U}, \mathcal{F})}$ since we can choose refining map r to be identity, and $H_{\mathcal{U}\mathcal{W}} = H_{\mathcal{V}\mathcal{W}} \circ H_{\mathcal{U}\mathcal{V}}$ for $\mathcal{U} < \mathcal{V} < \mathcal{W}$ since composition of two refining maps is again a refining map.

Definition 2.15 (Čech cohomology). Let \mathcal{F} be a sheaf of abelian groups on X and $k \geq 0$ be an integer. Then the k^{th} Čech cohomology group of \mathcal{F} on X is the group

$$\check{H}^k(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^k(\mathcal{U}, \mathcal{F})$$

where the direct limit¹⁴ is indexed over all the open covers of X with order relation induced by refinement, i.e. $\mathcal{U} < \mathcal{V}$ if \mathcal{V} is a refinement of \mathcal{U} .

Proposition 2.6. For any sheaf \mathcal{F} of X , we have $\check{H}^0(X, \mathcal{F}) \cong \mathcal{F}(X)$.

Proof. By Lemma 2.4 we know that at the \check{H}^0 level all the groups are isomorphic to $\mathcal{F}(X)$. Since all the maps $H_{\mathcal{U}\mathcal{V}}$ are compatible isomorphisms, using Proposition B.1 we conclude that the direct limit is also isomorphic to $\mathcal{F}(X)$. \square

Remark 2.16. What we have defined here is not the true definition of either Čech or sheaf cohomology [9, §IX.3] [3, pp. 38-40]. Čech cohomology can be defined either using the concept of *nerve* [14, §73][10, §3.4(a)], or *presheaf*¹⁵ [1, §10]. One can prove equivalence of both these definitions using the constant presheaf \underline{G} [19, §5.33]. Also note that Čech cohomology of the cover \mathcal{U} is a purely combinatorial object [1, Theorem 8.9].

Sheaf cohomology can be defined either using resolution of sheaf [20, Definition 3.10] [16, Definition 4.2.11] or axiomatically [19, §5.18]. The definition of Čech cohomology agrees with that of sheaf cohomology for smooth manifolds since Čech cohomology is isomorphic to sheaf cohomology for any sheaf on a paracompact Hausdorff space [19, §5.33]. This is all we need to obtain the desired proof, hence our definition of Čech cohomology of sheaves serves the purpose.

Remark 2.17. Another way of defining Čech cohomology groups with coefficients in sheaves is via sheafification. First step is to define the cohomology groups $\check{H}^k(\mathcal{U}, \mathcal{F})$ on an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X with coefficients in a presheaf \mathcal{F} . Then the cohomology groups $\check{H}^k(\mathcal{U}, \tilde{\mathcal{F}})$ of \mathcal{U} with coefficients in a sheaf $\tilde{\mathcal{F}}$ are defined to be the cohomology groups of \mathcal{U} with coefficients in the canonical presheaf \mathcal{F} of $\tilde{\mathcal{F}}$. Finally, the cohomology groups $\check{H}^k(X, \mathcal{F})$ and $\check{H}^k(X, \tilde{\mathcal{F}})$ are defined as the direct limit of all groups $\check{H}^k(\mathcal{U}, \mathcal{F})$ and $\check{H}^k(\mathcal{U}, \tilde{\mathcal{F}})$, respectively, as \mathcal{U} runs through all open coverings of X (directed by refinement) [4, §2.6].

¹³For its definition see Appendix B.

¹⁴For the definition of direct limit see Appendix B. To get the direct system $\{\check{H}^k(\mathcal{U}, \mathcal{F}), H_{\mathcal{U}\mathcal{V}}\}$, the “refinement” is defined to be the order relation for the directed set.

¹⁵For a discussion on the motivation behind this definition see [6, §2] and [2, §10.2].

2.2.1 Induced map of cohomology

Definition 2.16 (Induced map of cochains). If $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a map of sheaves on X , then the *induced map on cochains* is defined as

$$\begin{aligned} \phi_* : \check{C}^k(\mathcal{U}, \mathcal{F}) &\rightarrow \check{C}^k(\mathcal{U}, \mathcal{G}) \\ (f_{i_0, i_1, \dots, i_k}) &\mapsto (\phi_{U_{i_0, \dots, i_k}}(f_{i_0, i_1, \dots, i_k})) \end{aligned}$$

for any open covering \mathcal{U} of X .

Proposition 2.7. *The coboundary operator commutes with the induced map of cochains. That is, the following diagram commutes*

$$\begin{array}{ccc} \check{C}^k(\mathcal{U}, \mathcal{F}) & \xrightarrow{\delta} & \check{C}^{k+1}(\mathcal{U}, \mathcal{F}) \\ \downarrow \phi_* & & \downarrow \phi_* \\ \check{C}^k(\mathcal{U}, \mathcal{G}) & \xrightarrow{\delta} & \check{C}^{k+1}(\mathcal{U}, \mathcal{G}) \end{array}$$

Proof. The coboundary operator δ acts on each element via the group homomorphism ρ of the sheaf, and the induced map ϕ_* acts on each element via the group homomorphism $\phi_{U_{i_0, \dots, i_k}}$ of the sheaf map. By [Definition 2.4](#), we know that the group homomorphism of the sheaf and the group homomorphism of the sheaf map commute. \square

Corollary 2.1. *The induced map of cochains sends cocycles to cocycles, and coboundaries to coboundaries.*

Proof. Let f be a cocycle, i.e. $\delta(f) = 0$. From the previous proposition we know that $\delta(\phi_*(f)) = \phi_*(\delta(f)) = 0$. Hence $\phi_*(f)$ is also a cocycle. Next, let g be a coboundary, i.e. $g = \delta(h)$. From the previous proposition we know that $\phi_*(g) = \phi_*(\delta(h)) = \delta(\phi_*(h))$. Hence $\phi_*(g)$ is also a coboundary. \square

Proposition 2.8. *If $0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is an exact sequence of sheaves over X , then the induced sequence of cochains for any open cover \mathcal{U} of X*

$$0 \longrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{C}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} \check{C}^k(\mathcal{U}, \mathcal{F}'')$$

is also exact.

Proof. We can re-write the desired exact sequence of abelian groups as

$$0 \longrightarrow \prod_{(i_0, i_1, \dots, i_k)} \mathcal{F}'(U_{i_0, i_1, \dots, i_k}) \xrightarrow{\phi_*} \prod_{(i_0, i_1, \dots, i_k)} \mathcal{F}(U_{i_0, i_1, \dots, i_k}) \xrightarrow{\psi_*} \prod_{(i_0, i_1, \dots, i_k)} \mathcal{F}''(U_{i_0, i_1, \dots, i_k})$$

The exactness of the above sequence follows from [Lemma 2.2](#), since

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\phi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U)$$

is an exact sequence of abelian groups for all open sets U of X . \square

Definition 2.17 (Induced map of cohomology). Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a map of sheaves on X , then the *induced¹⁶ map of cohomology* is defined as

$$\begin{aligned} \Phi : \check{H}^k(\mathcal{U}, \mathcal{F}) &\rightarrow \check{H}^k(\mathcal{U}, \mathcal{G}) \\ \llbracket f \rrbracket &\mapsto \llbracket \phi_*(f) \rrbracket \end{aligned}$$

for $f \in \check{Z}^k(\mathcal{U}, \mathcal{F})$.

¹⁶It's well defined because of [Corollary 2.1](#).

Lemma 2.7. *The refining maps at the level of cohomology commute with any induced map of cohomology. That is, the following diagram commutes*

$$\begin{array}{ccc} \check{H}^k(\mathcal{U}, \mathcal{F}) & \xrightarrow{\Phi} & \check{H}^k(\mathcal{U}, \mathcal{G}) \\ \downarrow H_{\mathcal{U}\mathcal{V}} & & \downarrow H_{\mathcal{U}\mathcal{V}} \\ \check{H}^k(\mathcal{V}, \mathcal{F}) & \xrightarrow{\Phi} & \check{H}^k(\mathcal{V}, \mathcal{G}) \end{array}$$

Proof. The refining map $H_{\mathcal{U}\mathcal{V}}$ acts on each element via the group homomorphism ρ of the sheaf, and the induced map Φ acts on each element via the group homomorphism $\phi_{U_{i_0}, \dots, i_k}$ of the sheaf map. By [Definition 2.4](#), we know that the group homomorphism of the sheaf and the group homomorphism of the sheaf map commute. \square

Remark 2.18. This lemma implies that Φ is a map of direct systems $\{\check{H}^k(\mathcal{U}, \mathcal{F}), H_{\mathcal{U}\mathcal{V}}^{\mathcal{F}}\}$ and $\{\check{H}^k(\mathcal{U}, \mathcal{G}), H_{\mathcal{U}\mathcal{V}}^{\mathcal{G}}\}$. Hence $\phi : \mathcal{F} \rightarrow \mathcal{G}$ in fact induces the homomorphism at the level of Čech cohomology of X

$$\Phi : \check{H}^k(X, \mathcal{F}) \rightarrow \check{H}^k(X, \mathcal{G})$$

2.2.2 Long exact sequence of cohomology

In this subsection, proof of the fact that a short exact sequence of sheaves on paracompact Hausdorff space induces a long exact sequence of Čech cohomology will be presented following Serre [[17](#), §I.3] and Warner [[19](#), §5.33].

Theorem 2.1. *Let X be a paracompact Hausdorff space and*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves on X . Then there are connecting homomorphisms $\Delta : \check{H}^k(X, \mathcal{F}'') \rightarrow \check{H}^{k+1}(X, \mathcal{F}')$ for every $k \geq 0$ such that we have a long exact sequence of Čech cohomology groups

$$\dots \xrightarrow{\Phi} \check{H}^k(X, \mathcal{F}) \xrightarrow{\Psi} \check{H}^k(X, \mathcal{F}'') \xrightarrow{\Delta} \check{H}^{k+1}(X, \mathcal{F}') \xrightarrow{\Phi} \check{H}^{k+1}(X, \mathcal{F}) \xrightarrow{\Psi} \dots$$

Proof. Given to us is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

Then by [Proposition 2.8](#), for any open cover \mathcal{U} of X ,

$$0 \longrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{C}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} \check{C}^k(\mathcal{U}, \mathcal{F}'') \longrightarrow 0$$

is an exact sequence. However, if we replace $\check{C}^k(\mathcal{U}, \mathcal{F}'')$ by $\text{im } \psi_*$, we get a short exact sequence of abelian groups:

$$0 \longrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{C}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} \text{im } \psi_* \longrightarrow 0$$

To explicitly show the dependence of $\text{im } \psi_*$ on \mathcal{U} and k , let's write $I^k(\mathcal{U}, \mathcal{F}'') = \text{im } \psi_*$. Hence we have the following short exact sequence of cochain complexes^{[17](#)}

$$0 \longrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{C}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} I^k(\mathcal{U}, \mathcal{F}'') \longrightarrow 0$$

¹⁷All these are chain complexes since $\delta \circ \delta = 0$.

Then by the *zig-zag lemma*¹⁸ we get a long exact sequence in cohomology with respect to open cover \mathcal{U}

$$\dots \xrightarrow{\Phi} \check{H}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\Psi} \mathcal{I}^k(\mathcal{U}, \mathcal{F}'') \xrightarrow{\partial} \check{H}^{k+1}(\mathcal{U}, \mathcal{F}') \xrightarrow{\Phi} \check{H}^{k+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\Psi} \dots$$

where ∂ is the connecting homomorphism induced by the coboundary operator δ and

$$\mathcal{I}^k(\mathcal{U}, \mathcal{F}'') = \frac{\ker\{\delta : I^k(\mathcal{U}, \mathcal{F}'') \rightarrow I^{k+1}(\mathcal{U}, \mathcal{F}'')\}}{\text{im}\{\delta : I^{k-1}(\mathcal{U}, \mathcal{F}'') \rightarrow I^k(\mathcal{U}, \mathcal{F}'')\}}$$

Since direct limit is an exact functor¹⁹, we get the following long exact sequence in Čech cohomology

$$\dots \xrightarrow{\Phi} \check{H}^k(X, \mathcal{F}) \xrightarrow{\Psi} \mathcal{I}^k(X, \mathcal{F}'') \xrightarrow{\partial} \check{H}^{k+1}(X, \mathcal{F}') \xrightarrow{\Phi} \check{H}^{k+1}(X, \mathcal{F}) \xrightarrow{\Psi} \dots$$

where we have²⁰

$$\mathcal{I}^k(X, \mathcal{F}'') = \varinjlim_{\mathcal{U}} \mathcal{I}^k(\mathcal{U}, \mathcal{F}'')$$

Now to obtain the desired long exact sequence of Čech cohomology, it's sufficient to show that $\mathcal{I}^k(X, \mathcal{F}'') \cong \check{H}^k(X, \mathcal{F}'')$. Then the map $\Delta : \check{H}^k(X, \mathcal{F}'') \rightarrow \check{H}^{k+1}(X, \mathcal{F}')$ can be defined as the composition of the inverse of this isomorphism with $\partial : \mathcal{I}^k(X, \mathcal{F}'') \rightarrow \check{H}^{k+1}(X, \mathcal{F}')$.

We observe that the inclusion map $I^k(\mathcal{U}, \mathcal{F}'') \hookrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}'')$ induces a group homomorphism at the level of cohomology with respect to the cover (quotient group), which on passing through the limit induces a map at the level of Čech cohomology. Consider the quotient group

$$Q^k(\mathcal{U}, \mathcal{F}'') := \frac{\check{C}^k(\mathcal{U}, \mathcal{F}'')}{I^k(\mathcal{U}, \mathcal{F}'')}$$

Then we have the following short exact sequence of cochain complexes

$$0 \longrightarrow I^k(\mathcal{U}, \mathcal{F}'') \hookrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}'') \longrightarrow Q^k(\mathcal{U}, \mathcal{F}'') \longrightarrow 0$$

Then by the zig-zag lemma we get a long exact sequence in cohomology with respect to open cover \mathcal{U}

$$\dots \longrightarrow \check{H}^k(\mathcal{U}, \mathcal{F}'') \longrightarrow Q^k(\mathcal{U}, \mathcal{F}'') \xrightarrow{\partial} \mathcal{I}^{k+1}(\mathcal{U}, \mathcal{F}'') \longrightarrow \check{H}^{k+1}(\mathcal{U}, \mathcal{F}'') \longrightarrow \dots$$

where ∂ is the connecting homomorphism induced by the coboundary operator δ and

$$Q^k(\mathcal{U}, \mathcal{F}'') = \frac{\ker\{\delta : Q^k(\mathcal{U}, \mathcal{F}'') \rightarrow Q^{k+1}(\mathcal{U}, \mathcal{F}'')\}}{\text{im}\{\delta : Q^{k-1}(\mathcal{U}, \mathcal{F}'') \rightarrow Q^k(\mathcal{U}, \mathcal{F}'')\}}$$

Since direct limit is an exact functor, we get the following long exact sequence in Čech cohomology

$$\dots \longrightarrow \check{H}^k(X, \mathcal{F}'') \longrightarrow Q^k(X, \mathcal{F}'') \xrightarrow{\partial} \mathcal{I}^{k+1}(X, \mathcal{F}'') \longrightarrow \check{H}^{k+1}(X, \mathcal{F}'') \longrightarrow \dots$$

¹⁸For proof see [14, Lemma 24.1] and [18, Theorem 25.6].

¹⁹For proof see [Theorem B.2](#).

²⁰One needs to repeat the calculations done in [Lemma 2.6](#) to conclude that $\{\mathcal{I}^k(\mathcal{U}, \mathcal{F}''), H_{\mathcal{U}\mathcal{V}}\}$ is a direct system. Here also the indexing set is directed by refinement, i.e. $\mathcal{U} < \mathcal{V}$ is \mathcal{V} is a refinement of \mathcal{U} .

where we have

$$\mathcal{Q}^k(X, \mathcal{F}'') = \varinjlim_{\mathcal{U}} \mathcal{Q}^k(\mathcal{U}, \mathcal{F}'')$$

Now to obtain the desired isomorphism, it's sufficient to show that $\boxed{\mathcal{Q}^k(X, \mathcal{F}'') = 0}$. To prove this, we will use the fact that X is a paracompact Hausdorff space and ψ is surjective.

Claim: Let $\mathcal{U} = \{U_i\}_{i \in A}$ be an open cover of X , and $f = (f_{i_0, \dots, i_k})$ be an element of $\check{C}^k(\mathcal{U}, \mathcal{F}'')$. Then there exists a refinement $\mathcal{V} = \{V_j\}_{j \in B}$ along with a refining map $r : B \rightarrow A$ such that $V_j \subset U_{r(j)}$ and $\tilde{r}(f) \in I^k(\mathcal{V}, \mathcal{F}'')$, where \tilde{r} is the map defined in [Lemma 2.5](#). Therefore $\mathcal{Q}^k(X, \mathcal{F}'') = 0$.

Since X is paracompact, without loss of generality, assume \mathcal{U} to be locally finite. Also, by *shrinking lemma* ([Theorem A.1](#)) there exists a locally finite open covering $\mathcal{W} = \{W_i\}_{i \in A}$ of X such that $\overline{W_i} \subset U_i$ for each $i \in A$. For every $x \in X$, choose an open neighborhood V_x of x such that

1. If $x \in U_i$ then $V_x \subset U_i$ for all such i 's. If $x \in W_i$ then $V_x \subset W_i$ for all such i 's.
2. If $V_x \cap W_i \neq \emptyset$ then $V_x \subset U_i$ for all such i 's.
3. If $x \in U_{i_0, i_1, \dots, i_k}$ then there exists a $h \in \mathcal{F}(V_x)$ such that

$$\psi_{V_x}(h) = \rho_{U_{i_0, \dots, i_k}, V_x}^{\mathcal{F}''}(f_{i_0, \dots, i_k})$$

where by the first condition $V_x \subset U_{i_0, \dots, i_k}$.

The first condition can be satisfied because \mathcal{U} and \mathcal{W} are *point finite*²¹. Given the first condition, the second condition will be satisfied because $\overline{W_i} \subset U_i$. The third condition will be satisfied because \mathcal{U} is point finite and ψ is a surjective map of sheaves, i.e. there are only finitely many U_{i_0, \dots, i_k} which contain x and for every open set U_{i_0, \dots, i_k} containing x and every $f_{i_0, \dots, i_k} \in \mathcal{F}''(U_{i_0, \dots, i_k})$, there is an open subset $V_x \subset U_{i_0, \dots, i_k}$ containing x such that $\psi_{V_x}(h) = \rho_{U_{i_0, \dots, i_k}, V_x}^{\mathcal{F}''}(f_{i_0, \dots, i_k})$ for some $h \in \mathcal{F}(V_x)$ ([Remark 2.6](#)).

Choose a map $r : X \rightarrow A$ such that $x \in W_{r(x)}$. Then by the first condition, $V_x \subset W_{r(x)} \subset U_{r(x)}$ and $\mathcal{V} = \{V_x\}_{x \in X}$ is a refinement of \mathcal{U} . Now consider the map

$$\begin{aligned} \tilde{r} : \check{C}^k(\mathcal{U}, \mathcal{F}'') &\rightarrow \check{C}^k(\mathcal{V}, \mathcal{F}'') \\ f = (f_{i_0, \dots, i_k}) &\mapsto g = (g_{x_0, \dots, x_k}) \end{aligned}$$

where

$$g_{x_0, \dots, x_k} = \rho(f_{r(x_0), \dots, r(x_k)})$$

and $\rho : \mathcal{F}''(U_{r(x_0), \dots, r(x_k)}) \rightarrow \mathcal{F}''(V_{x_0, \dots, x_k})$ is the group homomorphism for the sheaf \mathcal{F}'' corresponding to the nested open subsets $V_{x_0, \dots, x_k} \subset U_{r(x_0), \dots, r(x_k)}$. It remains to show that $\tilde{r}(f) \in I^k(\mathcal{V}, \mathcal{F}'') = \psi_*(\check{C}^k(\mathcal{V}, \mathcal{F}''))$, i.e. there exists $h \in \mathcal{F}(V_{x_0, x_1, \dots, x_k})$ such that

$$\rho(f_{r(x_0), \dots, r(x_k)}) = \psi_{V_{x_0, x_1, \dots, x_k}}(h) \tag{2.9}$$

If $V_{x_0, \dots, x_k} = \emptyset$ then there is nothing to prove. If not, then we have $V_{x_0} \cap V_{x_\ell} \neq \emptyset$ for all $0 \leq \ell \leq k$. Since $V_{x_\ell} \subset W_{r(x_\ell)}$ we have $V_{x_0} \cap W_{r(x_\ell)} \neq \emptyset$ for all $0 \leq \ell \leq k$, then by the second condition we have $V_{x_0} \subset U_{r(x_\ell)}$ for all $0 \leq \ell \leq k$. Hence, $x_0 \in U_{r(x_0), \dots, r(x_k)}$ and we can use the third condition to conclude that there exists $h' \in \mathcal{F}(V_{x_0})$ such that

$$\psi_{V_{x_0}}(h') = \rho_{U_{r(x_0), \dots, r(x_k)}, V_{x_0}}^{\mathcal{F}''}(f_{r(x_0), \dots, r(x_k)})$$

²¹An open cover $\mathcal{U} = \{U_i\}_{i \in A}$ of X is *point finite* if each point of X is contained in U_i for only finitely many $i \in A$. Every locally finite cover is point finite, but the converse is not true. For example, $\{1/n\}_{n \in \mathbb{N}}$ is a point finite cover of \mathbb{R} , but is not locally finite at 0.

Now let $h = \rho_{V_{x_0}, V_{x_0, x_1, \dots, x_k}}^{\mathcal{F}''}(h')$ and use the fact that ψ commutes with ρ to get (2.9). Hence completing the proof. \square

Remark 2.19. By [Theorem XII](#) we know that manifolds are paracompact. Hence the above theorem can be applied to the sheaf of differential forms. In particular, by [Example 2.7](#) and [Example 2.8](#), we have the short exact sequence of sheaves on a smooth manifold M

$$0 \longrightarrow \mathcal{Z}^q \longleftarrow \Omega^q \xrightarrow{d} \mathcal{Z}^{q+1} \longrightarrow 0$$

This induces the following long exact sequence

$$\dots \longrightarrow \check{H}^k(M, \Omega^q) \longrightarrow \check{H}^k(M, \mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{H}^{k+1}(M, \mathcal{Z}^q) \longrightarrow \check{H}^{k+1}(M, \Omega^q) \longrightarrow \dots$$

2.2.3 Fine sheaves

In this subsection, the condition under which $\check{H}^k(X, \mathcal{F})$ vanishes for all $k \geq 1$ will be discussed following Hirzebruch [[4](#), §2.11] and Warner [[19](#), §5.10, 5.33].

Definition 2.18 (Sheaf partition of unity). Let \mathcal{F} be a sheaf of abelian groups over a paracompact Hausdorff space X . Given a locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X , the *partition of unity of \mathcal{F}* subordinate to the cover \mathcal{U} is a family of sheaf maps $\{\eta_i : \mathcal{F} \rightarrow \mathcal{F}\}$ such that

1. $\text{supp}(\eta_i) \subset U_i$ for each U_i
2. $\sum_{i \in I} \eta_i = \mathbf{1}_{\mathcal{F}}$ (the sum can be formed because \mathcal{U} is locally finite)

where $\text{supp}(\eta_i)$ is the closure of the set of those $x \in X$ for which $(\eta_i)_x : \mathcal{F}_x \rightarrow \mathcal{F}_x$ is not a zero map.

Definition 2.19 (Fine sheaf). A sheaf of abelian groups \mathcal{F} over a paracompact Hausdorff space X is *fine* if for any locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X there exists a partition of unity of \mathcal{F} subordinate to the covering \mathcal{U} .

Example 2.9. Since the multiplication by a continuous or differentiable globally defined function defines a sheaf map in a natural way. From [Theorem A.2](#) we conclude that the sheaf of continuous functions on a paracompact Hausdorff space is a fine sheaf. Also, by [Theorem XIII](#), the sheaf Ω^q of smooth q -forms on a smooth manifold M is a fine sheaf [[20](#), Example II.3.4].

Theorem 2.2. *Let \mathcal{F} be a fine sheaf over a paracompact Hausdorff space X . Then $\check{H}^k(X, \mathcal{F})$ vanishes for $k \geq 1$.*

Proof. Since X is paracompact, every open cover of X has a locally finite refinement, it suffices to prove that $\check{H}^k(\mathcal{U}, \mathcal{F}) = 0$ for all $k \geq 1$ if $\mathcal{U} = \{U_i\}_{i \in I}$ is any locally finite open cover of X . For $k \geq 1$, we define the homomorphism

$$\begin{aligned} \lambda_k : \check{C}^k(\mathcal{U}, \mathcal{F}) &\rightarrow \check{C}^{k-1}(\mathcal{U}, \mathcal{F}) \\ (f_{i_0, i_1, \dots, i_k}) &\mapsto (h_{i_0, i_1, \dots, i_{k-1}}) \end{aligned}$$

where

$$h_{i_0, i_1, \dots, i_{k-1}} = \sum_{i \in I} \eta_i (f_{i, i_0, \dots, i_{k-1}})$$

and $\{\eta_i : \mathcal{F} \rightarrow \mathcal{F}\}_{i \in I}$ is a partition of unity of \mathcal{F} subordinate to the covering \mathcal{U} . Also, let $\delta_k : \check{C}^k(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^{k+1}(\mathcal{U}, \mathcal{F})$ be the coboundary operator as in [Definition 2.10](#). Then from [Proposition 2.4](#) it follows that for $f = (f_{i_0, \dots, i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$ we have

$$\delta_{k-1}(\lambda_k(f)) = f \quad \text{for } k \geq 1$$

Therefore, $f \in \check{B}^k(\mathcal{U}, \mathcal{F})$ and $\check{H}^k(\mathcal{U}, \mathcal{F}) = 0$ for all $k \geq 1$.

We will check the claim just for the simplest case, when $k = 1$. For $f = (f_{i_0 i_1}) \in \check{Z}^1(\mathcal{U}, \mathcal{F})$ and $\delta(f) = (g_{i_0 i_1 i_2}) = 0$ we have [3, pp. 42]

$$\begin{aligned}
\delta_0(\lambda_1((f_{i_0 i_1}))) &= \delta_0\left(\left(\sum_{i \in I} \eta_i(f_{ii_0})\right)\right) \\
&= \left(\rho_{U_{i_1} U_{i_0 i_1}}\left(\sum_{i \in I} \eta_i(f_{ii_1})\right) - \rho_{U_{i_0} U_{i_0 i_1}}\left(\sum_{i \in I} \eta_i(f_{ii_0})\right)\right) \\
&= \left(\sum_{i \in I} \eta_i\left(\rho_{U_{i_1} U_{ii_1 i_0}}(f_{ii_1})\right) - \sum_{i \in I} \eta_i\left(\rho_{U_{i_0} U_{ii_1 i_0}}(f_{ii_0})\right)\right) \\
&= \left(\sum_{i \in I} \eta_i\left(\rho_{U_{i_1} U_{ii_1 i_0}}(f_{ii_1}) - \rho_{U_{i_0} U_{ii_1 i_0}}(f_{ii_0})\right)\right) \\
&= \left(\sum_{i \in I} \eta_i\left(\rho_{U_{i_1 i_0} U_{ii_1 i_0}}(f_{i_0 i_1})\right)\right) \\
&= \left(\rho_{U_{i_1 i_0} U_{i_1 i_0}}\left(\sum_{i \in I} \eta_i(f_{i_0 i_1})\right)\right) \\
&= (f_{i_0 i_1})
\end{aligned}$$

since sheaf map η_i commutes with ρ , ρ_{UU} is identity, $\{\eta_i\}$ is partition of unity and by [Proposition 2.4](#) we have

$$\begin{aligned}
0 = g_{ii_1 i_0} &= \rho_{U_{i_1 i_0} U_{ii_1 i_0}}(f_{i_1 i_0}) - \rho_{U_{i_0} U_{ii_1 i_0}}(f_{ii_0}) + \rho_{U_{i_1} U_{ii_1 i_0}}(f_{ii_1}) \\
\rho_{U_{i_1 i_0} U_{ii_1 i_0}}(f_{i_0 i_1}) &= -\rho_{U_{i_0} U_{ii_1 i_0}}(f_{ii_0}) + \rho_{U_{i_1} U_{ii_1 i_0}}(f_{ii_1})
\end{aligned}$$

□

Remark 2.20. We can apply this theorem to the the sheaf of smooth q -forms on a smooth manifold M , hence $\check{H}^k(M, \Omega^q) = 0$ for all $k \geq 1$.

Chapter 3

de Rham-Čech isomorphism

“Since the de Rham cohomology of a C^∞ manifold is defined using differential forms, it would seem to depend essentially on the differentiable structure of M . However, in reality, it is determined only by the properties of M as a topological space. It is the de Rham theorem that expresses this fact concretely.”

— Shigeyuki Morita, *Geometry of Differential Forms*, p. 113

Theorem 3.1. *Let M be a smooth manifold. Then for each $k \geq 0$ there exists a group isomorphism*

$$H_{dR}^k(M) \cong \check{H}^k(M, \mathbb{R})$$

Proof. For $k = 0$, from [Proposition 1.2](#) and [Proposition 2.6](#), we know that both $H_{dR}^0(M)$ and $\check{H}^0(M, \mathbb{R})$ are isomorphic to the group of locally constant real valued functions on M . That is

$$H_{dR}^0(M) \cong \check{H}^0(M, \mathbb{R})$$

Now let’s restrict our attention to $k \geq 1$. From [Example 2.8](#) we know that the Poincaré lemma implies the existence of the following long exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathbb{R} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

Then, as noted in [Remark 2.19](#), we get a family of short exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \hookrightarrow & \Omega^0 & \xrightarrow{d} & \mathcal{Z}^1 & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathcal{Z}^1 & \hookrightarrow & \Omega^1 & \xrightarrow{d} & \mathcal{Z}^2 & \longrightarrow & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & \mathcal{Z}^q & \hookrightarrow & \Omega^q & \xrightarrow{d} & \mathcal{Z}^{q+1} & \longrightarrow & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

Since a smooth manifold is a paracompact Hausdorff space, we can apply [Theorem 2.1](#) to get a family of long exact sequence of Čech cohomology

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \check{H}^k(M, \Omega^0) & \longrightarrow & \check{H}^k(M, \mathcal{Z}^1) & \xrightarrow{\Delta} & \check{H}^{k+1}(M, \mathbb{R}) & \longrightarrow & \check{H}^{k+1}(M, \Omega^0) & \longrightarrow & \dots \\ \dots & \longrightarrow & \check{H}^k(M, \Omega^1) & \longrightarrow & \check{H}^k(M, \mathcal{Z}^2) & \xrightarrow{\Delta} & \check{H}^{k+1}(M, \mathcal{Z}^1) & \longrightarrow & \check{H}^{k+1}(M, \Omega^1) & \longrightarrow & \dots \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \\ \dots & \longrightarrow & \check{H}^k(M, \Omega^q) & \longrightarrow & \check{H}^k(M, \mathcal{Z}^{q+1}) & \xrightarrow{\Delta} & \check{H}^{k+1}(M, \mathcal{Z}^q) & \longrightarrow & \check{H}^{k+1}(M, \Omega^q) & \longrightarrow & \dots \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

Now let's study one of these long exact sequence of Čech cohomology. By [Proposition 2.6](#) we have $\check{H}^0(M, \Omega^q) \cong \Omega^q(M)$ and $\check{H}^0(M, \mathcal{Z}^q) \cong \mathcal{Z}^q(M)$. Also by [Remark 2.20](#) we have $\check{H}^k(M, \Omega^q) = 0$ for all $k \geq 1$ and $q \geq 0$. Hence for any $q \geq 0$ we get the exact sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{Z}^q(M) & \hookrightarrow & \Omega^q(M) & \xrightarrow{d} & \mathcal{Z}^{q+1}(M) & \xrightarrow{\Delta} & \check{H}^1(M, \mathcal{Z}^q) & \longrightarrow & 0 \\ & & & & & & & & & & \downarrow \Delta \\ & & & & & & & & & & \check{H}^1(M, \mathcal{Z}^{q+1}) \\ & & & & & & & & & & \downarrow \Delta \\ \dots & \longleftarrow & 0 & \longleftarrow & \check{H}^3(M, \mathcal{Z}^q) & \xleftarrow{\Delta} & \check{H}^2(M, \mathcal{Z}^{q+1}) & \longleftarrow & 0 & \longleftarrow & \check{H}^2(M, \mathcal{Z}^q) \end{array}$$

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^q(M) \hookrightarrow \Omega^q(M) \xrightarrow{d} \mathcal{Z}^{q+1}(M) \xrightarrow{\Delta} \check{H}^1(M, \mathcal{Z}^q) \longrightarrow 0$$

Since this sequence is exact, the map $\Delta : \mathcal{Z}^{q+1}(M) \rightarrow \check{H}^1(M, \mathcal{Z}^q)$ is a surjective group homomorphism and $\text{im}\{d : \Omega^q(M) \rightarrow \mathcal{Z}^{q+1}(M)\} = \ker(\Delta)$. Hence by the *first isomorphism theorem* we get

$$\check{H}^1(M, \mathcal{Z}^q) \cong \frac{\mathcal{Z}^{q+1}(M)}{\ker(\Delta)} \quad \text{for all } q \geq 0$$

Since $\text{im}\{d : \Omega^q(M) \rightarrow \mathcal{Z}^{q+1}(M)\} = \text{im}\{d : \Omega^q(M) \rightarrow \Omega^{q+1}(M)\} = \mathcal{B}^{q+1}(M)$, we get

$$\check{H}^1(M, \mathcal{Z}^q) \cong H_{dR}^{q+1}(M) \quad \text{for all } q \geq 0 \quad (3.1)$$

Note that $\mathcal{Z}^0 = \underline{\mathbb{R}}$, hence from [\(3.1\)](#) we get

$$\boxed{\check{H}^1(M, \underline{\mathbb{R}}) \cong H_{dR}^1(M)}$$

Next we consider the remaining parts of the long exact sequence, i.e. for $k \geq 1$ and $q \geq 0$ we have

$$0 \longrightarrow \check{H}^k(M, \mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{H}^{k+1}(M, \mathcal{Z}^q) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{H}^{k+1}(M, \mathcal{Z}^q) \cong \check{H}^k(M, \mathcal{Z}^{q+1}) \quad \text{for all } k \geq 1, q \geq 0 \quad (3.2)$$

Again substituting $\mathcal{Z}^0 = \underline{\mathbb{R}}$ and restricting our attention to $k \geq 2$, we apply [\(3.2\)](#) recursively to get

$$\begin{aligned} \check{H}^k(M, \underline{\mathbb{R}}) &\cong \check{H}^{k-1}(M, \mathcal{Z}^1) \\ &\cong \check{H}^{k-2}(M, \mathcal{Z}^2) \\ &\vdots \\ &\cong \check{H}^1(M, \mathcal{Z}^{k-1}) \end{aligned}$$

Then using [\(3.1\)](#) we get

$$\boxed{\check{H}^k(M, \underline{\mathbb{R}}) \cong H_{dR}^k(M)} \quad \text{for all } k \geq 2$$

Hence completing the proof. \square

Remark 3.1. One can use Weil's method involving generalized Mayer-Vietoris principle for the Čech-de Rham complex to directly show the isomorphism between Čech cohomology with values in \mathbb{R} and de Rham cohomology of smooth manifold M , without using sheaf theory. There are two versions of the proof depending on the definition of Čech cohomology used, see [\[10, Theorem 3.19\]](#) if defined using nerve and [\[1, Proposition 10.6\]](#) if defined using presheaf.

Appendix A

Paracompact spaces

In this appendix some definitions and facts from [12, §39 and 41] will be stated. Here X denotes a topological space.

Definition A.1 (Locally finite collection). Let X be a topological space. A collection \mathcal{U} of subsets of X is said to be *locally finite* in X if every point of X has a neighborhood that intersects only finitely many elements of \mathcal{U} .

Lemma A.1. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite collection of subsets of X . Then

1. any subcollection of \mathcal{U} is locally finite.
2. the collection $\mathcal{V} = \{\overline{U_\alpha}\}_{\alpha \in A}$ of the closures of the elements of \mathcal{U} is locally finite.
3. $\overline{\bigcup_{\alpha \in A} U_\alpha} = \bigcup_{\alpha \in A} \overline{U_\alpha}$

Definition A.2 (Refinement of a collection). Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a collection of subsets of X . A collection $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ of subsets of X is said to be a *refinement* of \mathcal{U} if for each element V_β of \mathcal{V} , there is an element U_α of \mathcal{U} containing V_β .

Remark A.1. If elements of \mathcal{V} are open sets, the \mathcal{V} is called an *open refinement* of \mathcal{U} ; if they are closed, \mathcal{V} is called a *closed refinement*.

Definition A.3 (Paracompact space). The space X is *paracompact* if every open covering \mathcal{U} of X has a locally finite open refinement \mathcal{V} that covers X .

Remark A.2. In most algebraic geometry textbooks, following the lead of Bourbaki, the requirement that the space be Hausdorff is included as part of the definition of the term *compact* and *paracompact*. We shall not follow this convention.

Theorem A.1 (Shrinking lemma). Let X be a paracompact Hausdorff space; let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed family of open sets covering X . Then there exists a locally finite indexed family $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ of open sets covering X such that $\overline{V_\alpha} \subseteq U_\alpha$ for each α .

Definition A.4 (Continuous partition of unity). Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed open covering of X . An indexed family of continuous functions $\{\phi_\alpha : X \rightarrow [0, 1]\}$ is said to be a *continuous partition of unity* on X , dominated by $\{U_\alpha\}$, if

1. $\text{supp}(\phi_\alpha) \subseteq U_\alpha$ for each α
2. the indexed family $\{\text{supp}(\phi_\alpha)\}_{\alpha \in A}$ is locally finite
3. $\sum_{\alpha \in A} \phi_\alpha(x) = 1$ for each $x \in X$.

where $\text{supp}(\phi_\alpha)$ is the closure of the set of those $x \in X$ for which $\phi_\alpha(x) \neq 0$.

Theorem A.2. Let X be a paracompact Hausdorff space; let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed open covering of X . Then there exists a continuous partition of unity on X dominated by $\{U_\alpha\}$

Appendix B

Direct limit

In this appendix some definitions and facts from [14, §73] and [15, §IV.2] will be stated.

Definition B.1 (Directed set). A directed set A is a set with relation $<$ such that

1. $\alpha < \alpha$ for all $\alpha \in A$
2. $\alpha < \beta$ and $\beta < \gamma$ implies $\alpha < \gamma$
3. Given α and β , there exists δ such that $\alpha < \delta$ and $\beta < \delta$. The element δ is called an *upper limit* for α and β .

Definition B.2 (Direct system). A *direct system* of abelian groups and group homomorphisms, corresponding to the directed set A , is an indexed family $\{G_\alpha\}_{\alpha \in A}$ of abelian groups, along with the family of homomorphisms $\{f_{\alpha\beta} : G_\alpha \rightarrow G_\beta\}_{\alpha, \beta \in A, \alpha < \beta}$ such that

1. $f_{\alpha\alpha} : G_\alpha \rightarrow G_\alpha$ is identity
2. If $\alpha < \beta < \gamma$ then $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$; i.e. the following diagram commutes:

$$\begin{array}{ccc} G_\alpha & \xrightarrow{f_{\alpha\gamma}} & G_\gamma \\ & \searrow f_{\alpha\beta} & \nearrow f_{\beta\gamma} \\ & & G_\beta \end{array}$$

Definition B.3 (Direct limit). Given a directed set A and the associated direct system of abelian groups and homomorphisms $\{(G_\alpha, f_{\alpha\beta})\}$, the *direct limit* is defined to be the quotient

$$\varinjlim_{\alpha \in A} G_\alpha = \coprod_{\alpha \in A} G_\alpha / \sim$$

where, given $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$, $g_\alpha \sim g_\beta$ if there exists an upper bound δ of α and β such that $f_{\alpha\delta}(g_\alpha) = f_{\beta\delta}(g_\beta)$. Also, $g_\alpha \sim g_\beta$ implies that they belong to same equivalence class, i.e. $\llbracket g_\alpha \rrbracket = \llbracket g_\beta \rrbracket$. The direct limit is again an abelian group under addition defined as

$$\llbracket g_\alpha \rrbracket + \llbracket g_\beta \rrbracket := \llbracket f_{\alpha\delta}(g_\alpha) + f_{\beta\delta}(g_\beta) \rrbracket$$

for some upper bound δ of α and β .

Remark B.1. Just as in case of definition of sheaf, the definition of direct limit can be generalized to any category like groups, rings, modules, and algebras instead of abelian groups.

Proposition B.1. *Given a directed set A and the associated direct system $\{(G_\alpha, f_{\alpha\beta})\}$ of abelian groups and homomorphisms such that all the maps $f_{\alpha\beta}$ are isomorphisms, then $\varinjlim G_\alpha$ is isomorphic to any one of the groups G_α .*

Proposition B.2. *Given a directed set A and the associated direct system $\{(G_\alpha, f_{\alpha\beta})\}$ of abelian groups and homomorphisms such that all the maps $f_{\alpha\beta}$ are zero-homomorphisms, then $\varinjlim G_\alpha$ is the trivial group. More generally, if for each α there is a β such that $\alpha < \beta$ and $f_{\alpha\beta}$ is the zero homomorphism, then $\varinjlim G_\alpha$ is the trivial group.*

Definition B.4 (Map of direct systems). Let A and B be two directed sets. Let $\{(G_\alpha, f_{\alpha\beta})\}$ and $\{(G'_\gamma, f'_{\gamma\delta})\}$ be the associated direct systems of abelian groups and homomorphisms, respectively. A *map of direct systems* $\Phi = (\phi, \{\phi_\alpha\}) : \{(G_\alpha, f_{\alpha\beta})\} \rightarrow \{(G'_\gamma, f'_{\gamma\delta})\}$ is a collection of maps such that

1. the set map $\phi : A \rightarrow B$ that preserves order relation
2. for each $\alpha \in A$, $\phi_\alpha : G_\alpha \rightarrow G'_{\phi(\alpha)}$ is a group homomorphism such that the following diagram commutes

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\phi_\alpha} & G'_\gamma \\ \downarrow f_{\alpha\beta} & & \downarrow f'_{\gamma\delta} \\ G_\beta & \xrightarrow{\phi_\beta} & G'_\delta \end{array}$$

for $\alpha < \beta$, $\gamma = \phi(\alpha)$ and $\delta = \phi(\beta)$

Definition B.5 (Direct limit of direct system homomorphisms). The map of direct systems $\Phi : \{(G_\alpha, f_{\alpha\beta})\} \rightarrow \{(G'_\gamma, f'_{\gamma\delta})\}$ induces a homomorphism, called the *direct limit of the homomorphisms* ϕ_α

$$\Phi : \varinjlim_{\alpha \in A} G_\alpha \rightarrow \varinjlim_{\gamma \in B} G'_\gamma$$

It maps the equivalence class of $g_\alpha \in G_\alpha$ to the equivalence class of $\phi_\alpha(g_\alpha)$.

Theorem B.1 (Universal property of direct limits). *Let A be a directed set and $\{(G_\alpha, f_{\alpha\beta})\}$ be the associated direct system of abelian groups and homomorphisms. If $G = \varinjlim_{\alpha \in A} G_\alpha$, then the inclusion $i_\alpha : G_\alpha \hookrightarrow \coprod_{\alpha \in A} G_\alpha$ induces a family of group homomorphisms $\{\chi_\alpha : G_\alpha \rightarrow G\}_{\alpha \in A}$. If H is an abelian group such that for each $\alpha \in A$ there is a group homomorphism $\psi_\alpha : G_\alpha \rightarrow H$ satisfying $\psi_\alpha = \psi_\beta \circ f_{\alpha\beta}$, whenever $\alpha < \beta$. Then there exists a unique group homomorphism*

$$\Psi : G \rightarrow H$$

satisfying $\psi_\alpha = \Psi \circ \chi_\alpha$ for all $\alpha \in A$.

Remark B.2. We observe that this universal property is a special case of the preceding construction, in which second direct system consists of the single group H . Hence, we have $\Psi = \underline{\Psi}$. One can also observe that the family of group homomorphisms $\{\chi_\alpha : G_\alpha \rightarrow G\}_{\alpha \in A}$ satisfies the condition $\chi_\alpha = \chi_\beta \circ f_{\alpha\beta}$ for all $\alpha < \beta$ since the following diagram commutes

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\chi_\alpha} & G \\ \downarrow f_{\alpha\beta} & & \downarrow \mathbb{1}_G \\ G_\beta & \xrightarrow{\chi_\beta} & G \end{array}$$

Theorem B.2 (Direct limit is as an exact functor). Let A be a directed set¹. Let $\{(G'_\alpha, f'_{\alpha\beta})\}$, $\{(G_\alpha, f_{\alpha\beta})\}$ and $\{(G''_\alpha, f''_{\alpha\beta})\}$ be three direct systems of abelian groups and homomorphisms associated with A , with the maps of direct systems

$$\Phi : \{(G'_\alpha, f'_{\alpha\beta})\} \rightarrow \{(G_\alpha, f_{\alpha\beta})\} \quad \text{and} \quad \Psi : \{(G_\alpha, f_{\alpha\beta})\} \rightarrow \{(G''_\alpha, f''_{\alpha\beta})\}$$

such that the sequence of abelian groups

$$G'_\alpha \xrightarrow{\phi_\alpha} G_\alpha \xrightarrow{\psi_\alpha} G''_\alpha$$

is exact for every $\alpha \in A$. Then the induced sequence

$$\varinjlim_{\alpha \in A} G'_\alpha \xrightarrow{\Phi} \varinjlim_{\alpha \in A} G_\alpha \xrightarrow{\Psi} \varinjlim_{\alpha \in A} G''_\alpha$$

is also exact.

Proof. Let $G' = \varinjlim_{\alpha \in A} G'_\alpha$, $G = \varinjlim_{\alpha \in A} G_\alpha$ and $G'' = \varinjlim_{\alpha \in A} G''_\alpha$. We consider the commutative diagram, for all $\alpha \in A$

$$\begin{array}{ccccc} G'_\alpha & \xrightarrow{\phi_\alpha} & G_\alpha & \xrightarrow{\psi_\alpha} & G''_\alpha \\ \downarrow \chi'_\alpha & & \downarrow \chi_\alpha & & \downarrow \chi''_\alpha \\ G' & \xrightarrow{\Phi} & G & \xrightarrow{\Psi} & G'' \end{array}$$

where $\chi'_\alpha, \chi_\alpha$ and χ''_α are the homomorphisms induced by the inclusion maps into the respective disjoint union (as in [Theorem B.1](#)). Given to us is that $\text{im } \phi_\alpha = \ker \psi_\alpha$ for all $\alpha \in A$.

Claim: $\text{im } \Phi = \ker \Psi$

($\ker \Psi \subseteq \text{im } \Phi$) Let $g \in G$, then by the definition of direct limit there exists $\alpha \in A$ such that for some $g_\alpha \in G_\alpha$ we have $\chi_\alpha(g_\alpha) = g$. Also, let $\Psi(g) = 0_{G''}$. By the commutative diagram above, we have

$$\chi''_\alpha(\psi_\alpha(g_\alpha)) = \Psi(\chi_\alpha(g_\alpha)) = \Psi(g) = 0_{G''}$$

The direct limit is a collection of equivalence classes, hence we have

$$\chi''_\alpha(\psi_\alpha(g_\alpha)) = [\psi_\alpha(g_\alpha)] = [0_{G''_\alpha}]$$

Since $\psi_\alpha(g_\alpha), 0_{G''_\alpha} \in G''_\alpha$, we have $f''_{\alpha\delta}(\psi_\alpha(g_\alpha)) = f''_{\alpha\delta}(0_{G''_\alpha}) = 0_{G''_\delta}$ for some δ such that $\alpha < \delta$. But $\psi_\delta \circ f_{\alpha\delta} = f''_{\alpha\delta} \circ \psi_\alpha$, hence we have $\psi_\delta(f_{\alpha\delta}(g_\alpha)) = 0_{G''_\delta}$. Hence $f_{\alpha\delta}(g_\alpha) \in \ker \psi_\delta = \text{im } \phi_\delta$. So there exist $h_\delta \in G'_\delta$ such that $\phi_\delta(h_\delta) = f_{\alpha\delta}(g_\alpha)$. Using $\chi_\alpha = \chi_\delta \circ f_{\alpha\delta}$ and commutativity of diagram we get we get

$$g = \chi_\alpha(g_\alpha) = \chi_\delta(f_{\alpha\delta}(g_\alpha)) = \chi_\delta(\phi_\delta(h_\delta)) = \Phi(\chi'_\delta(h_\delta))$$

($\text{im } \Phi \subseteq \ker \Psi$) Suppose $g \in \text{im } \Phi$. Then $g = \Phi(h)$, and by definition of direct limit we have $h = \chi'_\alpha(h_\alpha)$ for some $h_\alpha \in G'_\alpha$. Now by the commutativity of diagram we have

$$g = \Phi(\chi'_\alpha(h_\alpha)) = \chi_\alpha(\phi_\alpha(h_\alpha))$$

Since $\psi_\alpha \circ \phi_\alpha = 0_{G''_\alpha}$ by exactness, we have

$$\Psi(g) = \Psi(\chi_\alpha(\phi_\alpha(h_\alpha))) = \chi''_\alpha(\psi_\alpha(\phi_\alpha(h_\alpha))) = \chi''_\alpha(0_{G''_\alpha}) = 0_{G''}$$

Hence completing the proof. □

¹To avoid too many new symbols, let all the direct systems be associated with the same directed set, i.e. $A = B = C$ and $\phi = 1_A$.

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