Sheaf-theoretic de Rham isomorphism

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Outline

Differential form

Let M be a smooth manifold of dimension n.

Differential k-form

A *differential k-form* on M is a smooth section of the vector bundle $\pi : \Lambda^k(T^*M) \to M$.

We denote the group of all smooth k-forms on M by $\Omega^k(M)$. Also, a *differential 0-form* on M is a smooth real valued function on M, i.e. $\Omega^0(M) = C^\infty(M)$.

Lemma

If $(U, x_1, \ldots, x_n)$ is a coordinate chart on a smooth manifold M, then $\omega \in \Omega^k(U)$ can be uniquely written as

$$\omega = \sum_{[I]} a_I \, dx_I, \qquad a_I \in C^\infty(U)$$

where $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\{1, \ldots, n\}$.

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Lemma

If $(U, x_1, \ldots, x_n)$ is a coordinate chart on a smooth manifold M, then $\omega \in \Omega^k(U)$ can be uniquely written as

$$\omega = \sum_{[I]} a_I \, \mathrm{d}x_I, \qquad a_I \in C^\infty(U)$$

where $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\{1, \ldots, n\}$.

Exterior derivative

Differential of a k-form

The $\mathbb{R}$-linear map $\mathrm{d}_U : \Omega^k(U) \to \Omega^{k+1}(U)$ defined as

$$\mathrm{d}_U\omega = \sum_{[I]} \mathrm{d}a_I \wedge \mathrm{d}x_I$$

is called the exterior derivative of ω on U. Let $p \in U$, then $(\mathrm{d}_U\omega)_p$ is independent of the chart containing p. The *differential of a k-form* is defined by the linear operator

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$$

such that for $k \geq 0$ and $\omega \in \Omega^k(M)$ one has $(\mathrm{d}\omega)_p = (\mathrm{d}_U\omega)_p$ for all $p \in M$.

Exterior derivative

Properties

1. It is a local operator, i.e. for all $k \geq 0$,whenever a k-form $\omega \in \Omega^k(M)$ is such that $\omega_p = 0$ for all points p in an open set U of M, then $\mathrm{d}\omega \equiv 0$ on U. Equivalently, for all $k \geq 0$, whenever two k-forms $\omega, \eta \in \Omega^k(M)$ agree on an open set U, then $\mathrm{d}\omega \equiv \mathrm{d}\eta$ on U
2. $\mathrm{d} \circ \mathrm{d} = 0$

Closed and Exact forms

Closed forms

$\omega \in \Omega^k(U)$ for $k \geq 0$ is said to be *closed* if $d\omega = 0$.

We denote the group of all closed k-forms on M by $\mathcal{Z}^k(M)$.

Exact k-forms

$\omega \in \Omega^k(U)$ for $k \geq 1$ is said to be *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$.

We denote the group of all exact k-forms on M by $\mathcal{B}^k(M)$. Also, $\mathcal{B}^0(M)$ is defined to be the set consisting of only zero.

Proposition

Every exact form is closed.

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The k^{th} de Rham cohomology group of M is the quotient group

$$H^k_{dR}(M) := \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)}$$

Hence, $H^k_{dR}(M)$ measures the extent to which closed k-forms are not exact on M.

Proposition

If the smooth manifold M has ℓ connected components, then $H^0_{dR}(M) = \mathbb{R}^\ell$. An element of $H^0_{dR}(M)$ is specified by an ordered ℓ-tuple of real numbers, each real number representing a constant function on a connected component of M.

Hence, $H^0_{dR}(M)$ is the set of all real valued locally constant functions on M.

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Poincaré lemma

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Let U be a star-convex open set in $\mathbb{R}^n$. If $k \geq 1$, then $H^k_{dR}(U) = 0$, i.e. every closed k-form on U is exact.

In particular, if U is an open ball

$$B(p, \varepsilon) = \{x \in \mathbb{R}^n : ||x - p|| < \varepsilon\}$$

then $H^k_{dR}(U) = 0$ for $k \geq 1$.

Corollary

For all $p \in M$ there exists an open neighborhood U such that every closed k-form on U is exact for $k \geq 1$.

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The theorem

We wish to prove that de Rham cohomology is a topological invariant. To do this we will show that the de Rham cohomology of a smooth manifold is isomorphic to the Čech cohomology of that manifold with real coefficients.

de Rham isomorphism

Let M be a smooth manifold. Then for each $k \geq 0$ there exists a group isomorphism

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Sheaves

Presheaf

A *presheaf* $\mathcal{F}$ of abelian groups on a topological space X consists of an abelian group $\mathcal{F}(U)$ for every open subset $U \subset X$ and a group homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any two nested open subsets $V \subset U$ satisfying the following two conditions:

1. for any open subset U of X one has $\rho_{UU} = \mathbb{1}_{\mathcal{F}(U)}$
2. for open subsets $W \subset V \subset U$ one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho_{UW}} & \mathcal{F}(W) \\ & {\scriptstyle\rho_{UV}} \searrow \quad \nearrow {\scriptstyle\rho_{VW}} & \\ & \mathcal{F}(V) & \end{array}$$

Examples: (continuous/constant) real valued functions, differential forms.

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A presheaf $\mathcal{F}$ on a topologial space X is called a *sheaf* if for every collection $\{U_\alpha\}_{\alpha\in A}$ of open subsets of X with $U = \cup_{\alpha\in A} U_\alpha$ the following conditions are satisfied

1. (Uniqueness) If $f, g \in \mathcal{F}(U)$ and $\rho_{UU_\alpha}(f) = \rho_{UU_\alpha}(g)$ for all $\alpha \in A$, then $f = g$.
2. (Gluing) If for all $\alpha \in A$ we have $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $\rho_{U_\alpha, U_\alpha\cap U_\beta}(f_\alpha) = \rho_{U_\beta, U_\alpha\cap U_\beta}(f_\beta)$ for any $\alpha, \beta \in A$, then there exists a $f \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}(f) = f_\alpha$ for all $\alpha \in A$ (this f is unique by previous axiom).

Observe that, if X is disconnected then the gluing axiom doesn't hold for the presheaf of constant real valued functions on X. We therefore define a *constant sheaf* $\mathbb{R}$ on X to be the collection of *locally constant* real valued functions.

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Sheaf of Differential forms

If one has a presheaf of functions (or forms) on X which is defined by some property which is a local property like continuity and differentiability, then that presheaf is also a sheaf. This is because the agreement of functions (or forms) on the overlap intersections automatically gives a well defined unique function (or form) on the open set U. In particular:

If M is a smooth manifold then Ω^q is the sheaf of smooth q-forms on M such that for every open subset U of M we have the abelian group $\Omega^q(U)$ of smooth q-forms on U along with the natural restriction maps as the group homomorphisms ρ_{UV} for nested open subsets $V \subset U$.

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Cochains

Let $\mathcal{F}$ be sheaf of abelian groups on a topologial space X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X, and fix an integer $k \geq 0$.

Čech cochain

A Čech k-cochain for the sheaf $\mathcal{F}$ over the open cover $\mathcal{U}$ is an element of $\prod_{(i_0,i_1,\ldots,i_k)} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k})$ where Cartesian product is take over all collections of $k+1$ indices $(i_0, \ldots, i_k)$ from I.

To simplify the notation, we will write

$$U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k} := U_{i_0,i_1,\ldots,i_k} \quad \text{and} \quad \mathcal{F}(U_{i_0,i_1,\ldots,i_k}) = \{f_{i_0,i_1,\ldots,i_k}\}$$

Hence a Čech k-cochain is a tuple of the form $(f_{i_0,i_1,\ldots,i_k})$. The abelian group of Čech k-cochains for $\mathcal{F}$ over $\mathcal{U}$ is denoted by $\check{C}^k(\mathcal{U},\mathcal{F})$; thus

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The coboundary operator is defined as

$$\begin{aligned}\delta : \check{C}^k(\mathcal{U}, \mathcal{F}) &\to \check{C}^{k+1}(\mathcal{U}, \mathcal{F})\\ (f_{i_0,i_1,\ldots,i_k}) &\mapsto (g_{i_0,i_1,\ldots,i_{k+1}})\end{aligned}$$

where

$$g_{i_0,i_1,\ldots,i_{k+1}} = \sum_{\ell=0}^{k+1}(-1)^\ell \rho(f_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}})$$

and $\rho : \mathcal{F}(U_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}}) \to \mathcal{F}(U_{i_0,i_1,\ldots,i_{k+1}})$ is the group homomorphism for the sheaf $\mathcal{F}$ corresponding to the nested open subsets $U_{i_0,i_1,\ldots,i_{k+1}} \subset U_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}}$.

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Cocycles

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A Čech k-cochain $f = (f_{i_0,i_1,\ldots,i_k})$ with $\delta(f) = 0$ is called Čech k-cocycle.

The abelian group of k-cocycles is denoted by $\check{Z}^k(\mathcal{U}, \mathcal{F})$, i.e. kernel of δ at the k^{th} level.

Proposition (cocycles are skew-symmetric)

Let $f = (f_{i_0,\ldots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, then

1. $f_{i_0,\ldots,i_n} = 0$ if any two indices are equal.
2. $f_{\sigma(i_0),\sigma(i_1),\ldots,\sigma(i_k)} = \mathrm{sgn}(\sigma) f_{i_0,i_1,\ldots,i_k}$ if σ is a permutation of $\{i_0, \ldots, i_k\}$

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A Čech k-cochain $f = (f_{i_0,i_1,\ldots,i_k})$ which is the image of δ, i.e. there exists $(k-1)$-cochain $g = (g_{i_0,i_1,\ldots,i_{k-1}})$ such that $\delta(g) = f$, is called Čech k-coboundary.

The abelian group of k-coboundaries is denoted by $\check{B}^k(\mathcal{U}, \mathcal{F})$, i.e. image of δ at the $(k-1)^{th}$ level. Also, we define $\check{B}^0(\mathcal{U}, \mathcal{F}) = 0$ for any sheaf $\mathcal{F}$ and open cover $\mathcal{U}$.

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The k^{th} *Čech cohomology group* of $\mathcal{F}$ with respect to the open cover $\mathcal{U}$ is the quotient group

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Hence, the Čech cohomology with respect to a cover measures the extent to which cocycles are not coboundaries for a given open cover.

Proposition

For any sheaf $\mathcal{F}$ and open covering $\mathcal{U}$ of X, $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) \cong \mathcal{F}(X)$.

The group isomorphism is given by:

$$\begin{aligned} \tau : \mathcal{F}(X) &\to \check{Z}^0(\mathcal{U},\mathcal{F}) \\ f &\mapsto (\rho_{XU_i}(f)) \end{aligned}$$

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Čech cohomology

Refinement map

Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open coverings of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ along with the refining map $r : J \to I$ such that $V_j \subset U_{r(j)}$ for every $j \in J$. The induced map at the level of cohomology, called the *refinement map*, is given by

$$\begin{aligned} H_{\mathcal{U}\mathcal{V}} : \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{H}}^k(\mathcal{V}, \mathcal{F}) \\ [(f_{i_0,\dots,i_k})] &\mapsto [(g_{j_0,\dots,j_k})] \end{aligned}$$

for $(f_{i_0,\dots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$, where

$$g_{j_0,\dots,j_k} = \rho(f_{r(j_0),\dots,r(j_k)})$$

and $\rho : \mathcal{F}(U_{r(j_0),\dots,r(j_k)}) \to \mathcal{F}(V_{j_0,\dots,j_k})$ is the group homomorphism for the sheaf $\mathcal{F}$ corresponding to the nested open subsets $V_{j_0,\dots,j_k} \subset U_{r(j_0),\dots,r(j_k)}$.

Čech cohomology

Note that $\{\check{H}^k(\mathcal{U},\mathcal{F}), H_{\mathcal{U}\mathcal{V}}\}$ is a *direct system* since:

1. $H_{\mathcal{U}\mathcal{U}} = \mathbb{1}_{\check{H}^k(\mathcal{U},\mathcal{F})}$ (we can choose refining map r to be identity)
2. $H_{\mathcal{U}\mathcal{W}} = H_{\mathcal{V}\mathcal{W}} \circ H_{\mathcal{U}\mathcal{V}}$ for $\mathcal{U} < \mathcal{V} < \mathcal{W}$

Čech cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups on X and $k \geq 0$ be an integer. Then the k^{th} *Čech cohomology group* of $\mathcal{F}$ on X is the direct limit of the direct system $\{\check{H}^k(\mathcal{U},\mathcal{F}), H_{\mathcal{U}\mathcal{V}}\}$ indexed over all the open covers of X with order relation induced by refinement, i.e. $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$

$$\check{H}^k(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^k(\mathcal{U},\mathcal{F})$$

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$$\check{\mathrm{H}}^k(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F})$$

$H^0_{dR}(M) \cong \check{H}^0(M, \mathbb{R})$

We know that at the $\check{H}^0$ level all the groups are isomorphic to $\mathcal{F}(X)$. Since all the maps $H_{\mathcal{UV}}$ are compatible isomorphisms, the direct limit is also isomorphic to $\mathcal{F}(X)$.

Proposition

For any sheaf $\mathcal{F}$ of X, we have $\check{H}^0(X, \mathcal{F}) \cong \mathcal{F}(X)$.

In particular, for $X = M$ and $\mathcal{F} = \mathbb{R}$, we have

$$\check{H}^0(M, \mathbb{R}) \cong \mathbb{R}(M)$$

Hence, $\check{H}^0(M, \mathbb{R})$ is isomorphic to the group of real valued locally constant functions on M. But we know that $H^0_{dR}(M)$ is also isomorphic to the group of real valued locally constant functions on M. Hence

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Map of sheaves

Map of sheaves

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on a topological space X. A maps of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X is given by a collection of group homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any open subset U of X, which commute with the group homomorphisms ρ for the two sheaves, i.e. for $V \subset U$ the following diagram commutes

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\
\downarrow{\scriptstyle \rho^{\mathcal{F}}_{UV}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{UV}} \\
\mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V)
\end{array}$$

Exterior derivative is a sheaf map

As seen earlier, for the sheaf of functions (or forms) the natural restriction map is the group homomorphism ρ_{UV} for nested open subsets $V \subset U$. Since the exterior derivative is a local operator, it commutes with restriction, i.e. the following diagram commutes for $V \subset U$

$$\begin{array}{ccc} \Omega^q(U) & \xrightarrow{\mathrm{d}_U} & \Omega^{q+1}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho_{UV}} \\ \Omega^q(V) & \xrightarrow{\mathrm{d}_V} & \Omega^{q+1}(V) \end{array}$$

$\mathrm{d} : \Omega^q \to \Omega^{q+1}$ is a map of sheaves, where Ω^q and Ω^{q+1} are sheaves of smooth q-forms and $(q+1)$-forms, respectively, defined on a smooth manifold M.

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Sheaf of closed forms

Associated presheaf

Given a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ on X, we have the *associated presheaves* $\ker(\phi), \mathrm{im}(\phi)$, and $\mathrm{coker}(\phi)$ defined in the obvious way, i.e. $\ker(\phi)(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ with group homomorphism ρ inherited from $\mathcal{F}$.

Lemma

$\ker(\phi)$ is a sheaf.

If M is a smooth manifold and $\mathrm{d} : \Omega^q \to \Omega^{q+1}$ is the exterior derivative. Then $\ker(\mathrm{d}) = \mathcal{Z}^q$ is the sheaf of closed q-forms on X.

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Exact sequence of sheaves

Stalk

Let $\mathcal{F}$ be a sheaf on X, and let $x \in X$. Then the *stalk* of $\mathcal{F}$ at x is the direct limit of the direct system $\{\mathcal{F}(U), \rho_{UV}\}$ indexed by the open subsets containing x, with order relation induced by reverse inclusion, i.e. $U < V$ if $V \subset U$:

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

The image of $f \in \mathcal{F}(U)$ in $\mathcal{F}_x$ under the group homomorphism induced by the inclusion map $\mathcal{F}(U) \hookrightarrow \coprod_{U \ni x} \mathcal{F}(U)$ is denoted by f_x.

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Since the map of sheaves is a map of direct systems

$$\phi : \{(\mathcal{F}(U), \rho^{\mathcal{F}}_{UV})\} \to \{(\mathcal{G}(U), \rho^{\mathcal{G}}_{UV})\}$$

the map of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is the direct limit of the homomorphisms ϕ_U.

Exact sequence of sheaves

A sequence of sheaves $\mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is said to be exact if $\mathcal{F}'_x \xrightarrow{\phi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{F}''_x$ is an exact sequence of abelian groups for every $x \in X$.

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Exact sequence of differential forms

By Poincaré lemma we know that for every point x in a smooth manifold M there exists an open subset U containing x such that

$$\Omega^0(U) \xrightarrow{d_U} \Omega^1(U) \xrightarrow{d_U} \Omega^2(U) \xrightarrow{d_U} \cdots$$

is an exact sequence of abelian groups. Also, since $\mathbb{R}(U)$ is the group of closed 0-forms, for all $x \in M$ we have a long exact sequence at the level of stalks

$$0 \longrightarrow \mathbb{R}_x \hookrightarrow \Omega^0_x \xrightarrow{d_x} \Omega^1_x \xrightarrow{d_x} \Omega^2_x \xrightarrow{d_x} \cdots$$

Therefore, the sequence of sheaves of differential forms on a smooth manifold

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Induced map of cohomology

Induced map of cochains

If $\phi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves on X, then the *induced map on cochains* is defined as

$$\begin{aligned} \phi_* : \check{C}^k(\mathcal{U}, \mathcal{F}) &\to \check{C}^k(\mathcal{U}, \mathcal{G}) \\ (f_{i_0,i_1,\ldots,i_k}) &\mapsto (\phi_{U_{i_0,\ldots,i_k}}(f_{i_0,i_1,\ldots,i_k})) \end{aligned}$$

for any open covering $\mathcal{U}$ of X.

Lemma

If $0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is an exact sequence of sheaves over X, then the induced sequence of cochains for any open cover $\mathcal{U}$ of X $0 \longrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{C}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} \check{C}^k(\mathcal{U}, \mathcal{F}'')$ is also exact.

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Induced map of cohomology

We observe that the induced map of cochains sends cocycles to cocycles, and coboundaries to cobundaries.

Induced map of cohomology with respect to a cover

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves on X, then the *induced map of cohomology* is defined as

$$\begin{aligned} \Phi : \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) &\to \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{G}) \\ [f] &\mapsto [\phi_*(f)] \end{aligned}$$

for $f \in \check{Z}^k(\mathcal{U}, \mathcal{F})$.

In fact, we have a map of direct systems

$$\Phi : \{\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}), H^{\mathcal{F}}_{\mathcal{U}\mathcal{V}}\} \to \{\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{G}), H^{\mathcal{G}}_{\mathcal{U}\mathcal{V}}\}$$

Thus $\phi : \mathcal{F} \to \mathcal{G}$ induces a homomorphism at the level of cohomology

$$\underrightarrow{\Phi} : \check{\mathrm{H}}^k(X, \mathcal{F}) \to \check{\mathrm{H}}^k(X, \mathcal{G})$$

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Long exact sequence of cohomology

Serre's theorem

Let X be a paracompact Hausdorff space and

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves on X. Then there are connecting homomorphisms $\Delta : \check{\mathrm{H}}^k(X, \mathcal{F}'') \to \check{\mathrm{H}}^{k+1}(X, \mathcal{F}')$ for every $k \geq 0$ such that we have a long exact sequence of Čech cohomology groups

$$\cdots \xrightarrow{\Phi} \check{\mathrm{H}}^k(X, \mathcal{F}) \xrightarrow{\Psi} \check{\mathrm{H}}^k(X, \mathcal{F}'') \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(X, \mathcal{F}') \xrightarrow{\Phi} \cdots$$

Long exact sequence of cohomology

Given to us is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

Then by the preceding lemma, for any open cover $\mathcal{U}$ of X,

$$0 \longrightarrow \check{C}^k(\mathcal{U}, \mathcal{F}') \xrightarrow{\phi_*} \check{C}^k(\mathcal{U}, \mathcal{F}) \xrightarrow{\psi_*} \check{C}^k(\mathcal{U}, \mathcal{F}'')$$

is an exact sequence. However, if we replace $\check{C}^k(\mathcal{U}, \mathcal{F}'')$ by $\mathrm{im}\,\psi_* = I^k(\mathcal{U}, \mathcal{F}'')$, we get a short exact sequence of cochain complexes

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Then by the *zig-zag lemma* we get a long exact sequence in cohomology with respect to open cover $\mathcal{U}$

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where ∂ is the connecting homomorphism induced by the coboundary operator δ and

$$\mathcal{I}^k(\mathcal{U},\mathcal{F}'') = \frac{\ker\{\delta : I^k(\mathcal{U},\mathcal{F}'') \to I^{k+1}(\mathcal{U},\mathcal{F}'')\}}{\mathrm{im}\{\delta : I^{k-1}(\mathcal{U},\mathcal{F}'') \to I^k(\mathcal{U},\mathcal{F}'')\}}$$

Since direct limit is an *exact functor*, we get the long exact sequence in Čech cohomology

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Long exact sequence of cohomology

Now to obtain the desired long exact sequence of Čech cohomology, it's sufficient to show that $\mathcal{I}^k(X, \mathcal{F}'') \cong \check{\mathrm{H}}^k(X, \mathcal{F}'')$.

We observe that the inclusion map $I^k(\mathcal{U}, \mathcal{F}'') \hookrightarrow \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}'')$ induces an exact sequence of cochain complexes

$$0 \longrightarrow I^k(\mathcal{U}, \mathcal{F}'') \hookrightarrow \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}'') \longrightarrow Q^k(\mathcal{U}, \mathcal{F}'') \longrightarrow 0$$

where

$$Q^k(\mathcal{U}, \mathcal{F}'') := \frac{\check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}'')}{I^k(\mathcal{U}, \mathcal{F}'')}$$

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Long exact sequence of cohomology

We can now apply zig-zag lemma to get a long exact sequence in cohomology with respect to the open cover $\mathcal{U}$

$$\cdots \longrightarrow \check{\mathsf{H}}^k(\mathcal{U}, \mathcal{F}'') \longrightarrow \mathcal{Q}^k(\mathcal{U}, \mathcal{F}'') \xrightarrow{\partial} \mathcal{I}^{k+1}(\mathcal{U}, \mathcal{F}'') \longrightarrow \cdots$$

where ∂ is the connecting homomorphism induced by the coboundary operator δ and

$$\mathcal{Q}^k(\mathcal{U}, \mathcal{F}'') = \frac{\ker\{\delta : Q^k(\mathcal{U}, \mathcal{F}'') \to Q^{k+1}(\mathcal{U}, \mathcal{F}'')\}}{\mathrm{im}\{\delta : Q^{k-1}(\mathcal{U}, \mathcal{F}'') \to Q^k(\mathcal{U}, \mathcal{F}'')\}}$$

Since direct limit is an exact functor, we get the following long exact sequence in Čech cohomology

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Long exact sequence of cohomology

Now to obtain the desired isomorphism, it's sufficient to show that $\mathcal{Q}^k(X, \mathcal{F}'') = 0$. To prove this, we will use the fact that X is a paracompact Hausdorff space and ψ_x is surjective for all $x \in X$.

Let $\mathcal{U} = \{U_i\}_{i \in A}$ be an open cover of X, and $f = (f_{i_0, \ldots, i_k})$ be an element of $\check{C}^k(\mathcal{U}, \mathcal{F}'')$. Then there exists a refinement $\mathcal{V} = \{V_j\}_{j \in B}$ along with a refining map $r : B \to A$ such that $V_j \subset U_{r(j)}$ and $\widetilde{r}(f) \in I^k(\mathcal{V}, \mathcal{F}'')$, where $\widetilde{r}$ is the refining map at the level of cochains. Therefore $\mathcal{Q}^k(X, \mathcal{F}'') = 0$.

Since X is paracompact, without loss of generality, assume $\mathcal{U}$ to be locally finite. By *shrinking lemma* there exists a locally finite open covering $\mathcal{W} = \{W_i\}_{i \in A}$ of X such that $\overline{W_i} \subset U_i$ for each $i \in A$.

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Long exact sequence of cohomology

For every $x \in X$, we can choose an open neighborhood V_x of x such that

1. If $x \in U_i$ then $V_x \subset U_i$ for all such i's. If $x \in W_i$ then $V_x \subset W_i$ for all such i's.
2. If $V_x \cap W_i \neq \emptyset$ then $V_x \subset U_i$ for all such i's.
3. If $x \in U_{i_0,i_1,\ldots,i_k}$ then there exists a $h \in \mathcal{F}(V_x)$ such that

$$\psi_{V_x}(h) = \rho^{\mathcal{F}''}_{U_{i_0,\ldots,i_k},V_x}(f_{i_0,\ldots,i_k})$$

where by the first condition $V_x \subset U_{i_0,\ldots,i_k}$.

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Long exact sequence of cohomology

Choose a map $r : X \to A$ such that $x \in W_{r(x)}$. Then by the first condition, $V_x \subset W_{r(x)}$ and $\mathcal{V} = \{V_x\}_{x \in X}$ is a refinement of $\mathcal{U}$.
Now consider the map

$$\begin{aligned} \widetilde{r} : \check{C}^k(\mathcal{U}, \mathcal{F}'') &\to \check{C}^k(\mathcal{V}, \mathcal{F}'') \\ f = (f_{i_0,\ldots,i_k}) &\mapsto g = (g_{x_0,\ldots,x_k}) \end{aligned}$$

where

$$g_{x_0,\ldots,x_k} = \rho(f_{r(x_0),\ldots,r(x_k)})$$

and ρ is the group homomorphism for the sheaf $\mathcal{F}''$ corresponding to the nested open subsets $V_{x_0,\ldots,x_k} \subset U_{r(x_0),\ldots,r(x_k)}$.

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Long exact sequence of cohomology

It remains to show that $\widetilde{r}(f) \in I^k(\mathcal{V}, \mathcal{F}'') = \psi_*(\check{C}^k(\mathcal{V}, \mathcal{F}''))$, i.e. there exists $h \in \mathcal{F}(V_{x_0,x_1,\ldots,x_k})$ such that

$$\rho(f_{r(x_0),\ldots,r(x_k)}) = \psi_{V_{x_0,x_1,\ldots,x_k}}(h) \tag{1}$$

If $V_{x_0,\ldots,x_k} = \emptyset$ then there is nothing to prove. If not, then we have $V_{x_0} \cap V_{x_\ell} \neq \emptyset$ for all $0 \leq \ell \leq k$. Since $V_{x_\ell} \subset W_{r(x_\ell)}$ we have $V_{x_0} \cap W_{r(x_\ell)} \neq \emptyset$ for all $0 \leq \ell \leq k$, then by the second condition we have $V_{x_0} \subset U_{r(x_\ell)}$ for all $0 \leq \ell \leq k$. Hence, $x_0 \in U_{r(x_0),\ldots,r(x_k)}$ and we can use the third condition to conclude that there exists $h' \in \mathcal{F}(V_{x_0})$ such that

$$\psi_{V_{x_0}}(h') = \rho^{\mathcal{F}''}_{U_{r(x_0),\ldots,r(x_k)}, V_{x_0}}(f_{r(x_0),\ldots,r(x_k)})$$

Now let $h = \rho^{\mathcal{F}''}_{V_{x_0}, V_{x_0,x_1,\ldots,x_k}}(h')$ and use the fact that ψ commutes with ρ to get (1). Hence completing the proof.

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Long exact sequence induced by differential forms

Since manifolds are paracompact, the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{Z}^q \hookrightarrow \Omega^q \xrightarrow{\mathrm{d}} \mathcal{Z}^{q+1} \longrightarrow 0$$

for $q \geq 0$, with $\mathcal{Z}^0 = \underline{\mathbb{R}}$, induces the long exact sequence

$$\cdots \longrightarrow \check{\mathrm{H}}^k(M, \Omega^q) \longrightarrow \check{\mathrm{H}}^k(M, \mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M, \mathcal{Z}^q) \longrightarrow \cdots$$

Vanishing cohomology

Sheaf partition of unity

Let $\mathcal{F}$ be a sheaf of abelian groups over a paracompact Hausdorff space X. Given a locally finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, the *partition of unity of* $\mathcal{F}$ subordinate to the cover $\mathcal{U}$ is a family of sheaf maps $\{\eta_i : \mathcal{F} \to \mathcal{F}\}$ such that

1. $\operatorname{supp}(\eta_i) \subset U_i$ for each U_i
2. $\sum_{i\in I} \eta_i = \mathbb{1}_{\mathcal{F}}$ (the sum can be formed because $\mathcal{U}$ is locally finite)

where $\operatorname{supp}(\eta_i)$ is the closure of the set of those $x \in X$ for which $(\eta_i)_x : \mathcal{F}_x \to \mathcal{F}_x$ is not a zero map.

Since the multiplication by a continuous or differentiable globally defined function defines a sheaf map in a natural way. The sheaf Ω^q of smooth q-forms on a smooth manifold M admits a sheaf partition of unity.

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Theorem

Let $\mathcal{F}$ be a sheaf over a paracompact Hausdorff space X which admits partition of unity. Then $\check{\mathrm{H}}^k(X, \mathcal{F})$ vanishes for $k \geq 1$.

Since X is paracompact, every open cover of X has a locally finite refinement, it suffices to prove that $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) = 0$ for all $k \geq 1$ if $\mathcal{U} = \{U_i\}_{i \in I}$ is any locally finite open cover of X. For $k \geq 1$, we define the homomorphism

$$\lambda_k : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{C}}^{k-1}(\mathcal{U}, \mathcal{F})$$
$$(f_{i_0, i_1, \ldots, i_k}) \mapsto (h_{i_0, i_1, \ldots, i_{k-1}})$$

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Vanishing cohomology

Let $\delta_k : \check{\mathrm{C}}^k(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{C}}^{k+1}(\mathcal{U}, \mathcal{F})$ be the coboundary operator. Since the cocycles are skew-symmetric, for $f = (f_{i_0,\ldots,i_k}) \in \check{Z}^k(\mathcal{U}, \mathcal{F})$ we have

$$\delta_{k-1}\left(\lambda_k(f)\right) = f \quad \text{for } k \geq 1$$

Therefore, $f \in \check{B}^k(\mathcal{U}, \mathcal{F})$ and $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) = 0$ for all $k \geq 1$.

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$H^1_{dR}(M) \cong \check{H}^1(M, \mathbb{R})$

We have $\check{H}^0(M, \Omega^q) \cong \Omega^q(M)$ and $\check{H}^0(M, \mathcal{Z}^q) \cong \mathcal{Z}^q(M)$. Hence for any $q \geq 0$ we have the long exact sequence

$$0 \longrightarrow \mathcal{Z}^q(M) \hookrightarrow \Omega^q(M) \xrightarrow{d} \mathcal{Z}^{q+1}(M) \xrightarrow{\Delta} \check{H}^1(M, \mathcal{Z}^q) \longrightarrow 0 \longrightarrow \check{H}^1(M, \mathcal{Z}^{q+1})$$

$$\downarrow \Delta$$

$$\cdots \longleftarrow 0 \longleftarrow \check{H}^3(M, \mathcal{Z}^q) \xleftarrow{\Delta} \check{H}^2(M, \mathcal{Z}^{q+1}) \longleftarrow 0 \longleftarrow \check{H}^2(M, \mathcal{Z}^q)$$

Now consider the following part of the above sequence

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Since this sequence is exact, the map $\Delta : \mathcal{Z}^{q+1}(M) \to \check{H}^1(M, \mathcal{Z}^q)$ is a surjective group homomorphism and

$$\mathrm{im}\{d : \Omega^q(M) \to \mathcal{Z}^{q+1}(M)\} = \ker(\Delta)$$

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$H^1_{dR}(M) \cong \check{\mathrm{H}}^1(M, \mathbb{R})$

We have $\check{\mathrm{H}}^0(M, \Omega^q) \cong \Omega^q(M)$ and $\check{\mathrm{H}}^0(M, \mathcal{Z}^q) \cong \mathcal{Z}^q(M)$. Hence for any $q \geq 0$ we have the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathcal{Z}^q(M) \hookrightarrow \Omega^q(M) \xrightarrow{\mathrm{d}} \mathcal{Z}^{q+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^q) \longrightarrow 0 \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{Z}^{q+1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \Delta \\ \cdots \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^3(M, \mathcal{Z}^q) \xleftarrow{\Delta} \check{\mathrm{H}}^2(M, \mathcal{Z}^{q+1}) \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^2(M, \mathcal{Z}^q) \end{array}$$

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^q(M) \hookrightarrow \Omega^q(M) \xrightarrow{\mathrm{d}} \mathcal{Z}^{q+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^q) \longrightarrow 0$$

Since this sequence is exact, the map $\Delta : \mathcal{Z}^{q+1}(M) \to \check{\mathrm{H}}^1(M, \mathcal{Z}^q)$ is a surjective group homomorphism and

$$\mathrm{im}\{\mathrm{d} : \Omega^q(M) \to \mathcal{Z}^{q+1}(M)\} = \ker(\Delta)$$

$H^1_{dR}(M) \cong \check{\mathrm{H}}^1(M, \underline{\mathbb{R}})$

By the *first isomorphism theorem* we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^q) \cong \frac{\mathcal{Z}^{q+1}(M)}{\ker(\Delta)} \quad \text{for all } q \geq 0$$

Since $\mathrm{im}\{\mathrm{d} : \Omega^q(M) \to \mathcal{Z}^{q+1}(M)\} = \mathrm{im}\{\mathrm{d} : \Omega^q(M) \to \Omega^{q+1}(M)\} = \mathcal{B}^{q+1}(M)$, we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^q) \cong H^{q+1}_{dR}(M) \quad \text{for all } q \geq 0 \tag{2}$$

Note that $\mathcal{Z}^0 = \underline{\mathbb{R}}$, thus we have

$$\boxed{\check{\mathrm{H}}^1(M, \underline{\mathbb{R}}) \cong H^1_{dR}(M)}$$

$H^1_{dR}(M) \cong \check{\mathrm{H}}^1(M, \mathbb{R})$

By the *first isomorphism theorem* we get

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$H^k_{dR}(M) \cong \check{\mathrm{H}}^k(M, \underline{\mathbb{R}})$ for $k \geq 2$

Consider the remaining parts of the long exact sequence, i.e. for $k \geq 1$ and $q \geq 0$ we have

$$0 \longrightarrow \check{\mathrm{H}}^k(M, \mathcal{Z}^{q+1}) \xrightarrow{\Delta} \check{\mathrm{H}}^{k+1}(M, \mathcal{Z}^q) \longrightarrow 0$$

Here, the group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{\mathrm{H}}^{k+1}(M, \mathcal{Z}^q) \cong \check{\mathrm{H}}^k(M, \mathcal{Z}^{q+1}) \quad \text{for all } k \geq 1, q \geq 0 \tag{3}$$

Again substituting $\mathcal{Z}^0 = \underline{\mathbb{R}}$ and restricting our attention to $k \geq 2$, we apply (3) recursively to get

$$\check{\mathrm{H}}^k(M, \underline{\mathbb{R}}) \cong \check{\mathrm{H}}^{k-1}(M, \mathcal{Z}^1) \cong \check{\mathrm{H}}^{k-2}(M, \mathcal{Z}^2) \cdots \cong \check{\mathrm{H}}^1(M, \mathcal{Z}^{k-1})$$

Then using (2) we get

$$\boxed{\check{\mathrm{H}}^k(M, \underline{\mathbb{R}}) \cong H^k_{dR}(M)} \quad \text{for all } k \geq 2$$

$H^k_{dR}(M) \cong \check{H}^k(M, \mathbb{R})$ for $k \geq 2$

Consider the remaining parts of the long exact sequence, i.e. for $k \geq 1$ and $q \geq 0$ we have

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