For what algebraic curves do rational points form a group?

[+11] [2] mick

[2012-10-31 15:55:15]

[group-theory number-theory algebraic-geometry rational-numbers]

[https://math.stackexchange.com/questions/226127/for-what-algebraic-curves-do-rational-points-form-a-group]

For what real algebraic curves do rational points form a group ? How does this relate to Jacobian Varieties ?

(2) The *set* of rational points always has a group structure because any non-empty set has a group structure. But some group structures are more natural than others – for example, it is well-known that the rational points of an elliptic curve form a group... - **Zhen Lin**

The question should be to classify 1-dimensional algebraic groups (over $\mathbb{Q}$). In the projective case, I think that any 1-dimensional abelian variety has genus 1, so it is an elliptic curve (see answer below). For the affine case, you have $\mathbb{G}_a, \mathbb{G}_m$; that's all over an algebraically closed field, but over $\mathbb{Q}$ you have many non-split tori (see the conics discussed below). - **Watson**

[+11] [2012-11-01 19:02:58] user23365 [✔ACCEPTED]

There are lots of curves whose set of rational points have a group structure with a geometric interpretation - which is the question that should have been asked. In addition to the projective group laws on elliptic curves there are group laws on many curves of genus 0, in particular Pell conics (degenerate or not; actually any conic will do, but the group structure is "canonical" only if you have a canonical rational point acting as an identity). And of course you can always get group structure on higher dimensional varieties such as products of Pell conics.

Geometrically, these group laws com from "generalized Jacobians". These are nicely described in the thesis of Isabelle Dechene, which can be found here [1]. On a much more elementary level I will try to put a few things together here [2].

[1] http://www.math.mcgill.ca/darmon/theses/dechene/thesis.pdf
[2] http://www.rzuser.uni-heidelberg.de/~hb3/pell.html

(1) Is is true that the geometric structures in this case correspond to (one-dimensional norm) tori ? - **user18119**
(2) @franz lemmermeyer, this book you linked is very exciting, well be looking forward to seeing it develop! - **sperners lemma**

1

[+5] [2012-10-31 17:32:02] user29743

The question is a little imprecise. For any algebraic curve X the set of rational points $X(\mathbb{Q})$ can be given a group structure, because it's just a set.

If you want X to be a projective algebraic group over $\mathbb{Q}$, which means that the identity, inverse, and multiplication map are defined over $\mathbb{Q}$, then its genus had better be one (this has nothing to do with the arithmetic and is already true for projective curves over an algebrically closed field - over $\mathbb{C}$ you can see this topologically since π_1 had better be abelian, which eliminates higher genus curves, and $\mathbb{P}^1$ is just S^2).

For any Y of genus bigger than zero, there is an embedding of Y into another variety J, its Jacobian, which is an algebraic group. The embedding is arithmetic in the sense that if Y is defined over a field, or for that matter a ring, then so is J and so is the embedding, with the caveat that $Y(R)$ needs to be non-empty for the embedding to be defined over R. The Jacobian won't be a curve; rather, the dimension of J is the genus of Y.

2
