Deuring Correspondence and Public Key Cryptography

Oral Comprehensive Examination

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December 07, 2023

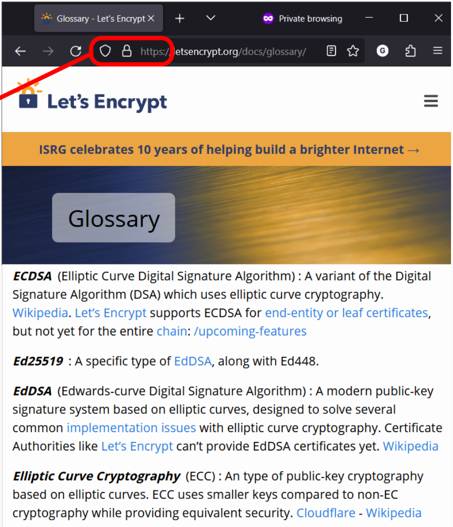
The University of Arizona

Table of contents

Appetizer: Discrete logarithm

Public Key Cryptography

You are securely connected to this website. **Key exchange** allows two parties to agree on a common secret using only publicly exchanged information. **Digital signature** allows parties to authenticate themselves.



ECDSA (Elliptic Curve Digital Signature Algorithm) : A variant of the Digital Signature Algorithm (DSA) which uses elliptic curve cryptography. Wikipedia. Let's Encrypt supports ECDSA for end-entity or leaf certificates, but not yet for the entire chain: /upcoming-features

Ed25519 : A specific type of EdDSA, along with Ed448.

EdDSA (Edwards-curve Digital Signature Algorithm) : A modern public-key signature system based on elliptic curves, designed to solve several common implementation issues with elliptic curve cryptography. Certificate Authorities like Let's Encrypt can't provide EdDSA certificates yet. Wikipedia

Elliptic Curve Cryptography (ECC) : An type of public-key cryptography based on elliptic curves. ECC uses smaller keys compared to non-EC cryptography while providing equivalent security. Cloudflare - Wikipedia

Public Key Cryptography

Let's Encrypt is the world's largest certificate authority with over 2.53 billion certificates issued.

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Public Key Cryptography

Let's Encrypt

ISRG celebrates 10 years of helping build a brighter Internet →

Glossary

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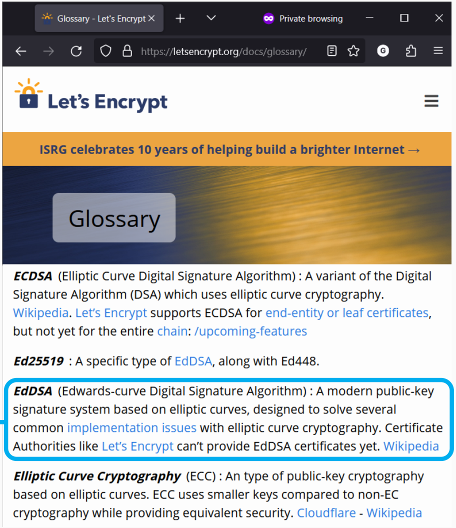
For 128-bit security, DSA (based on DLP) needs 4096-bit keys, but ECDSA (based on ECDLP) only needs 256-bit key.

Public Key Cryptography



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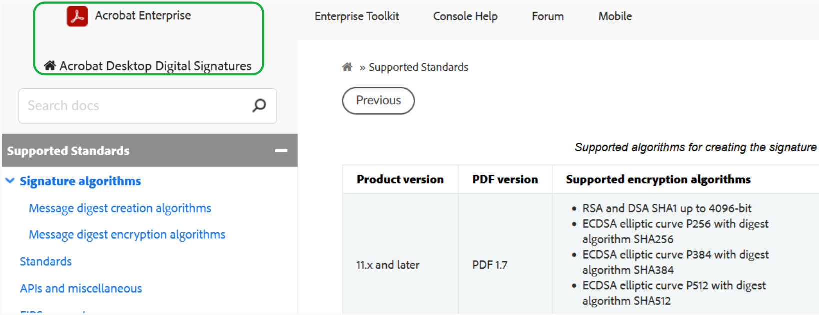
EdDSA is not ECDSA over a different curve. Rather, it is a *Schnorr signature* implemented for the Ed25519 Edwards curve.

Preparation for Schnorr

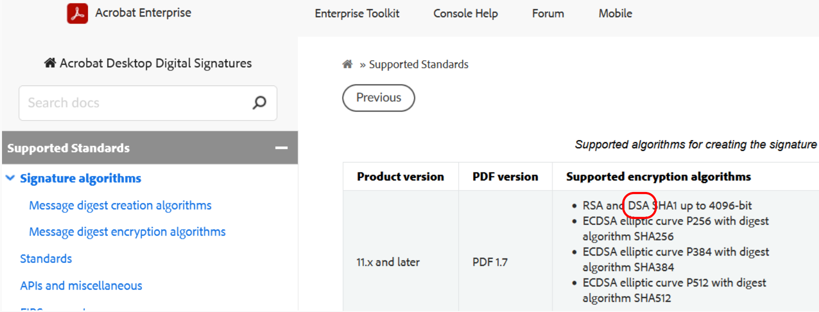


Supported algorithms for creating the signature

Product version	PDF version	Supported encryption algorithms
11.x and later	PDF 1.7	• RSA and DSA SHA1 up to 4096-bit • ECDSA elliptic curve P256 with digest algorithm SHA256 • ECDSA elliptic curve P384 with digest algorithm SHA384 • ECDSA elliptic curve P512 with digest algorithm SHA512

Most of us have used Adobe Acrobat Sign to digitally sign PDF documents.

Preparation for Schnorr



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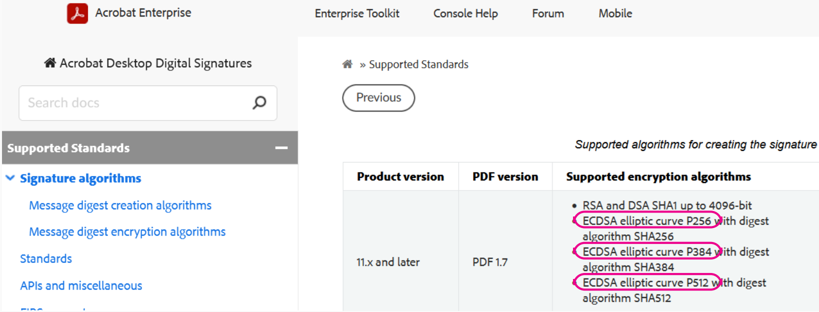
Let $(\mathbb{G}, \cdot)$ be a finite abelian group of prime order ℓ. The *discrete logarithm problem* (DLP) in $\mathbb{G}$ is: given $\langle g \rangle = \mathbb{G}$ and $h \in \mathbb{G}$, find an integer $k \in \{0, \ldots, \ell - 1\}$ such that $g^k = h$.

Number of operations for generic $\mathbb{G}$ is $\sqrt{\#\mathbb{G}}$.

Preparation for Schnorr



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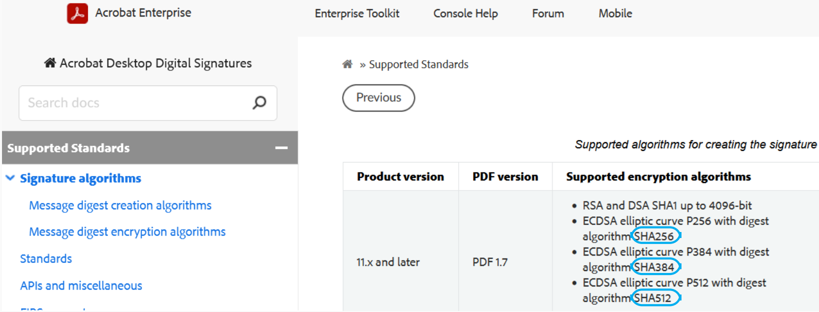
Let p be a prime larger than 3 and $q = p^n$ for $n > 0$. An elliptic curve E over finite field $\mathbb{F}_q$ can be written as $E : y^2 = x^3 + ax + b$ where $a, b \in \mathbb{F}_q$ and $4a^3 + 27b^2 \neq 0$, along with an extra point $\mathcal{O}_E$. Points on E form a group with $\mathcal{O}_E$ as the neutral element. P be a point on E of prime order ℓ, then $\mathbb{G} = \langle P \rangle$ with the exponentiation replaced with scalar point multiplication, we get ECDLP.

Preparation for Schnorr



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A *cryptographic hash function* H takes arbitrary length bit strings as input and produces a fixed-length bit string as output, such that it is preimage resistant (can't find input of given output), second preimage resistant (can't find a different input leading to given output), and collision resistant (can't find two inputs with same output).

Schnorr signature, Step 1: Σ-protocol

Let $\mathbb{G} = \langle g \rangle$ where $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ is an element of prime order ℓ.

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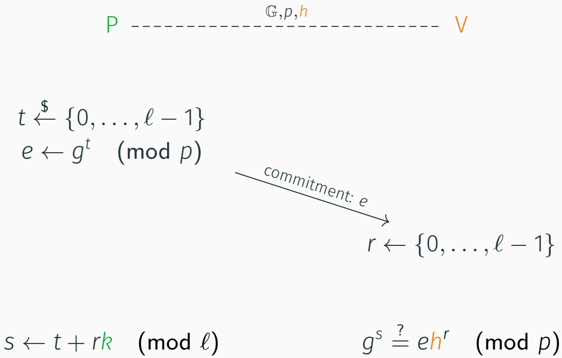

$$t \xleftarrow{\$} \{0, \dots, \ell - 1\}$$
$$e \leftarrow g^t \pmod p$$

$$r \leftarrow \{0, \dots, \ell - 1\}$$

$$s \leftarrow t + rk \pmod \ell$$

$$g^s \overset{?}{=} eh^r \pmod p$$

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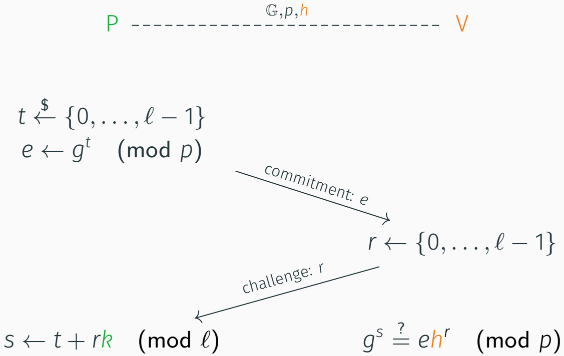

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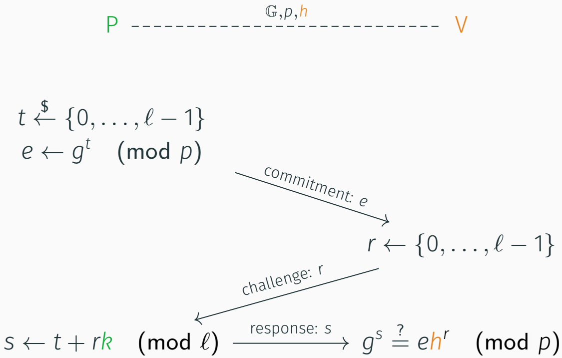

P $\mathbb{G},p,h$ V

$t \xleftarrow{\$} \{0, \dots, \ell - 1\}$

$e \leftarrow g^t \pmod{p}$

commitment: e

$r \leftarrow \{0, \dots, \ell - 1\}$

challenge: r

$s \leftarrow t + rk \pmod{\ell} \xrightarrow{\text{response: } s} g^s \overset{?}{=} eh^r \pmod{p}$

Schnorr signature, Step 2: Fiat-Shamir transformation

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Signing $(\mathbb{G}, g, k, \mathrm{H}, m)$

1. $t \xleftarrow{\$} \{1, \ldots, \ell - 1\}$
2. $e \leftarrow g^t \pmod{p}$
3. $r \leftarrow \mathrm{H}(m \| e)$
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Verification $(\mathbb{G}, g, h, \mathrm{H}, m, \sigma)$

1. $r \leftarrow \mathrm{H}(m \| e)$
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Secure?

On Dec 20, 2016, NIST initiated the process.

Neal Koblitz | University of Washington
Alfred Menezes | University of Waterloo

In August 2015, the NSA released a major policy statement on the need for postquantum cryptography. Certain peculiarities in its wording and timing have puzzled many people and given rise to speculation concerning the NSA, elliptic curve cryptography, and quantum-safe cryptography. Of the various theories that have been proposed, some seem more plausible than others, but a definitive explanation is elusive.

> "It is a riddle wrapped in a mystery inside an enigma; but perhaps there is a key." —Winston Churchill, 1939 (in reference to the Soviet Union)

In August 2015, the US government's NSA released a major policy statement on the need to develop standards for postquantum cryptography (PQC).[1] The NSA, like many other organizations, believes that the time is right to make a major push to design public-key cryptographic protocols whose security depends on hard problems that can't be solved efficiently by a quantum computer. Ever since Peter Shor's pioneering work more than 20 years ago,[2] it has been known that both the integer factorization problem, upon which RSA is based, and the elliptic curve discrete logarithm problem (ECDLP), upon which elliptic curve cryptography (ECC) is based, can be solved in polynomial time by a quantum computer.

The NSA announcement will give a tremendous boost to efforts to develop, standardize, and commercialize quantum-safe cryptography. While standards for new postquantum algorithms are several years away, in the immediate future the NSA is encouraging vendors to add quantum resistance to existing protocols by means of conventional symmetric-key tools such as the Advanced Encryption Standard (AES). Given the NSA's strong interest in PQC, the demand for quantum-safe cryptographic solutions by governments and industry will likely grow dramatically in the coming years.

Most of the NSA statement was unexceptionable. However, one passage was puzzling and unexpected:[1]

> For those partners and vendors that have not yet made the transition to Suite B algorithms, we recommend not making a significant expenditure to do so at this point but instead to prepare for the upcoming quantum resistant algorithm transition.… Unfortunately, the growth of elliptic curve use has bumped up against the fact of continued progress in the research on quantum computing, necessitating a re-evaluation of our cryptographic strategy.

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In 1994, quantum algorithm for solving the DLP. Unfortunately, that is the hard-problem used by state-of-the-art digital signatures.

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November/December 2016 Copublished by the IEEE Computer and Reliability Societies 1540-7993/16/$33.00 © 2016 IEEE

Entrée: Supersingular isogeny graph

Supersingular elliptic curves

Recall, p is a prime larger than 3 and $q = p^n$ for $n > 0$. For $E/\mathbb{F}_q$ we have $\#E(\mathbb{F}_q) = q + 1 - t$, where $|t| \leq 2\sqrt{q}$.

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Example

$E_1 : y^2 = x^3 + x$ over $\mathbb{F}_{23}$ is a supersingular elliptic curve because $\#E(\mathbb{F}_{23}) = 24$ and $t = 0$.

Isogeny

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Example

For $E_1 : y^2 = x^3 + x$ and $E_2 : y^2 = x^3 + 19x$ over $\mathbb{F}_{23}$ we have

$$\begin{aligned} \phi &: E_1 \to E_2 \\ (x, y) &\mapsto \left(\frac{x^2+1}{x}, \frac{x^2y-y}{x^2}\right) \end{aligned}$$

Degree of isogeny

Degree of (separable) isogeny

The degree of a (separable) isogeny $\phi : E \rightarrow E'$ over $\mathbb{F}_q$, is the number of points on E, taken over any extension field of $\mathbb{F}_q$, mapping to $\mathcal{O}_{E'}$.

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Degree is multiplicative: $\deg(\phi \circ \psi) = \deg(\phi)\deg(\psi)$

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Example

For $E_1 : y^2 = x^3 + x$ and $E_3 : y^2 = x^3 + 2x$ over $\mathbb{F}_{23}$ we have

$$\begin{aligned} \tau : E_1 &\to E_3 \\ (x, y) &\mapsto (-5x, -6y) \end{aligned}$$

Isomorphism class label

j-invariant

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If E is supersingular, then we can replace “algebraic closure of $\mathbb{F}_q$” with $\mathbb{F}_{p^2}$.

Supersingular isomorphism classes

The number of supersingular isomorphism classes over an algebraic closure of $\mathbb{F}_p$, with representative curves defined over $\mathbb{F}_{p^2}$, is $S_p := \left\lfloor \frac{p}{12} \right\rfloor + \epsilon$ where $\epsilon \in \{0, 1, 2\}$.

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Example

$S_{23} = 3$ with the classes represented by the j-invariants $0, 3, 19$.

Supersingular ℓ-isogeny graph

Let ℓ be a prime different from p. The supersingular ℓ-isogeny graph over an algebraic closure of $\mathbb{F}_q$ is the directed multigraph $G_\ell(p)$ whose vertices belong to the set of isomorphism classes of supersingular elliptic curves $\{j(E_1), \ldots, j(E_s)\}$ with $s = S_p$ and $E_i/\mathbb{F}_{p^2}$; there is a directed edge $[E_i, E_{i'}]$ for each equivalence class (same kernel) of ℓ-isogenies from E_i to $E_{i'}$.

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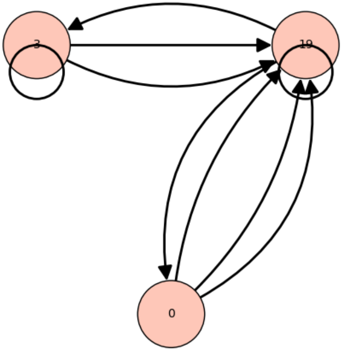
Example: $G_2(23)$



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Endomorphism ring

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The endomorphism ring of E, $\mathrm{End}_{\mathbb{F}_q}(E)$, is the set of $\mathbb{F}_q$-isogenies from E to itself, together with the zero map $[0] : E \to E$ given $[0](P) = \mathcal{O}_E$. In particular, for isogenies over an algebraic closure of $\mathbb{F}_q$, we write $\mathrm{End}(E)$.

Example

For $E_1 : y^2 = x^3 + x$ over $\mathbb{F}_{23}$, we have

$$\mathrm{End}(E_1) = \mathbb{Z}\,\mathrm{id} + \mathbb{Z}\iota + \mathbb{Z}\frac{\iota + \pi}{2} + \mathbb{Z}\frac{\mathrm{id} + \iota \circ \pi}{2}$$

where $\pi, \iota \in \mathrm{End}(E_1)$ such that $\pi(x, y) = (x^{23}, y^{23})$ and $\iota(x, y) = (-x, \alpha y)$ is an isomorphism over $\mathbb{F}_{23^2} = \mathbb{F}_{23}(\alpha)$ with $\alpha^2 + 1 = 0$.

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Maximal orders

Deuring correspondence - I

E is a supersingular elliptic curve over $\mathbb{F}_q$ if and only if $\mathbf{End}(E)$ is isomorphic to a maximal order in the quaternion algebra $B_{p,\infty}$.

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Quaternion algebra

A quaternion algebra over $\mathbb{Q}$ is of the form $\mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle = \mathbb{Q} + \mathbb{Q}\mathbf{i} + \mathbb{Q}\mathbf{j} + \mathbb{Q}\mathbf{ij}$, where $\mathbf{i}^2, \mathbf{j}^2 \in \mathbb{Q}^\times$, and $\mathbf{ij} = -\mathbf{ji}$. In particular, we have

$$B_{p,\infty} = \begin{cases} \mathbf{i}^2 = -1, \mathbf{j}^2 = -1 & \text{if } p = 2 \\ \mathbf{i}^2 = -1, \mathbf{j}^2 = -p & \text{if } p \equiv 3 \pmod 4 \\ \mathbf{i}^2 = -2, \mathbf{j}^2 = -p & \text{if } p \equiv 5 \pmod 8 \\ \mathbf{i}^2 = -\ell, \mathbf{j}^2 = -p & \text{if } p \equiv 1 \pmod 8 \end{cases}$$

where $\ell \equiv 3 \pmod 4$ is a prime quadratic non-residue mod p.

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Quaternion (maximal) order

$O \subseteq \mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$ is called an *order* if O is a ring whose elements are integral, $\mathbb{Z} \subseteq O$, and contains a basis for $\mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$ as $\mathbb{Q}$-vector space. Moreover, an order $O \subsetneq B$ is called *maximal* if it is not properly contained in another order.

Maximal orders

Deuring correspondence - I

E is a supersingular elliptic curve over $\mathbb{F}_q$ if and only if $\mathbf{End}(E)$ is isomorphic to a maximal order in the quaternion algebra $B_{p,\infty}$.

Example

In $B_{23,\infty} = \langle \mathbf{i}, \mathbf{j} \mid \mathbf{i}^2 = -1, \mathbf{j}^2 = -23, \mathbf{ij} = -\mathbf{ji} \rangle$, two examples of maximal orders are

$$O_1 = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\frac{\mathbf{i}+\mathbf{j}}{2} + \mathbb{Z}\frac{1+\mathbf{ij}}{2}; \text{ and}$$
$$O_2 = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\frac{1+\mathbf{j}}{2} + \mathbb{Z}\frac{\mathbf{i}(1+\mathbf{j})}{2}$$

Note that O_1 is isomorphic to $\mathbf{End}(E_1)$ we saw above.

Maximal orders

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Mestre-Oesterle-Ribet

$$\left\{\begin{array}{l}\text{isomorphism classes}\\ \text{of supersingular}\\ \text{elliptic curves over } \overline{\mathbb{F}}_p\end{array}\right\}\Big/\mathbf{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longleftrightarrow \left\{\begin{array}{l}\text{maximal orders}\\ \text{of } B_{p,\infty}\end{array}\right\}/\cong$$

That is, there is one-to-one correspondence if $j(E) \in \mathbb{F}_p$ and two-to-one correspondence if $j(E) \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$.

Left ideals

Deuring correspondence - II

Fix, E, a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ with $\mathrm{End}(E) \cong O \subseteq B_{p,\infty}$. There is a bijection between isomorphism classes over $\overline{\mathbb{F}}_p$ and the left class set $\mathrm{Cls}_{\mathrm{L}}(O)$.

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Quaternion (left) O-ideal

$I \subseteq \mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$ is called an *ideal* if I is a $\mathbb{Z}$-module that contains a basis for $\mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$ as $\mathbb{Q}$-vector space. Furthermore, given an order O of $\mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$, I is called a *left O-ideal* if $\alpha I \subseteq I$ for all $\alpha \in O$.

Left ideals

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Left-ideal class set

We say ideals I, J are in the *same left class*, $I \sim_{\mathrm{L}} J$, if there exists $\alpha \in B^\times$ such that $I\alpha = J$. Furthermore, the left equivalence class is denoted by $[I]_{\mathrm{L}}$. In particular, we have

$$\mathrm{Cls}_{\mathrm{L}}(O) := \{[I]_{\mathrm{L}} \mid I \text{ is an } \textit{invertible} \text{ left } O\text{-ideal}\}$$

$\mathrm{Cls}_{\mathrm{L}}(O)$ has has the structure of a *pointed set* with distinguished element $[O]_{\mathrm{L}} \in \mathrm{Cls}_{\mathrm{L}}(O)$.

Left ideals

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Example

Let $O_2 \subset B_{23,\infty}$ as above. Then we have $\mathrm{Cls}_{\mathrm{L}}(O_2) = \{[I_1]_{\mathrm{L}}, [I_2]_{\mathrm{L}}, [I_3]_{\mathrm{L}}\}$ with

$$\begin{aligned} I_1 &= 2\mathbb{Z}(1+\mathbf{j}) + 2\mathbb{Z}\mathbf{i}(1+\mathbf{j}) + 4\mathbb{Z}\mathbf{j} + 4\mathbb{Z}\mathbf{ij}, \\ I_2 &= 2\mathbb{Z}(1+3\mathbf{j}) + 2\mathbb{Z}\mathbf{i}(1+3\mathbf{j}) + 8\mathbb{Z}\mathbf{j} + 8\mathbb{Z}\mathbf{ij}, \text{ and} \\ I_3 &= 2\mathbb{Z}(1+3\mathbf{j}+4\mathbf{ij}) + 2\mathbb{Z}(\mathbf{i}+4\mathbf{j}+3\mathbf{ij}) + 16\mathbb{Z}\mathbf{j} + 16\mathbb{Z}\mathbf{ij} \end{aligned}$$

Here $[I_1]_{\mathrm{L}}$, $[I_2]_{\mathrm{L}}$, and $[I_3]_{\mathrm{L}}$ correspond to the isomorphism classes of supersingular curves represented by j-invariants 3, 19, and 0, respectively.

Left ideals

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Waterhouse

$E[I] := \{P \in E(\overline{\mathbb{F}}_p) \mid \phi(P) = 0 \ \forall \ \text{separable } \phi \in I\}$, where I is a nonzero left $\mathbf{End}(E)$-ideal. $\phi_I : E \to E/E[I]$ with $\mathbf{deg}(\phi_I) = \#E[I]$.

$I(H) := \{\phi \in \mathbf{End}(E) \mid \phi(P) = 0 \text{ for all } P \in H\}$, where $H \leq E(\overline{\mathbb{F}}_p)$ is finite. If $\phi : E \to E'$ an isogeny, then $I_\phi := I(\mathbf{ker}(\phi))$ a left $\mathbf{End}(E)$-ideal and right $\mathbf{End}(E')$-ideal (connecting ideal).

Left ideals

Deuring correspondence - II

Fix, E, a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ with $\mathrm{End}(E) \cong \mathcal{O} \subseteq B_{p,\infty}$. There is a bijection between isomorphism classes over $\overline{\mathbb{F}}_p$ and the left class set $\mathrm{Cls}_{\mathrm{L}}(\mathcal{O})$.

Waterhouse

- $E[I(H)] = H$ and $I(E[I]) = I$ (overloaded notation).
- If $I \sim_{\mathrm{L}} J$ then $E/E[I] \cong E/E[J]$.
- $\phi_{I.J} = \tau_J \circ \phi_I$ and $I_{\tau \circ \phi} = I_\phi \cdot I_\tau$
- $\phi_{\overline{I}} = \widehat{\phi_I}$ (dual isogeny) and $I_{\widehat{\phi}} = \overline{I_\phi}$
- $\mathrm{deg}(\phi_I) = \mathrm{nrd}(I)$ and $\mathrm{nrd}(I_\phi) = \mathrm{deg}(\phi)$

Eichler orders

Deuring correspondence - III

(E, C) is a pair of supersingular elliptic curve over $\overline{\mathbb{F}}_p$ and cyclic subgroup of order M with $\mathbf{gcd}(p, M) = 1$ iff $\mathrm{End}(E, C) \cong O(M) \subseteq B_{p,\infty}$ an Eichler order of level M.

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Eichler order

An *Eichler order* $O \subset B$ is the intersection of two (not necessarily distinct) maximal orders. Therefore, maximal orders are also Eichler orders.

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Level of an Eichler order

The level of an Eichler order O, is defined as the ratio of the reduced discriminant of order O and the discriminant of the quaternion algebra $B = \mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$.

$$\mathrm{lev}(O) = \frac{\mathrm{discrd}(O)}{\mathrm{disc}(B)}$$

From the definition of (reduced) discriminants it follows that maximal orders are Eichler orders of level 1.

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Example

$O_3 = O_1 \cap O_2 = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\mathbf{j} + \mathbb{Z}\frac{1 + \mathbf{i} + \mathbf{j} + \mathbf{ij}}{2}$ is an Eichler order of level 2, because $\mathrm{discrd}(O_3) = 26$ and $\mathrm{disc}(B_{23,\infty}) = 23$. Therefore, if $\phi \in \mathrm{End}(E, C) \cong O_3$ then $\phi \in \mathrm{End}(E)$ such that $\phi(C) = C$ with $\#C = 2$.

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Kohel

Fix a base point (E_0, C_0), where $C_0 \leq E(\overline{\mathbb{F}}_p)$ is a cyclic subgroup of order M. Then $\text{End}(E_0, C_0)$, the subring of $\text{End}(E_0)$ that maps C_0 to itself, is an Eichler order of level M and reduced discriminant pM.

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Kohel

Let $\mathcal{S}_M$ be the category of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ equipped with a cyclic M-isogeny (under isogenies identifying the cyclic subgroups).
Let $\mathcal{I}_M$ be the category of left $\mathrm{End}(E_0, C_0)$-ideals (under module homomorphisms).

Then the functor $\mathrm{Hom}(-, (E_0, C_0))$ from $\mathcal{S}_M$ to $\mathcal{I}_M$ is an equivalence of categories.

Spectral graph theory

Deuring correspondence lets us use the relationship between quaternion algebras and modular forms to study the eigenvalues of the adjacency matrix of $G_\ell(p)$.

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1. $G_\ell(p)$ is connected with diameter $O(\log p)$, where the constant in the bound is independent of ℓ. That is, the largest number of vertices which must be traversed in order to travel from one vertex to another when paths which backtrack, detour, or loop are excluded from consideration is $O(\log p)$.

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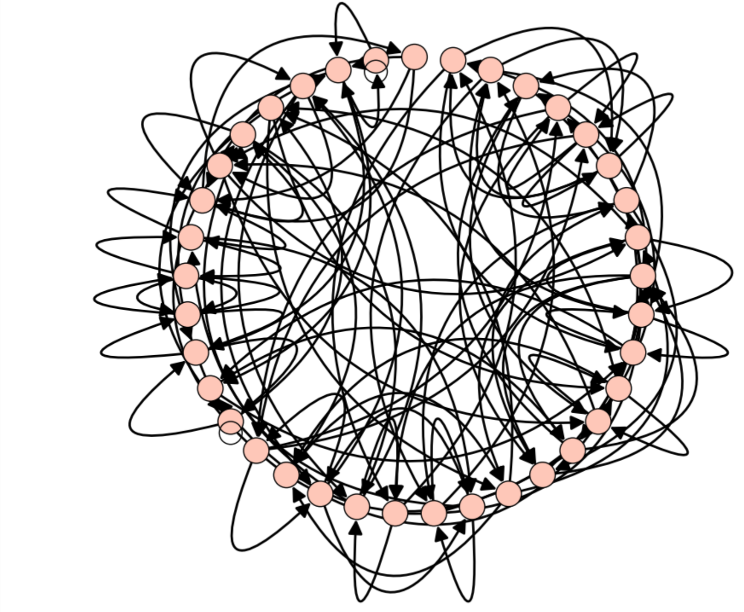
2. $G_\ell(p)$ is an *expander graph*, i.e. simultaneously sparse and highly connected. Therefore, the natural random walk on $G_\ell(p)$ converges to its limiting distribution as rapidly as possible.

$G_2(431)$ with 37 vertices and diameter 7



Problems about supersingular elliptic curves

Difficult problems

1. Given $E/\mathbb{F}_{p^2}$, find a maximal order $O \subseteq B_{p,\infty}$ such that $O \cong \mathrm{End}(E)$.

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3. Given maximal orders $O, O' \subseteq B_{p,\infty}$ find an ideal I that is left O-ideal and right O'-ideal.
4. Given p, determine (all) supersingular j-invariants in $\mathbb{F}_{p^2}$.

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Equivalent and quantum-safe

- All three problems are known to be equivalent.

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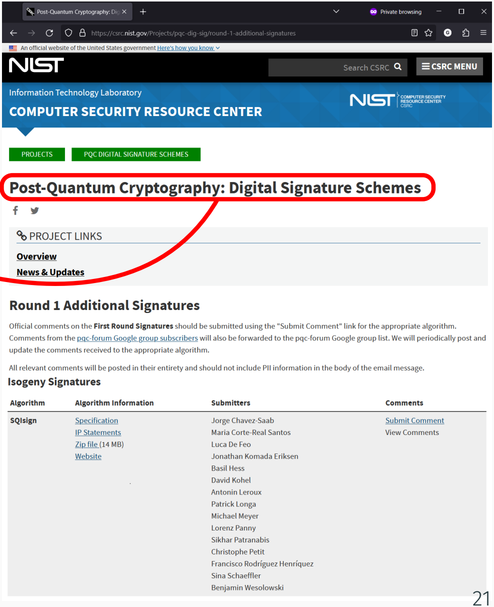
Equivalent and quantum-safe

- All three problems are known to be equivalent.
- The fact that $\mathrm{End}(E)$ is non-commutative makes these problems resistant to known quantum algorithms.
- We can rewrite these problems in terms of cyclic M-isogenies and Eichler orders of level M. For SQIsign, we assume that given $E/\mathbb{F}_{p^2}$ it is *difficult* to find a (non-trivial) cyclic endomorphism of E of smooth degree.

Dessert: Quantum-safe signature

NIST list

In 2022, NIST selected two lattice-based signatures (*CRYSTALS-Dilithium* and *FALCON*) and one hash-based signature (*SPHINCS+*)



Post-Quantum Cryptography: Digital Signature Schemes

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Round 1 Additional Signatures

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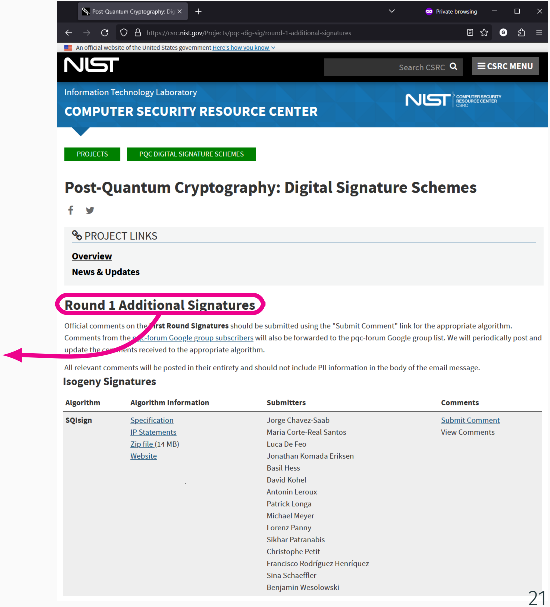
All relevant comments will be posted in their entirety and should not include PII information in the body of the email message.

Isogeny Signatures

Algorithm	Algorithm Information	Submitters	Comments
SQIsign	Specification IP Statements Zip file (14 MB) Website	Jorge Chavez-Saab Maria Corte-Real Santos Luca De Feo Jonathan Komada Eriksen Basil Hess David Kohel Antonin Leroux Patrick Longa Michael Meyer Lorenz Panny Sikhar Patranabis Christophe Petit Francisco Rodríguez Henríquez Sina Schaeffler Benjamin Wesolowski	Submit Comment View Comments

NIST list

Signature schemes with short signatures and fast verification; not based on structured lattices.



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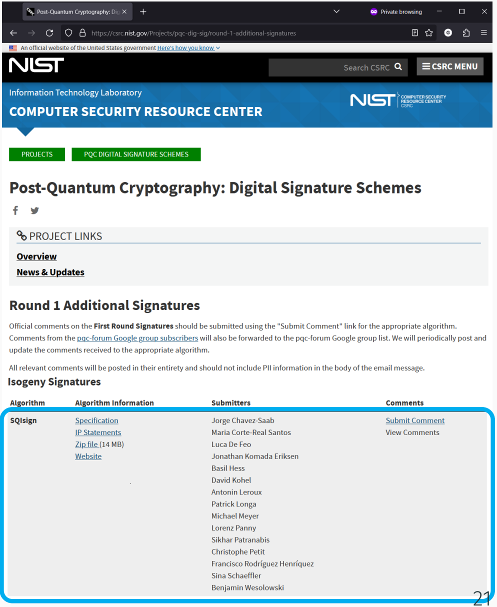
Only submission based on isogeny; shortest signatures; fast verification; complex signing procedure



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Preparation for SQIsign

Let λ be the security parameter.

- Fix a prime $p \equiv 3 \pmod 4$ with $\log_2(p) \approx 2\lambda.$ such that the $N2^f$-torsion subgroup is defined over a small extension of $\mathbb{F}_{p^2}$ for smooth number $N \simeq p^{5/4}$ and f is as big as possible.

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- Fix a prime $p \equiv 3 \pmod 4$ with $\log_2(p) \approx 2\lambda.$ such that the $N2^f$-torsion subgroup is defined over a small extension of $\mathbb{F}_{p^2}$ for smooth number $N \simeq p^{5/4}$ and f is as big as possible.
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- Fix $E_0 : y^2 = x^3 + x$ with known endomorphism ring $O_0 := \mathrm{End}(E_0)$.

SQIsign, Step 1: Σ-protocol

The prover P chooses a random isogeny $\phi : E_0 \to E_1$ such that $\mathbf{deg}(\phi)$ is a prime smaller than $2^{\lambda/2}$, leading to a random elliptic curve E_1. P keeps ϕ secret and publishes E_1. Now, P can prove “knowledge” of $O_1 \coloneqq \mathbf{End}(E_1)$ to a verifier V:

P

V

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$$E_1' = \phi'(E_0)$$

$$C \leq E_1'(\overline{\mathbb{F}}_p), C \cong \mathbb{Z}/M\mathbb{Z}$$

$$\tau \leftarrow \mathbf{Hom}((E_1', C), -)$$

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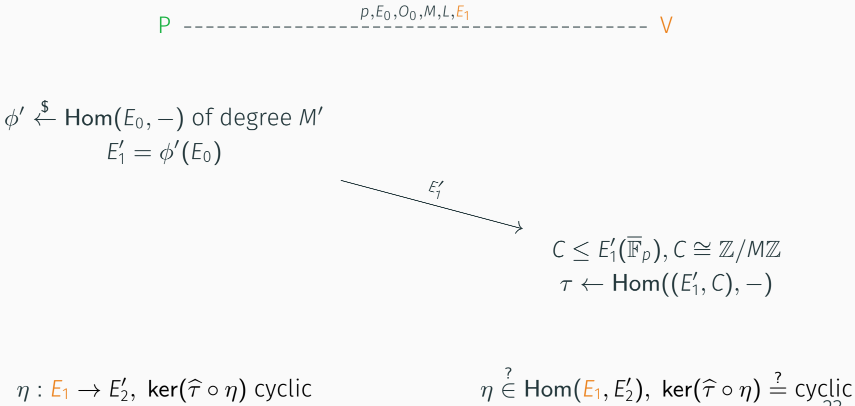

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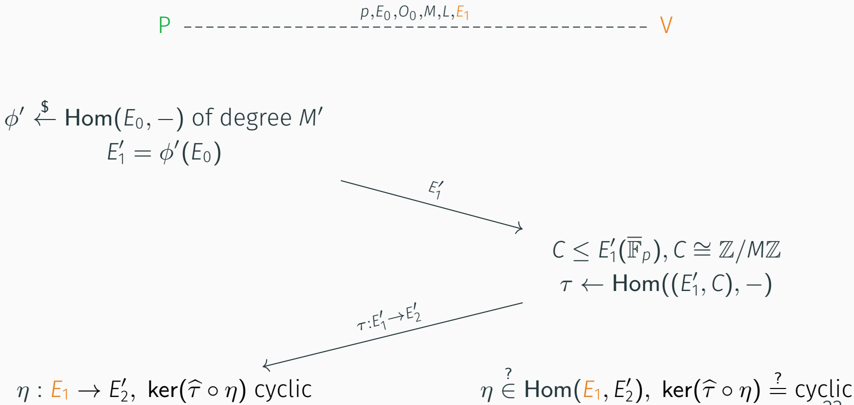

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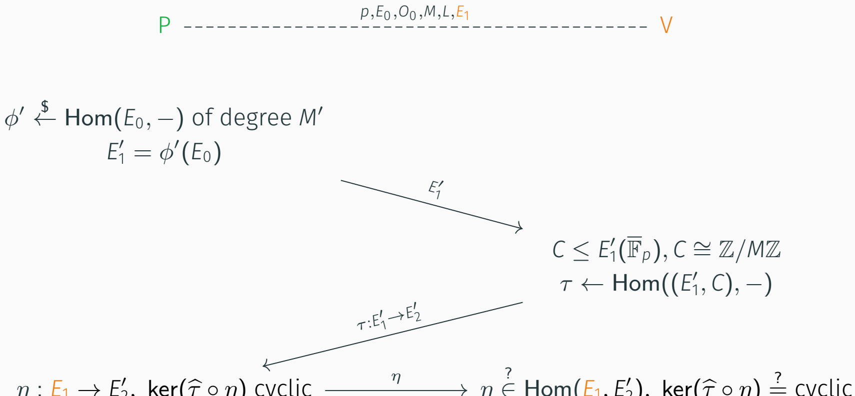

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$$\begin{tikzcd}
E_0 \arrow[r, "\text{secret: } \phi"] \arrow[d, "\text{secret: } \phi'"'] & E_1 \arrow[d, squiggly, "\text{response}"', "\eta"] \\
E_1' \arrow[r, dashed, "\tau", "\text{challenge}"'] & E_2'
\end{tikzcd}$$

Computing L-isogeny $\eta : E_1 \to E_2'$

1. Translate isogeny $\tau \circ \phi' \circ \widehat{\phi}$ to left O_1-ideal $I := \bar{I}_\phi \cdot I_{\phi'} \cdot I_\tau$ (`isogeny-to-kernel-to-ideal`).
2. From I, I_ϕ get $J \in [I]_{\mathrm{L}}$ with $\mathbf{nrd}(J) = L$.
3. Translate left O_1-ideal J to η (`ideal-to-kernel-to-isogeny`)

SQIsign, Step 2: Fiat-Shamir transformation

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Signing $(M', M, L, O_1, \mathrm{H}, m)$

1. $\phi' \overset{\$}{\leftarrow} \mathbf{Hom}(E_0, -)$ of degree M'
2. $E_1' = \phi'(E_0)$
3. $b = \mathrm{H}(m \| j(E_1'))$
4. $\tau = \mathbf{Decompress}(E_1', b)$
5. $\eta : E_1 \to E_2'$, $\mathbf{ker}(\widehat{\tau} \circ \eta)$ cyclic
6. return $\sigma \coloneqq (E_1', \eta)$

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```

Verification $(M, L, E_1, \mathrm{H}, m, \sigma)$

```
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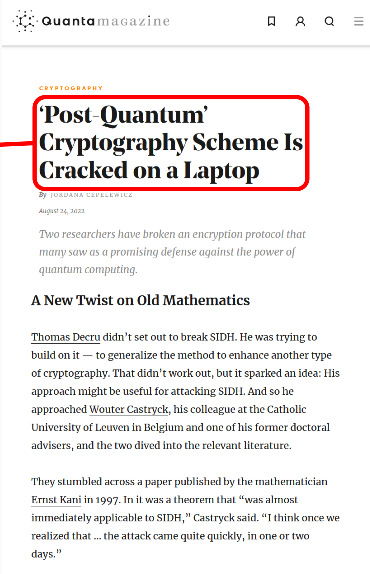
Quantum-safe?

SIDH (2011-2022) reached Round 4 of NIST’s quantum-safe KEM list.



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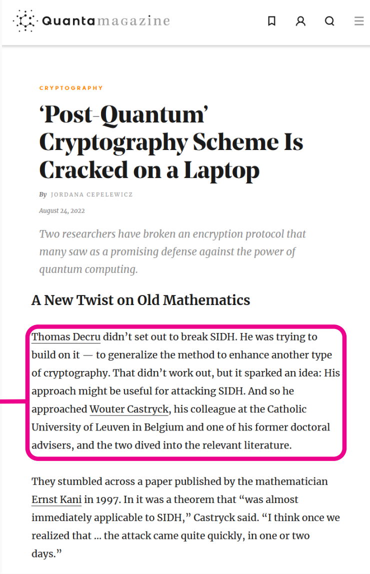
They stumbled across a paper published by the mathematician Ernst Kani in 1997. In it was a theorem that “was almost immediately applicable to SIDH,” Castryck said. “I think once we realized that ... the attack came quite quickly, in one or two days.”

Quantum-safe?

On August 5, 2022, Castryck and Decru posted a preprint outlining an efficient classical key recovery algorithm against SIDH.



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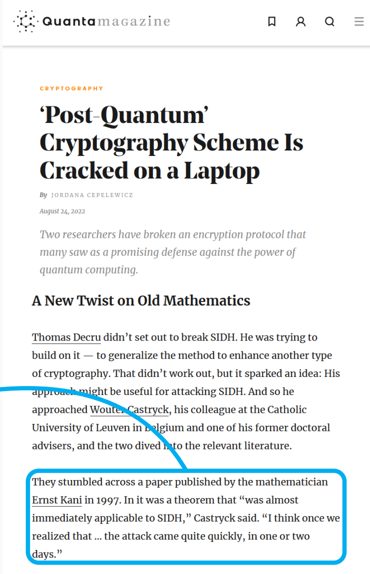

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SQISignHD uses this constructively: easier to generate public parameters & simpler signing procedure; but needs efficient implimentation of 4D isogeny.

Questions?

ECDSA

Choose a cryptographic hash function H with appropriate domain and codomain. The key generation algorithm G outputs a pair (k, Q) such that $Q = [k]P$, where k is the *secret signing key* and Q is the *public verification key*.

Signing $(\mathbb{G}, P, k, \mathrm{H}, m)$

```
1. $t \xleftarrow{\$} \{1, \ldots, \ell - 1\}$
2. $R \leftarrow [t]P$
3. $r \leftarrow x(R) \pmod{\ell}$
4. if $r = 0$ then goto Step 1.
5. $e \leftarrow \mathrm{H}(m)$
6. $s \leftarrow (e + kr)t^{-1} \pmod{\ell}$
7. if $s = 0$ then goto Step 1.
8. return $\sigma := (r, s)$
```

Verification $(\mathbb{G}, P, Q, \mathrm{H}, m, \sigma)$

```
1. $e \leftarrow \mathrm{H}(m)$
2. $u_1 \leftarrow es^{-1} \pmod{\ell}$, $u_2 \leftarrow rs^{-1} \pmod{\ell}$
3. $T \leftarrow [u_1]P + [u_2]Q$
4. return $r \overset{?}{=} x(T) \pmod{\ell}$
```

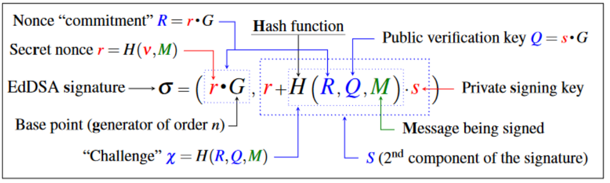
EdDSA, a footnote

Twisted Edwards model

A twisted Edwards curve defined over $\mathbb{F}_q$ is the curve

$$C : ax^2 + y^2 = 1 + dx^2y^2,\ a, d \in \mathbb{F}_q,\ \text{and } ad(a-d) \neq 0$$

with two singular points. It is birationally equivalent to $E : v^2 = u^3 + 2(a+d)u^2 + (a-d)^2u$ such that every point has order divisible by 4.



NIST IR 8214B, Notes on Threshold EdDSA/Schnorr Signatures