Appendix

Curves and Their Jacobians

D. Mumford: LNM 1358, pp. 225–291, 1999.
© Springer-Verlag Berlin Heidelberg 1999

Preface[13]

This appendix is a slightly expanded version of the series of four Ziwet Lectures which I gave in November 1974 at The University of Michigan, Ann Arbor. The aim of the lectures and of this volume is to introduce people in the mathematical community at large – professors in other fields and graduate students beyond the basic courses – to what I find one of the most beautiful and what objectively speaking is at least one of the oldest topics in algebraic geometry: curves and their Jacobians. Because of time constraints, I had to avoid digressions on any foundational topics and to rely on the standard definitions and intuitions of mathematicians in general. This is not always simple in algebraic geometry since its foundational systems have tended to be more abstract and apparently more idiosyncratic than in other fields such as differential or analytic geometry, and have therefore not become widely known to non-specialists. My idea was to get around this problem by imitating history: i.e., by introducing all the characters simultaneously in their complex analytic and algebraic forms. This did mean that I had to omit discussion of the characteristic p and arithmetic sides. However it also meant that I could immediately compare the strictly analytic constructions (such as Teichmüller Space) with the varieties which we were principally discussing.

When I first started doing research in algebraic geometry, I thought the subject attractive for two reasons: firstly, because it dealt with such down-to-earth and really concrete objects as projective curves and surfaces; secondly, because it was a small, quiet field where a dozen people did not leap on each new idea the minute it became current. As it turned out, the field seems to have acquired the reputation of being esoteric, exclusive and very abstract with adherents who are secretly plotting to take over all the rest of mathematics! In one respect this last point is accurate: algebraic geometry is a subject which relates frequently with a very large number of other fields – analytic and differential geometry, topology, K-theory, commutative algebra, algebraic groups and number theory, for instance – and both gives and receives theorems, techniques and examples with all of them. And certainly Grothendieck's work contributed to the field some very abstract and very powerful ideas which are quite hard to digest. But this subject, like all subjects, has a dual aspect in that all these abstract ideas would collapse of their own weight were it not for the underpinning supplied by concrete classical geometry. For me it has been a real adventure to perceive

[13] Written in 1975.

the interactions of all these aspects, and to learn as much as I could about the theorems both old and new of algebraic geometry.

Dafydd ap Gwilym's *The Lark* seems to me like the muse of mathematics:

High you soar, Wind's own power,
And on high you sing each song,
Bright spell near the wall of stars,
A far high-turning journey.

If this book entices a few to go on to learn these "spells", I'll be very pleased. I'd like to thank Fred Gehring, Peter Duren and the many people I met at Ann Arbor for their warm hospitality and their willingness to listen. A final point: in order not to interrupt the text, we have omitted almost all references and attributions in the lectures themselves, and instead written a separate section at the end giving references as well as suggestions for good places to learn various topics.

Lecture I: What is a Curve and How Explicitly Can We Describe Them?

In these lectures we shall deal entirely with algebraic geometry over the complex numbers $\mathbb{C}$, leaving aside the fascinating arithmetic and characteristic p side of the subject. In this first lecture, I would like to recapitulate some classical algebraic geometry, giving a leisurely tour of the zoo of curves of low genus, pointing out various features and their generalizations, and leading up to my first main point: the "general" curve of genus g, for g large, is very hard to describe explicitly.

The beginning of the subject is the AMAZING SYNTHESIS, which surely overwhelmed each of us as graduate students and should really not be taken for granted. Starting in 3 distinct fields of mathematics, we can consider 3 types of objects:

a) *Algebra*: consider *field extensions* $K \supset \mathbb{C}$, where K is finitely generated and of transcendence degree 1 over $\mathbb{C}$.
b) *Geometry*: First fix some notations: we denote by $\mathbb{P}^n$ the projective space of complex $(n+1)$-tuples $(X_0, \ldots, X_n)$, not all zero, mod scalars. $X_0, \ldots, X_n$ are called homogeneous coordinates. $\mathbb{P}^n$ is covered by $(n+1)$-affine pieces. $U_i =$ (pts where $X_i \neq 0$) and $x_0 = X_0/X_i, \ldots, x_n = X_n/X_i$ (x_i omitted) are the affine coordinates on U_i. Consider *algebraic curves* $C \subset \mathbb{P}^n$: loci defined by a finite set of homogeneous equations $f_\alpha(X_0, \ldots, X_n) = 0$, and such that for every $x \in X$, C is "locally defined by $n-1$ equations with independent differentials", i.e., $\exists\, f_{\alpha_1}, \ldots f_{\alpha_{n-1}}$ plus g, with $g(x) \neq 0$, such that for all α,

$$g f_\alpha \equiv \sum_{i=1}^{n-1} h_{\alpha,i} f_{\alpha_i}, \quad \text{some polynomials } h_{\alpha,i}$$

and

$$rk\,(\partial f_{\alpha_i}/\partial X_j(x)) = n-1.$$

c) *Analysis*: consider compact Riemann surfaces[14].

The result is that there are canonical bijections between the set of isomorphism classes of objects of either type. [A word about isomorphism in case (b): the simplest and oldest way to describe isomorphism in the algebraic category is that $C_1 \subset \mathbb{P}^{n_1}$ and $C_2 \subset \mathbb{P}^{n_2}$ are isomorphic if there is a bijective algebraic

[14] Perversely, algebraic geometers persist in talking about curves and analysts about surfaces when they mean essentially the same object!

correspondence between C_1 and C_2, i.e., there is a curve $D \subset C_1 \times C_2$ defined by bihomogeneous equations $g_\alpha(X_0, \dots X_{n_1};\ Y_0, \dots Y_{n_2}) = 0$ which projects bijectively onto C_1 and onto C_2.] To go back and forth between objects of type a), b), c), for instance, we

1) associate to a curve C the field K of functions $f : C \to \mathbb{C} \cup (\infty)$ given by restricting to C rational functions $p((X_0, \dots, X_n)/q(X_0, \dots, X_n)))$, $\deg p = \deg q$; and the Riemann surface just given by C with the induced complex structure from $\mathbb{P}^n$.
2) associate to a Riemann surface X its field of meromorphic functions; and any curve C which is the image of holomorphic embedding of X in $\mathbb{P}^n$.
3) from the field K, we recover C or X as point set just as the set of valuation rings R, $\mathbb{C} \subset R \subset K$.

To X or C or K we can associate a *genus* g as usual:

$$g = \text{ no. of handles of } X$$

or

$$g = \dim \text{ of } \begin{cases} [\text{space of holomorphic differentials } \omega \text{ on } X] \\ \qquad \qquad \| \\ [\text{space of rational differentials } \omega = a\,dx, \\ \quad (a, x \in K,\ x \notin \mathbb{C}) \text{ on } C \text{ with no poles}] \end{cases}$$

or

$$2g - 2 = (\text{no. of zeroes}) - (\text{no. of poles}), \quad \text{of any differential } \omega.$$

For each g, we shall let $\mathfrak{M}_g$ denote the set of isomorphism classes of X or C or K of genus g: we shall discuss the structure of $\mathfrak{M}_g$ in the second lecture.

So much for generalities. Most of what I shall say later is best understood by considering the computable explicit cases of low genus. Let's take these up and see what we have:

$\boxed{g = 0:}$ there is only one object here:
$X =$ Riemann sphere $\mathbb{C} \cup (\infty)$
$C = \mathbb{P}^1$ itself
$K = \mathbb{C}(X)$.

$\boxed{g = 1:}$ Here we have the famous theory of elliptic curves:
$X = \mathbb{C}/L$, L a lattice which may be taken to be $\mathbb{Z} + \mathbb{Z} \cdot \omega$, $\operatorname{Im} \omega > 0$.
$C =$ any non-singular plane cubic curve, i.e., $C \subset \mathbb{P}^2$ defined by $f(x, y, z) = 0$, f homogeneous of degree 3, with some partial non-zero at each root; in affine coordinates, x, y, C is given as the zeroes of a cubic polynomial $f(x, y) = 0$.
$K = \mathbb{C}\left(X, \sqrt{f(X)}\right)$, where f is a polynomial of degree 3 with distinct roots.

The connections between these are given as follows: given X, form the Weierstrass $\wp$-function:

$$\wp(z) = \frac{1}{z^2} - \sum_{\substack{a \in L \\ a \neq 0}} \left[\frac{1}{(z-a)^2} - \frac{1}{a^2} \right]$$

and map $\mathbb{C}/L$ into $\mathbb{P}^2$ by

$$\begin{array}{lll} z & \longmapsto & (1,\ \wp(z),\ \wp'(z)), & z \notin L \\ z & \longmapsto & (0,\ 0,\ 1), & z \in L, \end{array}$$

(i.e., $(\mathbb{C} - L)/L$ is mapped to the affine piece $X_0 \neq 0$ by $\wp$ and $\wp'$, and the one point L/L is mapped to the "line at infinity" for this affine piece.) Then $\wp$ and $\wp'$ generate the field K of X and since $\wp'^2$ is a cubic polynomial in $\wp$, K is as above. Or starting with any plane cubic C, take affine coordinates x, y so that the line at infinity is a line of inflexion. Then C is readily normalized to the form:

$$y^2 = f(x), \quad \deg f = 3.$$

Therefore the field of rational functions on C is $\mathbb{C}\left(x, \sqrt{f(x)}\right)$. To go back to X, look at the abelian line integral

$$w = \int_{(x_0,y_0)}^{(x,y)} \frac{dx}{y}$$

taken on C; then

$$(x, y) \longmapsto w$$

is well-defined up to a period which lies in a lattice L, hence defines:

$$C \xrightarrow{\approx} \mathbb{C}/L.$$

A few comments on this set-up: $\mathbb{C}/L$ is clearly a group, and hence so is C – here the group law is characterized geometrically by the beautiful:

$$x + y + z = 0 \Longleftrightarrow x, y, z \text{ collinear.}$$

For instance, $3x = 0 \Leftrightarrow x$ a point of inflexion. Since $3x = 0 \Leftrightarrow x \in \frac{1}{3}L/L$, there will be 9 of these. Now via $X \approx \mathbb{C}/L$, we get flat metrics on X with curvature $\equiv 0$. But if we instead look at a metric on X induced from the standard metric on $\mathbb{P}^2$ via $X \cong C \subset \mathbb{P}^2$, we get a metric whose curvature at the 9 points of inflexion equals that of $\mathbb{P}^2$, which is positive; and by the Gauss-Bonnet theorem, it must be negative at other points. The wobbly curvature points up the fact that X *does not fit symmetrically in* $\mathbb{P}^2$ – we will discuss this further in Lecture III. Another indication of the antagonism between $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\cdot\omega$ and C is the Gelfand-Schneider result: with a few exceptions for very special ω's, (i.e., $\omega \in \mathbb{Q}(\sqrt{-n})$), ω and the coefficients of any isomorphic cubic C are never simultaneously algebraic.

$\boxed{g=2:}$ Start with the fields K: these are all of the form

$$K = \mathbb{C}\left(X, \sqrt{f(X)}\right), \text{ where degree } f = 5.$$

What this means is that the corresponding curve C admits a $2-1$ mapping onto $\mathbb{P}^1$ ramified at 6 points: the 5 roots of f and the point at infinity. This does not quite give us C *embedded in* $\mathbb{P}^n$ though. We can do 2 things: let

$$\pi : C \longrightarrow \mathbb{P}^1$$

be the above map. Fix x_1, x_2 with $\pi x_1 = \pi x_2$. Then one can prove that C can be mapped to a plane quartic curve C_0 bijectively *except* that x_1 and x_2 are identified to a double point of C_0. This means that at the double point C_0 is given by an equation

$$0 = xy + f_3(x,y) + f_4(x,y)$$

where the double point equals the origin. In this form, $\pi(x,y) = x/y$; or geometrically, $\pi : C \to \mathbb{P}^1$ is defined by "projecting from $(0,0)$." This still doesn't represent C embedded in $\mathbb{P}^n$! In fact, to do this, you need $n = 3$, and at least 3 equations too. You start with a line $\ell \subset \mathbb{P}^3$, then take quadric and cubic surfaces $F, G \subset \mathbb{P}^3$ containing ℓ. Then $F \cap G$ will fall into 2 components – ℓ plus a quintic curve C, and it can be proven that every curve of genus 2 occurs as such a C.

Given such a C, there is only one $2-1$ map $\pi : C \longrightarrow \mathbb{P}^1$ and the most important points on C are the 6 points x_i where it ramifies. They have 2 significances

a) they are the Weierstrass points of C, i.e., the points $x \in C$ such that there is a rational function f on C with a double pole at x and no other poles, (if t is the coordinate on $\mathbb{P}^1$, let $f = (t - t(x))^{-1}$)
b) they represent the "odd theta-characteristics," i.e., look for differentials ω with no poles and zeroes only with even multiplicities: one writes this

$$(\omega) = 2\mathfrak{A}$$

if (ω) = divisor of zeroes and poles of ω. In this case, there are ω_i with one double zero at x_i, i.e.,

$$(\omega_i) = 2x_i$$

and no others (in fact if $a_i = t(x_i)$, $\omega_i = \sqrt{\frac{t-a_i}{\prod_{j\neq i} t-a_j}}\, dt$).

Analytically, C can be represented by a Fuchsian group:

$$C \cong H/\Gamma$$

where:

$$\begin{aligned} H &= \{z \mid \operatorname{Im} z > 0\} \\ \Gamma &= \text{discrete subgroup of } SL(2,\mathbb{R})/(\pm 1) \end{aligned}$$

or by various Kleinian groups:

$$C \cong D/\Gamma$$

where

$$\begin{aligned} D &= \text{open subset of } \mathbb{C} \cup (\infty) \\ \Gamma &= \text{discrete subgroup of } SL(2,\mathbb{C})/(\pm I) \\ &\quad\ \text{which acts discontinuously on } D. \end{aligned}$$

I want to make only one remark on these representations in connection with my main question of how explicitly one can describe C. Start with a Fuchsian Γ. Choosing a standard basis of $\pi_1(C)$, Γ is generated by hyperbolic transformations A_1, B_1, A_2, B_2 satisfying

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} = e.$$

It is quite clear from the work of Fricke-Klein and of Purzitsky and Keen that there is a small number of inequalities on the traces of small words in $A_1, \ldots, B_2$ which are always satisfied for Fuchsian Γ's, such that *conversely* if $A_1, \ldots, B_2 \in SL(2,\mathbb{R})$ satisfy these inequalities, they generate a Fuchsian Γ. (It would be nice to know these inequalities precisely.) This means that one can actually find all Fuchsian Γ's quite explicitly. For Kleinian Γ's no such simple inequalities are known and presumably do not exist. In the simplest case, the problem arises – describe explicitly the set of pairs $(A, B) \in SL(2,\mathbb{C})^2$ which generate a free group of only hyperbolic elements acting discontinuously at some $z_0 \in \mathbb{C}$, i.e., the Schottky groups. This looks very hard.[15] $\boxed{g = 3:}$ Here we encounter first the phenomenon of not having one easy description of all C's at once: "almost all" C's can be described one way, but some are a special case and must be described a different way. The general type are the C's which are non-singular plane quartic curves. The embedding of C in $\mathbb{P}^2$ is canonical and is given in the following simple way: let $\varphi_1, \varphi_2, \varphi_3$ be a basis of the differentials of first kind (= with no poles) on C. For all $x \in C$, let dt be a differential near x with no zero at x so that $\varphi_i(x) = a_i(x)dt$, a_i a function. Define:

$$C \longrightarrow \mathbb{P}^2$$

by

$$x \longmapsto (a_1(x), a_2(x), a_3(x)).$$

This is independent of the choice of dt because changing dt multiplies the triple by a scalar. This procedure works in any genus and defines the so-called canonical map

$$\Phi : C \longrightarrow \mathbb{P}^{g-1},$$

given, loosely speaking, by:

[15] Cf. forthcoming book, "Indra's Pearls" by D. Wright, C. Series and myself, Camb. Univ. Press.

$$x \longmapsto (\varphi_1(x), \ldots, \varphi_g(x)),$$

where $\{\varphi_i\}$ is a basis of differentials of 1^{st} kind.

[Note that there is a natural correspondence between linear functions in the homogeneous coordinates on $\mathbb{P}^{g-1}$ and arbitrary differentials $\sum \lambda_i \varphi_i$ of 1^{st} kind on C.] As is well known, there are 2 types of C's: those for which Φ is an embedding (i.e., Φ injective *and* $\Phi(C)$ non-singular), and those for which Φ is $2-1$, and the image is isomorphic to $\mathbb{P}^1$. All C's which admit $2-1$ maps to $\mathbb{P}^1$ fall into the 2^{nd} category and are called *hyperelliptic.* Thus for $g = 3$, either $C \cong \Phi(C)$ – then because each φ_i has $2g-2 = 4$ zeroes, each line in $\mathbb{P}^2$ meets $\Phi(C)$ in 4 points and $\Phi(C)$ is a quartic – or $\Phi(C)$ is a non-singular conic "with multiplicity 2," i.e., Φ is $2-1$. As all non-singular conics are isomorphic to $\mathbb{P}^1$, C is then hyperelliptic. In general, in the non-hyperelliptic case, $\Phi(C)$ will have degree $2g-2$, because the hyperplanes $H \subset \mathbb{P}^{g-1}$ correspond to differentials φ of 1^{st} kind in such a way that:

$$\Phi \ \text{(zeroes of } \varphi) = H \cap \Phi(C).$$

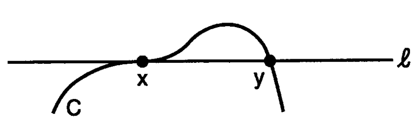
Plane quartic curves C are intricate objects. They have lots of special points on them:

a) their 24 points of inflexion are the Weierstrass points of C: the points x such that there is a function f on C with a triple pole at x and no other[16],
b) their 28 bitangents – lines tangent to C at 2 points – correspond to the odd theta-characteristics. Because if ℓ is tangent to C at x and y, then the differential φ corresponding to ℓ has a double zero at x and y:

$$(\varphi) = 2x + 2y.$$

In fact, projective geometry yields a vast constellation of "*higher Weierstrass points*" too, such as the 108 points x for which there is a conic D touching at x with contact of order 6. More generally, for any degree d, look at the points x for which there is a curve D of degree d touching C at X with contact "one more than is expected," i.e., one more than is possible at most points of D. One can think of this as some kind of analog on C of the finite set of points of order d, $\frac{1}{d}L/L \subset \mathbb{C}/L$ in the genus 1 case. This analogy goes quite far. For

[16] You can find the function as follows: let x be a point of inflexion, let ℓ be the tangent line to C at x. Then ℓ meets C at one further point y:



Let u, v be affine coordinates such that y is the origin $u = v = 0$, and ℓ is the coordinate axis $u = 0$. Consider the function $f = v/u$. Since u and v are zero at y, f is regular at y; but at x, f has a triple pole.

instance, as $d \to \infty$, one can show that these points are dense in C and even fairly evenly distributed in the "Bergman metric," i.e., for any curve C, choose a basis $\varphi_1, \ldots, \varphi_g$ of differentials of 1st kind for which

$$\int_C \varphi_i \wedge \overline{\varphi_j} = \delta_{ij}.$$

Then using such a basis, we can normalize our canonical embedding

$$\Phi : C \longrightarrow \mathbb{P}^{g-1}$$

up to unitary transformations, in which case the standard metric ds^2 on $\mathbb{P}^{g-1}$ has a restriction ds^2_B to C independent of the choice of the φ_i's: this is the Bergman metric.

An interesting question that arises in this connection is the relationship between the Bergman metric ds^2_B and the Poincaré metric ds^2_P of constant negative curvature induced from the standard metric on H:

$$ds^2 = dx^2 + dy^2/y^2, \quad z = x + iy$$

via the Fuchsian uniformization $C = H/\Gamma$. Kazdan suggested that if $\Gamma_n \subset \Gamma$ are subgroups of finite index and cofinal among such subgroups, if $C_n = H/\Gamma_n$, and if $ds^2_{B_n}$ is the Bergman metric on C_n pulled back to H, then with suitable scalars λ_n,

$$\lim_{n\to\infty} \lambda_n \, ds^2_{B_n} = ds^2_p.$$

We won't say much about the hyperelliptic case: in genus g, if $C \to \mathbb{P}^1$ is $2-1$, then there are exactly $2g+2$ branch points, and the corresponding fields are just $\mathbb{C}\left(X, \sqrt{f(X)}\right)$, where $\deg f = 2g+1$ or $2g+2$. [If f has degree $2g+2$, by a linear fractional transformation in X, taking some root to ∞, $\mathbb{C}\left(X, \sqrt{f}\right) \cong \mathbb{C}\left(X', \sqrt{f'}\right)$ where $\deg f' = 2g+1$.] These curves are special however in the following precise sense: one can build a big algebraic family of curves of genus g:

$$f : X \to S$$

such that all curves of genus g occur as fibres $X_s = f^{-1}(s)$. Then the set of s such that X_s is not hyperelliptic will form a dense Zariski-open subset of S.

$\boxed{g = 4:}$ Let $\Phi : C \to \mathbb{P}^3$ be the canonical map. If C is not hyperelliptic, we saw that $\Phi(C)$ was a space curve of degree 6. In fact, $\Phi(C)$ is the complete intersection $F \cap G$ of a quadric and cubic surface meeting transversely. One could also ask, however, is C a plane curve or is there a map $\pi : C \to \mathbb{P}^1$ of low degree? The answer to the first question is that C must be given singularities before it can be put in $\mathbb{P}^2$: the simplest way is to identify 2 pairs of points making C into a plane quintic C_0 with 2 double points; as for π, one can always find a π of degree 3.

As the genus g grows, it gets harder and harder to represent the general curve C of genus g either as a plane curve with relatively few singular points, or as a covering of fairly low degree of $\mathbb{P}^1$. For instance, it can be shown that the lowest degree curve representing such a C has degree

$$d = \left\lceil \frac{2g+8}{3} \right\rceil.$$

In general, its singularities will only be double points but the number of these will be

$$\delta = \frac{(d-1)(d-2)}{2} - g$$

which is asymptotic to $2/9(g^2)$. If $g \leq 10$, one can work backwards and write down all equations $f(X_0, X_1, X_2)$ defining curves of this degree d and this number δ of double points, hence having genus g. This is because the vector space of such f's has dimension $(d+1)(d+2)/2$ (count the coefficients), and for any point (a_0, a_1, a_2), if we require the coefficients of f to satisfy the 3 linear equations:

$$\frac{\partial f}{\partial X_i}(a_0, a_1, a_2) = 0,$$

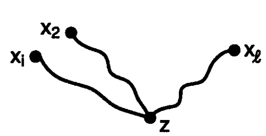
then $f = 0$ has a singularity at (a_0, a_1, a_2). Now if $g \leq 10$, then $3\delta < \frac{(d+1)(d+2)}{2}$ (see table below), hence we can pick an *arbitrary* set of δ points $P_i = \left(a_0^{(i)}, a_1^{(i)}, a_2^{(i)}\right)$ in $\mathbb{P}^2$ and always find at least one curve C of degree d with singular points $P_1, \ldots, P_\delta$; in general these will be double points and C will have genus g. However if $g \geq 11$, if we choose the δ singular points generically, there will be no such f, i.e., the coordinates of the δ double points will always satisfy some obscure identities. The upshot is that there is no reasonably explicit way to write down the equations of these plane curves: one is in a realm of unexplicitness almost as bad as with Kleinian groups.

Next, it can be shown that the lowest degree map $\pi : C \to \mathbb{P}^1$ has degree

$$d = \left\lceil \frac{g+3}{2} \right\rceil.$$

(This is equivalently the smallest number of poles of any non-constant function on the general curve C.) This also, to my knowledge, does not lead to any explicit polynomial presentation of C, but it does lead to a very explicit topological presentation of C. Namely, assuming the branch points of π are all simple, then one can reconstruct C in 5 steps:

a) Choose the branch points $\{x_i\}$ arbitrarily: there are $2(g+d-1)$ of them.
b) Choose a set of "cuts" joining x_i to a base point z:



c) Choose $2(g+d-1)$ transpositions σ_i acting on $\{1,\ldots,d\}$ such that $(\sigma_1 \cdot \sigma_2 \cdot \ldots) = e$.
d) Make a topological covering space C_0 of $\mathbb{P}^1$ by glueing together d copies of $\mathbb{P}^1$ via the transposition σ_i on the i^{th} cut.
e) By Riemann's existence theorem, C_0 has a unique algebraic structure, i.e., there is a unique curve C and map $\pi : C \to \mathbb{P}^1$ such that C is homeomorphic to C_0 as covering of $\mathbb{P}^1$.

Unfortunately, step b) is essentially topological and seems very deep from an algebraic point of view. For instance, if you want to algebraize this construction, you are led to ask: given prescribed branch points, cuts and transpositions, find an explicit multi-valued algebraic function with these branch points and transpositions. Thus if $d = 2$, $\sqrt{\prod(x - x_i)}$ is a function; if $d = 3$ or 4, the solvability of S_3, S_4 (the permutation groups) allows one to find such explicit functions too. But I don't know of any general method for larger d. We summarize these discussions in the table on the next page.

For general g, the simplest explicit polynomial presentation of C seems to be one due to K. Petri in a paper that was until recently almost forgotten. He was M. Noether's last student and collaborated with E. Noether and appears to have written only 2 papers. I want to conclude this lecture by describing his results in one of these published in 1922. This is unavoidably a bit messy, but just to be able to brag, I think it is a good idea to be able to say "I have seen *every* curve once."

Let C be a non-hyperelliptic curve of genus g. Petri starts by choosing g points $x_1, \ldots, x_g$ on C in a reasonably general position (we won't worry about this). Let $\varphi_1, \ldots, \varphi_g$ be a dual basis of differential forms, i.e.,

$$\begin{aligned} \varphi_i(x_j) &= 0 \quad \text{if} \quad i \neq j \\ &\neq 0 \quad \text{if} \quad i = j. \end{aligned}$$

Let $X_1, \ldots, X_g$ be the corresponding homogeneous coordinates in $\mathbb{P}^{g-1}$ for the canonical map $\Phi : C \to \mathbb{P}^{g-1}$. Also, if $3 \leq i \leq g$, write $\varphi_i = dt_i$, t_i a local coordinate at x_i and then expand

$$\begin{aligned} \varphi_1 &= \lambda_i t_i dt_i + \ldots \\ \varphi_2 &= \mu_i t_i dt_i + \ldots . \end{aligned}$$

(We may assume $\lambda_i \neq 0$ if $3 \leq i \leq g$.) Then Petri's first step is to write down a *basis* for the vector space of *k-fold* holomorphic differential forms on C for every k: these are differential forms $a(x)(dx)^k$ with no poles. For $k \geq 2$, they form a vector space of dimension $(2k-1)(g-1)$. The table below summarizes his results. Look at it carefully – each column displays a basis for k-fold differentials, $1 \leq k \leq 5$. Within each column however, we group the differentials in rows according to the multiplicity of their zeroes on $\mathfrak{A} \underset{\text{def}}{=} x_3 + \ldots x_g$. Thus the first row is always $\varphi_3^k, \ldots, \varphi_g^k$ as each of these has no zero at one of the x_i, whereas all other monomials in the φ_i's will be zero at least to 1^{st} order at each point of $\mathfrak{A}$. The second column arises like this:

a) one checks that every quadratic differential which is 0 on $\mathfrak{A}$ is on the form $\varphi_1 \cdot (\ ) + \varphi_2 \cdot (\ )$,
b) hence if $3 \leq i < j \leq g$, $\varphi_i \varphi_j$ can be rewritten as $\varphi_1 \cdot (\ ) + \varphi_2 \cdot (\ )$.
c) Omitting these $\varphi_i \varphi_j$, the remaining $3g - 3$ monomials form a basis as indicated.

Table of representations of the general curve C of genus g

g	degree of map $\pi : C \to \mathbb{P}^1$	no. of branch points	degree d of plane curve $C_0 \subset \mathbb{P}^2$	no. double points δ of C_0	canonical curve	3δ	vs.	$\frac{(d+1)(d+2)}{2}$	
0	1	0	1	0	–	0	vs.	3	
1	2	4	3	0	–	0	vs.	10	
2	2	6	4	1	–	3	vs.	15	
3	3	10	4	0	$C_4 \subset \mathbb{P}^2$	0	vs.	15	
4	3	12	5	2	$C_6 \subset \mathbb{P}^3$	6	vs.	21	
5	4	16	6	5	$C_8 \subset \mathbb{P}^4$	15	vs.	28	
6	4	18	6	4	$C_{10} \subset \mathbb{P}^5$	12	vs.	28	
⋯	⋯	⋯	⋯						
10	6	30	9	18	$C_{18} \subset \mathbb{P}^9$	54	vs.	55	$\longleftarrow$
11	7	34	10	25	$C_{20} \subset \mathbb{P}^{10}$	75	vs.	66	$\longleftarrow$
⋯	⋯	⋯	⋯						
100	51	300	69	2178	$C_{198} \subset \mathbb{P}^{99}$				

The third column arises like this:

a) one checks the triple differentials $\varphi_1^2(\ ) + \varphi_1\varphi_2(\ ) + \varphi_2^2(\ )$ are of codimension 1 in the vector space of triple differentials ω, with double zeroes on $\mathfrak{A}$! This is a reflection of the "fundamental class on C": the condition for such an ω to be formed out of $\varphi_1^2, \varphi_1\varphi_2, \varphi_2^2$ alone is that

$$(*) \qquad \sum_{\substack{\text{all zeroes} \\ y \text{ of } \varphi_1 \\ \text{except } x_3,\dots,x_g}} \operatorname{Res}_y(\omega/\varphi_1\varphi_2) = 0.$$

b) Writing η_i as indicated, this has a double zero on $\mathfrak{A}$ and every difference $\eta_i - \eta_j$ satisfies $(*)$.
c) Hence $\eta_i - \eta_j$ can be rewritten as indicated and this leaves exactly $5g - 5$ remaining triple differentials as a basis.

The remaining columns are quite mechanical: the 2 ways of rewriting differentials reduce us to the attached list, and, by counting, leave us with exactly the right number to be a basis!

Let us write out the 2 sets of identities by which these reductions are made. They will be:

$$\begin{aligned} \varphi_i\varphi_j &= \sum_{k=3}^{g} \alpha_{ijk}(\varphi_1, \varphi_2)\varphi_k + \nu_{ij}\varphi_1\varphi_2 \\ \eta_i - \eta_j &= \sum_{k=3}^{g} \alpha'_{ijk}(\varphi_1, \varphi_2)\varphi_k + \nu'_{ij}\varphi_1^2\varphi_2 + \nu''_{ij}\varphi_1\varphi_2^2 \end{aligned}$$

(here the α is linear, the α' is quadratic, the ν's are scalars, and $3 \le i, j \le g$, $i \ne j$). But what this means in terms of equations in $\mathbb{P}^{g-1}$ is precisely that 2 sets of homogeneous equations:

$$\begin{aligned} f_{ij} &= x_i x_j - \sum_{k=3}^{g} \alpha_{ijk}(x_1, x_2)x_k - \nu_{ij}x_1x_2 \\ g_{ij} &= (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_j x_1 - \lambda_j x_2)x_j^2 \\ &\quad - \sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ij}x_1^2x_2 - \nu''_{ij}x_1x_2^2 \end{aligned}$$

of degrees 2 and 3 generate the ideal of C!

Petri now goes on to prove 3 beautiful results –

I) These equations are related by syzygies:
   a) $f_{ij} = f_{ji}$, $g_{ij} + g_{jk} = g_{ik}$

b) $x_k f_{ij} - x_j f_{ik} + \sum_{\substack{\ell=3 \\ \ell \neq k}}^{g} \alpha_{ij\ell} f_{k\ell} - \sum_{\substack{\ell=3 \\ \ell \neq j}}^{g} \alpha_{ik\ell} f_{j\ell} = \rho_{ijk} g_{jk}$

where $3 \leq i, j, k \leq g$, i, j, k distinct, and the ρ_{ijk}'s are scalars symmetric in i, j, and k.

II) There are 2 possibilities: either $\rho_{ijk} = \alpha_{ijk} = 0$ whenever i, j, k are distinct, and then C is very special – it is a triple covering of $\mathbb{P}^1$ or if $g = 6$ it may also be a non-singular plane quintic; or else most of the ρ's and α's are non-zero (precisely, one can write $\{3, \ldots, g\} = I_1 \cup I_2$ so that for all $j \in I_1$, $k \in I_2$, there exists an i with $\rho_{ijk} \neq 0$, $\alpha_{ijk} \neq 0$), and then the f_{ij}'s alone generate the ideal of C.

III) Given any set of f_{ij}'s, g_{ij}'s as above related by the syzygies in (I), where all $\lambda_i \neq 0$, and at least one $\rho_{ijk} \neq 0$, there exists a curve C of genus g whose canonical image in $\mathbb{P}^{g-1}$ is defined by these equations.

In my mind, (III) is the most remarkable: this means that we have a complete set of identities on the coefficients α, α', ν, ν', ν'', λ, μ, ρ characterizing those that give canonical curves. It would be marvellous to use this formidable and precise machine for applications.

Petri's basis for the canonical ring

	Here, use $\varphi_i\varphi_j = (-)\varphi_1 + (-)\varphi_2$ $3 \le i < j \le g$	Here, use $\eta_i - \eta_j = (-)\varphi_1^2$ $+(-)\varphi_1\varphi_2 + (-)\varphi_2^2$ $3 \le i < j \le g$	$\eta_i = (\mu_i\varphi_1 - \lambda_i\varphi_2)\varphi_i^2$ $3 \le i \le g$		
$\varphi_3, \dots, \varphi_g$	$\varphi_3^2, \dots, \varphi_g^2$	$\varphi_3^3, \dots, \varphi_g^3$	$\varphi_3^4, \dots, \varphi_g^4$	$\varphi_3^5, \dots, \varphi_g^5$	*not zero on* $\mathfrak{A}$
φ_1, φ_2	$\varphi_1\varphi_i, \varphi_2\varphi_i$ $3 \le i \le g$	$\varphi_1\varphi_3^2, \dots, \varphi_1\varphi_g^2$	$\varphi_1\varphi_3^3, \dots, \varphi_1\varphi_g^3$	$\varphi_1\varphi_3^4, \dots, \varphi_1\varphi_g^4$	*simple zero on* $\mathfrak{A}$
–	$\varphi_1^2, \varphi_1\varphi_2, \varphi_2^2$	η_3 $\varphi_1^2\varphi_i, \varphi_1\varphi_2\varphi_i, \varphi_2^2\varphi_i$ $3 \le i \le g$	$\varphi_1^2\varphi_3^2, \dots, \varphi_1^2\varphi_g^2$	$\varphi_1^2\varphi_3^3, \dots, \varphi_1^2\varphi_g^3$	*double zero on* $\mathfrak{A}$
–	–	$\varphi_1^3, \varphi_1^2\varphi_2, \varphi_1\varphi_2^2, \varphi_2^3$	$\varphi_1\eta_3, \varphi_2\eta_3$ $\varphi_1^3\varphi_i, \varphi_1^2\varphi_2\varphi_i, \varphi_1\varphi_2^2, \varphi_i$ $\varphi_2^3\varphi_i$, $3 \le i \le g$	$\varphi_1^3\varphi_3^2, \dots, \varphi_1^3\varphi_g^2$	*triple zero on* $\mathfrak{A}$
–	–	–	$\varphi_1^4, \varphi_1^3\varphi_2, \varphi_1^2\varphi_2^2, \varphi_1\varphi_2^3, \varphi_3^4$	$\varphi_1^2\eta_3, \varphi_1\varphi_2\eta_3, \varphi_2^2\eta_3$ $\varphi_1^4\varphi_i, \varphi_1^3\varphi_2\varphi_i, \varphi_1^2\varphi_2^2\varphi_i$ $\varphi_1\varphi_2^3\varphi_i, \varphi_2^4\varphi_i$, $3 \le i \le g$	*4-fold zero on* $\mathfrak{A}$
–	–	–	–	$\varphi_1^5, \varphi_1^4\varphi_2, \varphi_1^3\varphi_2^2, \varphi_1^2\varphi_2^3, \varphi_1\varphi_2^4, \varphi_2^5$	*5-fold zero on* $\mathfrak{A}$
g simple differentials	*$3g-3$ quadruple differentials*	*$5g-5$ triple differentials*	*$7g-7$ 4-tuple differentials*	*$9g-9$ 5-tuple differentials*	

Lecture II: The Moduli Space of Curves: Definition, Coordinatization, and Some Properties

In the previous lecture, we studied each curve separately. We now want to discuss in its own right the space of *all* curves of genus g, which we denote by $\mathfrak{M}_g$. Also very important is the allied space:

$$\mathfrak{M}_{g,n} = \left\{ \begin{array}{l} \text{isomorphism classes of objects } (C, x_1, \ldots, x_n) \\ \text{where } C \text{ is a curve of genus } g, \text{ and } x_1, \ldots, x_n \text{ are} \\ \text{distinct ordered points of } C. \end{array} \right\}$$

Let us begin as before by looking first at the simplest cases:

I) $\mathfrak{M}_{0,n} \cong \left[\mathbb{P}^1 - (0, 1, \infty)\right]^{n-3}$ – (all diagonals).

In fact, if we have n distinct points $x_1, \ldots, x_n \in \mathbb{P}^1$, a unique automorphism of $\mathbb{P}^1$ takes x_1 to 0, x_2 to 1, and x_3 to ∞. The remaining $n-3$ are arbitrary except for being distinct and not equal to 0 or 1 or ∞.

II) $\mathfrak{M}_{1,0} = \mathfrak{M}_{1,1} \cong \mathbb{A}^1_j$ (the affine line[17] with coordinate j).

Because curves of genus 1 are groups, their automorphisms act transitively on them, hence $\mathfrak{M}_{g,0} = \mathfrak{M}_{g,1}$. To determine this space, recall that all such curves are isomorphic to one of the plane cubics C_λ, defined by

$$y^2 = x(x-1)(x-\lambda).$$

Equivalently, C_λ is the double cover of $\mathbb{P}^1$ ramified at $0, 1, \infty, \lambda$. One proves easily that $C_{\lambda_1} \approx C_{\lambda_2}$ if and only if there is an automorphism of $\mathbb{P}^1$ carrying $\{0, 1, \infty, \lambda_1\}$ (unordered sets) to $\{0, 1, \infty, \lambda_2\}$. This happens if and only if

$$\lambda_2 = \lambda_1,\ 1-\lambda_1,\ 1/\lambda_1,\ (\lambda_1 - 1)/\lambda_1,\ \lambda_1/(\lambda_1 - 1),\ \text{or } 1/(1-\lambda_1)$$

[e.g., note that the map

$$(x, y) \longmapsto (1-x, y)$$

carries C_λ to $C_{1-\lambda}$; and the map

$$(x, y) \longmapsto (1/x, y/x)$$

[17] Out of habit, I find it more comfortable to call affine n-space $\mathbb{A}^n$ instead of $\mathbb{C}^n$: because $\mathbb{A}^n$ also denotes affine n-space over other ground fields.

carries C_λ to $C_{1/\lambda}$].

One must cook up an expression in λ invariant under these substitutions and no more. It is customary to use:

$$j = 256\,\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

(It is readily checked that this j is invariant under these 6 substitutions and since $6 = \max(\text{deg of numerator and denominator})$, no other λ's give the same j.)

We then get a bijection between the isomorphism classes of genus 1 curves C and the complex numbers $\mathbb{C}$ by taking C to $j(\lambda)$ if $C \approx C_\lambda$.

Analytically, if $C = \mathbb{C}/L$, the j-invariant of C can be calculated from L in the following way: define[18]

$$\begin{aligned} g_2 &= 60\cdot\sum_{\substack{\lambda\in L\\ \lambda\neq 0}} 1/\lambda^4 \\ g_3 &= 140\cdot\sum_{\substack{\lambda\in L\\ \lambda\neq 0}} 1/\lambda^6. \end{aligned}$$

Then it can be shown that:

$$j(C) = 1728\cdot g_2^3/\left(g_2^3 - 27g_3^2\right).$$

In particular, if $L = \mathbb{Z}+\mathbb{Z}\cdot\omega$, then $j(\omega) = j(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\cdot\omega)$ is the famous elliptic modular function. Its most important property is its invariance under $SL(2,\mathbb{Z})$, which can be explained from a moduli point of view as follows:

$$\left\{\begin{array}{l} \exists\ \alpha\in\mathbb{C}\ \text{ such that} \\ \alpha(\mathbb{Z}+\mathbb{Z}\cdot\omega_1) = \mathbb{Z}+\mathbb{Z}\cdot\omega_2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \exists\ \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})\ \text{ such that} \\ \omega_2 = \dfrac{a\omega_1+b}{c\omega_1+d} \end{array}\right\}.$$

(This is trivial to check.) But

$$\begin{array}{l} \left\{\begin{array}{l} \exists\ \text{isomorphism} \\ C/\mathbb{Z}+\mathbb{Z}\cdot\omega_1 \cong \mathbb{C}/\mathbb{Z}+\mathbb{Z}\cdot\omega_2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \exists\ \alpha\in\mathbb{C}\ \text{ such that} \\ \alpha(\mathbb{Z}+\mathbb{Z}\cdot\omega_1) = \mathbb{Z}+\mathbb{Z}\cdot\omega_2 \end{array}\right\} \\ \qquad\qquad\Updownarrow \\ j(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\cdot\omega_1) = j(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\cdot\omega_2). \end{array}$$

Hence:

$$j(\omega_1) = \omega_2 \Longleftrightarrow \left\{\begin{array}{l} \exists\ \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})\ \text{ such that} \\ \omega_2 = \dfrac{a\omega_1+b}{c\omega_1+d} \end{array}\right\}.$$

[18] The stream of funny constants can best be explained as making a certain Fourier expansion have *integral*, not just rational, coefficients. This makes the theory work well under "reduction modulo p".

III) $\mathfrak{M}_{2,0}$: this space was studied classically by Bolza among others and in recent years was analyzed completely by Igusa, and was attacked as follows: describe a curve C of genus 2 as a double cover of $\mathbb{P}^1$ ramified in 6 points $\lambda_1, \ldots, \lambda_6$. This sets up a bijection:

$$\left\{ \begin{array}{l} \text{Isom. classes of} \\ C \ \text{ of genus 2} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{unordered distinct 6-tuples} \\ \lambda_1, \ldots, \lambda_6 \in \mathbb{P}^1 \ \text{ modulo automorphisms} \\ \text{of } \mathbb{P}^1, \ \text{ i.e., } \ PGL(2, \mathbb{C}) \end{array} \right\}.$$

Describe an unordered 6-tuple $\{\lambda_i\}$ by its homogeneous equation $f(X_0, X_1)$ of degree 6, a so-called binary sextic, and we arrive at the problem: find polynomial functions of the coefficients of a binary sextic $f(X_0, X_1)$ invariant under linear substitutions in X_0, X_1 of determinant one. This is a problem worked out by the classical invariant theorists. These invariant functions are then coordinates on $\mathfrak{M}_{2,0}$. Without going into any more detail, suffice it to say that the simplest way to describe the answer you get is:

$$\mathfrak{M}_{2,0} \cong \mathbb{A}^3 \Big/ \underbrace{\begin{array}{l} \text{modulo } \mathbb{Z}/5\mathbb{Z} \ \text{ acting by} \\ (x,y,z) \mapsto (\zeta^1 x, \zeta^2 y \cdot \zeta^3 z) \\ \text{where } \zeta^5 = 1 \end{array}}_{\begin{array}{l} \text{this, in turn, may be embedded in } \mathbb{A}^8 \\ \text{by the 8 functions} \\ x^5, x^3y, xy^2, y^5, x^2z, xz^3, z^5, yz \end{array}}$$

For all $g \geq 3$, $\mathfrak{M}_{g,0}$ *has never been explicitly described*! This rather discouraging fact does not mean that the other $\mathfrak{M}_{g,n}$'s have not been studied however. The lack of an explicit description is rather a challenge i) to find one and ii) to find the properties of $\mathfrak{M}_{g,n}$ even without such a description!

The first point to be made about $\mathfrak{M}_{g,n}$ in general is why you call it a "space" and expect it to be a variety in the first place. Recall that a *projective variety* $X \subset \mathbb{P}^n$ is defined to be the complete set of zeroes of a set of homogeneous polynomials f_i which generate a prime ideal $\wp \subset \mathbb{C}[X_0, \ldots, X_n]$, and that a *quasi-projective variety* $X \subset \mathbb{P}^n$ is defined to be the difference $\overline{X} - (Y_1 \cup \ldots \cup Y_n)$ where $\overline{X}, Y_i$ are projective varieties. We then say that a normal[19] quasi-projective variety $M_{g,n}$ *is* the moduli space if

i) we are given a bijection between $\mathfrak{M}_{g,n}$ and the set of points of $M_{g,n}$,
ii) for every algebraic family of curves of genus g with n distinct points, i.e., every "proper smooth morphism $\pi : X \to S$ of varieties whose fibres are curves of genus g[20], plus n disjoint sections $\sigma_i : S \to X$," the induced set-theoretic map $\phi : S \to M_{g,n}$ defined by

[19] This means that the affine coordinate rings of $M_{g,n}$ are integrally closed in their quotient field. This is a mild condition needed only for technical reasons.

[20] Again it is not essential to know in detail what these terms mean: the idea is to generalize, for instance, the family of curves $y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, which would represent an algebraic family of curves of genus 2 parametrized by $\mathbb{A}^3$.

$$\phi(s) = \left[\begin{array}{l} \text{pt. of } M_{g,n} \text{ corresponding via (i) to the curve} \\ \pi^{-1}(s), \text{ and points } \sigma_i(s) \end{array} \right]$$

is a morphism of varieties.

It is not hard to show that any 2 such $M_{g,n}$, $M'_{g,n}$ are canonically isomorphic as varieties: hence we may speak of *the* variety $\mathfrak{M}_{g,n}$. It is a non-trivial theorem however that such a variety $M_{g,n}$ exists at all.

The second point is to explain the relationship between $\mathfrak{M}_{g,n}$ and the Teichmüller space $\mathfrak{T}_{g,n}$. Define

$$\Pi = \left\{ \begin{array}{l} \text{free group on } 2g+n \text{ generators } A_1, \ldots, A_g, B_1, \ldots, B_g, \\ C_1, \ldots, C_n \text{ mod one relation} \\ A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} C_1 \ldots C_n = e \end{array} \right\}.$$

Define set-theoretically:

$$\mathfrak{T}_{g,n} = \left\{ \begin{array}{l} \text{set of objects } (C, \alpha, x_1, \ldots, x_n), \text{ where } C \text{ is a} \\ \text{curve of genus } g, \ x_1, \ldots, x_n \text{ are distinct points} \\ \text{of } C \text{ and} \\ \qquad \alpha : \Pi \xrightarrow{\approx} \pi_1(C - \{x_1, \ldots, x_n\}) \\ \text{is an isomorphism such that } \alpha(C_i) \text{ is freely} \\ \text{homotopic to a small loop around } x_i \text{ in positive} \\ \text{sense [and if } n = 0, \alpha \text{ is ``orientation preserving,''} \\ \text{e.g., via the intersection pairing } (.), \\ (\alpha(A_1) \cdot \alpha(B_1)) = +1], \ \textit{modulo} \ (C, \alpha, x) \sim (C', \alpha', x') \\ \text{if there is an isomorphism } \phi : C \xrightarrow{\approx} C' \text{ such that} \\ \phi(x_i) = x'_i \text{ and such that } (\phi_*) \circ \alpha \text{ differs from } \alpha' \\ \text{by an inner automorphism} \end{array} \right\}.$$

Via the deformation theory of compact complex manifolds, it is easy to put a complex structure on $\mathfrak{T}_{g,n}$: this is the Teichmüller space. It is a deep theorem that $\mathfrak{T}_{g,n}$ is, in fact, a bounded, holomorphically convex domain in $\mathbb{C}^{3g-3+n}$. Let

$$\Gamma_{g,n} = \left\{ \begin{array}{l} \text{automorphisms } \sigma \text{ of } \Pi \text{ such} \\ \text{that } \sigma(C_i) = \text{ conjugate of } C_i \\ \text{(and if } n = 0, \sigma \text{ is orientation} \\ \text{preserving in a suitable sense)} \end{array} \right\} \Big/ \begin{array}{l} \text{Inner} \\ \text{automorphisms} \end{array}.$$

Then it follows easily that $\Gamma_{g,n}$ acts discontinuously on $\mathfrak{T}_{g,n}$ via

$$(C, \alpha, x_i) \longmapsto (C, \alpha \circ \sigma, x_i)$$

and that

$$\mathfrak{M}_{g,n} \cong \mathfrak{T}_{g,n} / \Gamma_{g,n}.$$

In the case $g = n = 1$, we just have again the situation mentioned above: viz.

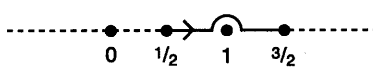
$$\begin{aligned}\mathfrak{I}_{1,1} &\cong \{\omega \in \mathbb{C} \mid \operatorname{Im} \omega > 0\}\\ \Gamma_{1,1} &\cong SL(2,\mathbb{Z})/(\pm I)\\ \mathfrak{M}_{1,1} &\cong \{j \in \mathbb{C}\} = \mathbb{A}^1_j.\end{aligned}$$

In fact, given $\omega \in \mathbb{C}$, define (C, α, x_1) as follows –

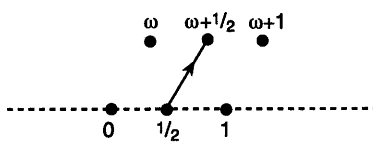
$$\begin{aligned} C &= \mathbb{C}/\mathbb{Z} + \mathbb{Z}\cdot\omega\\ x_1 &= \text{image of } 0\end{aligned}$$

and if we let the image y of $1/2 \in \mathbb{C}$ be the base point C, define $\alpha : \Pi \xrightarrow{\approx} \pi_1(C - x_1, y)$ by

$\alpha(A_1) =$ loop in C obtained by projecting:



$\alpha(B_1) =$ loop in C obtained by projecting:



The third point we want to discuss is how one proves that $\mathfrak{M}_{g,n}$ is, in fact, a quasi-projective variety, i.e., how one finds global homogeneous coordinates for $\mathfrak{M}_{g,n}$. To tie this in, for instance, with Petri's approach in Lecture I, one can view his ideas as leading to coordinates on some Zariski-open subset $U \subset \mathfrak{M}_{g,g}$: (i.e., not on all of $\mathfrak{M}_{g,g}$ because the curve C had to be non-hyperelliptic and the g auxiliary points $x_1, \ldots, x_g$ had to be carefully chosen not in too special a position). In general, the hard part of this problem is to make the coordinates work everywhere on $\mathfrak{M}_{g,n}$ and not just on a Zariski-open U however. These coordinates can be viewed as automorphic forms on the Teichmüller space $\mathfrak{I}_{g,n}$ with respect to the Teichmüller modular group $\Gamma_{g,n}$; however this approach to their construction has not been pursued. I know of 3 methods to obtain coordinates:

I. via "theta-null werte,"
II. via the cross-ratios of the higher Weierstrass points,
III. via invariants of the Chow form.

The first method will be discussed in Lecture IV, and we will pass over it for now.

Method II is like this: let C by any curve of genus g. For any $n \geq 3$, let $R_n(C)$ be the vector space of n-fold differential forms with no poles – it has dimension $d_n = (2n-1)(g-1)$ – and let $\omega_i^{(n)}$, $1 \leq i \leq d_n$, be a basis. Define

$$\Phi_n : C \longrightarrow \mathbb{P}^{d_n - 1}$$

by

$$x \longmapsto \left(\omega_1^{(n)}(x), \ldots, \omega_{d_n}^{(n)}(x)\right)$$

just like the usual canonical embedding. Regardless of whether C is hyperelliptic or not, these are all projective embeddings of C. On $\Phi_n(C)$, there is a finite set of points x of hyperosculation, i.e., points where for some hyperplane H, H touches $\Phi_n(C)$ at x with order $\geq d_n$. Allowing these x to be counted with suitable multiplicity, there are $e_n = gd_n^2$ of them: call them $x_i^{(n)}$, $1 \leq i \leq e_n$. These are the n-fold Weierstrass points (our definition here is slightly different from that of Lecture I, but is equivalent). Consider the $e_n \times d_n$-matrix giving the coordinates of the Weierstrass points:

$$\left(\omega_i^{(n)}\left(x_j^{(n)}\right)\right).$$

For every $I \subset \{1, \ldots, e_n\}$, $\#I = d_n$, consider the minor:

$$M_I = \det_{\substack{1 \leq i \leq d_n \\ j \in I}} \left[\omega_i^{(n)}\left(x_j^{(n)}\right)\right].$$

Note that the M_I's are not numbers, but rather products of differential forms at the various points $x_j^{(n)}$, $j \in I$. Now for large N look at monomials in these minors:

$$M_r = \prod_I M_I^{r_I}$$

where $r_I \geq 0$ and $\sum_{i \in I} r_I = N$ for all i. Then these monomials are products over *all* $x_j^{(n)}$ of nN-fold differentials at $x_j^{(n)}$. It follows that although the M_r's are not complex numbers, their *ratios* are! Or if there are μ possible choices of exponents r_I satisfying $r_I \geq 0$ and $\sum_{i \in I} r_I = N$, the set of values M_r, as r varies, is a well-defined point in $\mathbb{P}^{\mu - 1}$. Finally we must symmetrize under permutations of the $x_j^{(n)}$ which are not naturally ordered:

$$M'_r = \sum_{\substack{\text{perm. } \sigma \\ \text{of } \{1, \ldots, e_n\}}} \prod_I M_I^{r_{\sigma(I)}}.$$

Then the ratios M'_{r_1}/M'_{r_2} *depend only on* C, and not on the bases $\omega_i^{(n)}$ or on the ordering of the $x_j^{(n)}$'s. Thus we get

$$(\ldots, M'_r(C), \ldots) \in \mathbb{P}^{\mu-1}$$

depending only on C. One proves a) that not all $M'_r(C)$ are zero, and b) that if $C_1 \not\approx C_2$, $M'_r(C_1)$ is not proportional to $M'_r(C_2)$, all r. Thus we have coordinates on $\mathfrak{M}_{g,0}$. $\mathfrak{M}_{g,n}$ is very similar.

Method III is not so explicit. In general, for any curve $C \subset \mathbb{P}^m$, we can describe C by its "Chow form": let $X_0, \ldots, X_m$ be coordinates on $\mathbb{P}^m$ and consider 2 hyperplanes: H_u defined by $\sum u_i X_i = 0$ and H_v defined by $\sum v_i X_i = 0$. Then it turns out that there is *one* equation $F_C(u; v)$ such that

$$F_C(u; v) = 0 \Longleftrightarrow C \cap H_u \cap H_v \neq \emptyset.$$

F_C is called the Chow form of C and it determines C. (For curves in $\mathbb{P}^3$, this idea goes back to Cayley.) Consider the Chow form $F_{\Phi_n(C)}$. This depends on C and on the choice of basis $\omega_i^{(n)}$ of $R_n(C)$. However, changing the basis $\left\{\omega_i^{(n)}\right\}$ changes the Chow form $F_{\Phi_n(C)}(u; v)$ by the contragrediant linear substitution in u and v. Writing out

$$F(u, v) = \sum F_{\alpha\beta} u^\alpha v^\beta,$$

this means that there is a natural representation of $SL(d_n, \mathbb{C})$ in the space of forms F or of the space of coefficients $F_{\alpha\beta}$. One proves that there are "enough" invariant polynomials $c_i(F_{\alpha\beta})$ so that

a) for each curve C, at least one $c_i\left(F_{\Phi_n(C),\alpha\beta}\right)$ is not zero, and
b) if $C_1 \not\approx C_2$, then the set of numbers $c_i\left(F_{\Phi_n(C),\alpha\beta}\right)$ is not proportional to $c_i\left(F_{\Phi_n(C_2),\alpha\beta}\right)$. Thus again the map

$$C \longmapsto \left(\ldots, c_i\left(F_{\Phi_n(C),\alpha\beta}\right), \ldots\right)$$

embeds $\mathfrak{M}_g$ into projective space.

The fourth point I want to make about $\mathfrak{M}_{g,n}$ is that although it is not compact, because a sequence of curves may "degenerate", $\mathfrak{M}_{g,n}$ has a natural compactification $\overline{\mathfrak{M}}_{g,n}$ obtained by casting out your net further and attempting to make into a moduli space not only the non-singular curves, but also some singular ones too. In fact one looks at curves $C \subset \mathbb{P}^n$ which may have "ordinary double points" and may even have several components. To be precise, we mean either

a) that as an analytic set, C is connected and everywhere is isomorphic locally either to the unit disc Δ, or to 2 copies of the unit disc $\Delta_1 \cup \Delta_2$ crossing transversely

or equivalently

b) that in the Zariski topology, C is connected and everywhere is defined locally either by $n-1$ equations $f_1, \ldots, f_{n-1}$ with independent differentials df_i or by $n-1$ equations $g, f_2, \ldots, f_{n-1}$ where g vanishes to second order with leading term (x, y) and the f_i's vanish only to 1^{st} order, with $dx, dy, df_2, \ldots, df_{n-1}$ all independent.

For instance, we could take 2 non-singular curves and let them cross transversely at one or more points; or we could take 1 non-singular curve and map it to $\mathbb{P}^n$ so that it crosses itself transversely at one or more points. Or we combine both operations! Then $\overline{\mathfrak{M}}_{g,n}$ is to be the space of objects $(C, x_1, \ldots, x_n)$, up to isomorphism, where C is a projective curve with only ordinary double points as defined above and the x_i are distinct non-singular points of C and g is the sum of genuses of the components of C treated as non-singular curves, plus the number of double points, minus the number of components, plus one: $g = \sum(g_i - 1) + \delta + 1$; and finally if any component C_0 of C is isomorphic to $\mathbb{P}^1$, then there are at least 3 points of C_0 which are x_i's or where C_0 meets other components of C. It is a theorem that $\overline{\mathfrak{M}}_{g,n}$ is, in a natural way, a projective variety, esp. it is compact.

The last topic I would like to discuss at some length is the curious ambivalence in the variety $\mathfrak{M}_{g,n}$ to be in various senses somehow hyperbolic on the one hand, yet in other senses it wants to be elliptic. To explain this, it's best to go back first to $\mathfrak{M}_{1,1}$. We can factor the map:

$$\begin{array}{ccc} \mathfrak{J}_{1,1} & \longrightarrow & \mathfrak{M}_{1,1} \\ \| & & \| \\ H & & \mathbb{A}^1_j \cong H/SL(2,\mathbb{Z}) \end{array}$$

by considering subgroups $\Gamma \subset SL(2,\mathbb{Z})$ of finite index:

$$H \longrightarrow H/\Gamma \longrightarrow H/SL(2,\mathbb{Z}).$$

The curves H/Γ are finite coverings of $\mathfrak{M}_{1,1}$ and are called "higher level" moduli spaces: I'll denote H/Γ by $\mathfrak{M}^{\Gamma}_{1,1}$. It too can be naturally compactified by adding a finite set of points; so we get finally the diagram

$$\begin{array}{ccc} \mathfrak{J}_{1,1} & & \\ \alpha\big\downarrow & & \\ \mathfrak{M}^{\Gamma}_{1,1} & \subset & \overline{\mathfrak{M}}^{\Gamma}_{1,1} \\ \big\downarrow & & \big\downarrow\beta \\ \mathfrak{M}_{1,1} & \subset & \overline{\mathfrak{M}}_{1,1} \end{array}.$$

Now of course all curves lie in 3 classes:

Elliptic Class:	$g = 0$;	admits positively curved metric;	no holo. k-forms
Parabolic Class:	$g = 1$;	admits flat metric;	*one* holo. k-form for each k
Hyperbolic Class:	$g \geq 2$;	admits negatively curved metric;	*lots* of holo. k-forms giving proj. embedding

The point is that $\overline{\mathfrak{M}}_{1,1}$ is $\mathbb{P}^1$, hence is elliptic, while if Γ is moderately small, $\overline{\mathfrak{M}}_{1,1}^{\Gamma}$ is hyperbolic. The reason this flip is possible is that β is ramified: in fact there are 2 finite points $j = 0$ and $j = 12^3$ at which $\mathfrak{J}_{1,1} \longrightarrow \mathfrak{M}_{1,1}$ is respectively triply and doubly ramified, and 1 infinite point $j = \infty$ over which the β's are arbitrarily highly ramified. From another point of view, $\mathfrak{J}_{1,1}$ admits a canonical metric with negative curvature, i.e., $ds^2 = dx^2 + dy^2/y^2$, (if $z = x+iy \in H \simeq \mathfrak{J}_{1,1}$ is the coordinate). This induces a negatively curved metric on each $\mathfrak{M}_{1,1}^{\Gamma}$. In this metric, $\mathfrak{M}_{1,1}^{\Gamma}$ has finite volume, but the metric has singularities, a) at points where α is ramified and b) at points of $\overline{\mathfrak{M}}_{1,1}^{\Gamma} - \mathfrak{M}_{1,1}^{\Gamma}$. (If Γ is small enough, α will be unramified and only (b) occurs.)

It is this constellation of facts that to some extent generalizes to $\mathfrak{M}_{g,n}$. In our present state of knowledge, the generalization is very partial. To begin with, we get the same diagram:

$$\begin{array}{ccc} \mathfrak{J}_{g,n} & & \\ \big\downarrow{\scriptstyle\alpha} & & \\ \mathfrak{M}_{g,n}^{\Gamma} & \subset & \overline{\mathfrak{M}}_{g,n}^{\Gamma} \\ \big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \mathfrak{M}_{g,n} & \subset & \overline{\mathfrak{M}}_{g,n} \end{array} .$$

for each $\Gamma \subset \Gamma_{g,n}$ of finite index. Let me begin with the known elliptic-type properties which are unfortunately weak: we assume $n = 0$ for simplicity.

a) Assume also $g \geq 4$ for simplicity[21]. Then the singular set $S \subset \mathfrak{M}_g$ is the set of points of $\mathfrak{M}_g$ where $\mathfrak{J}_g \to \mathfrak{M}_g$ ramifies and is the set of points corresponding to curves C with automorphisms. Then $B_1(\mathfrak{M}_g - S)$, the first betti number, is zero, hence so is B_1 of $\mathfrak{M}_g$, $\overline{\mathfrak{M}}_g$ and *any non-singular blow-up* $\mathfrak{M}_g^*$ of $\overline{\mathfrak{M}}_g$. This means, e.g., that the so-called Albanese variety of $\mathfrak{M}_g^*$ is trivial.
b) $\mathfrak{M}_g$ has lots of rational curves in it. In fact for any algebraic surface X and rational function f on X, let $C_t \subset X$ be the curve $f(x) = t$, and let $[C_t] \in \mathfrak{M}_g$ denote the corresponding point. Then

$$t \longmapsto [C_t]$$

[21] If $g = 2$ or 3, $\mathrm{Sing}(\mathfrak{M}_g) \underset{\neq}{\subset} (\text{Ram.Pts. of } \mathfrak{J}_g \to \mathfrak{M}_g) \subseteq \{C \text{ with automorphisms}\}$. Always $B_1(\mathfrak{M}_g - \mathrm{Sing}\ \mathfrak{M}_g) = 0$, hence $B_1(\mathfrak{M}_g) = B_1(\overline{\mathfrak{M}}_g) = B_1(\mathfrak{M}_g^*) = 0$.

is a morphism
$$\mathbb{P}^1 \longrightarrow \overline{\mathfrak{M}}_g.$$

c) If $g \leq 10$, $\mathfrak{M}_g$ has the much stronger property of being *unirational.* This means equivalently that the field $\mathbb{C}(\mathfrak{M}_g)$ of rational functions is a *subfield* of $\mathbb{C}(t_1, t_2, \dots, t_n)$ for some n or that there is a Zariski-open set $U \subset \mathbb{A}^n$ and a morphism $f : U \to \mathfrak{M}_g$ with dense image[22]. In terms of moduli, $\mathfrak{M}_g$ being unirational means that one can write down a family of curves of genus g depending on parameters $t_1, \dots, t_n$ which can be arbitrary complex numbers satisfying some inequalities $f_i(t) \neq 0$, such that "almost all" curves of genus g appear in the family: e.g., if $g = 2$, take the family
$$y^2 = x^5 + t_1x^4 + t_2x^3 + t_3x^2 + t_4x + t_5$$
and if $g = 3$, take the family
$$\begin{aligned} &y^4 + y^3\left(t_1x + t_2\right) + y^2\left(t_3x^2 + t_4x + t_5\right) + y\left(t_6x^3 + t_7x^2 + t_4x + t_8\right) \\ &+ \left(t_9x^4 + t_{10}x^3 + t_{11}x^2 + t_{12}x + t_{13}\right) = 0. \end{aligned}$$
In fact, if $g \leq 10$ we may use the remarks in Lecture I about realizing curves as plane curves with double points to write down a family of plane curves of degree $d = \left[\frac{2g+8}{3}\right]$ with free parameters almost all of which represent curves of genus g and which include almost all curves of genus g.

Whether more $\mathfrak{M}_g$'s, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if g is quite large. Now consider the hyperbolic tendencies of $\mathfrak{M}_{g,n}$. First of all, we can put 2 types of metric on $\mathfrak{I}_{g,n}$: one of these is the famous *Teichmüller metric* ρ_T. This is a Finsler metric, so it's a bit messy. However, it equals the Kobayashi metric of $\mathfrak{I}_{g,n}$, so all holomorphic maps $f : \Delta \to \mathfrak{I}_{g,n}$ are distance decreasing for ρ_T and the Poincaré metric on Δ : a hyperbolic property. Its unit balls have been determined and are quite amazingly wrinkled and creased: this led Royden to prove the rigidity theorem that if $\dim \mathfrak{I}_{g,n} > 1$, $\Gamma_{g,n} = \mathrm{Aut}\ (\mathfrak{I}_{g,n})$; esp. $\mathfrak{I}_{g,n}$ is not at all a homogeneous domain in $\mathbb{C}^{3g-3+n}$. On the other hand, in this funny metric, $\mathfrak{I}_{g,n}$ is a straight space in the sense of Busemann, i.e., has unique indefinitely prolongable geodesics, but contrary to a conjecture does not have negative curvature in Busemann's sense (this fly in the ointment shows that my general picture is not entirely accurate!). $\mathfrak{I}_{g,n}$ carries another metric ρ_{P-W}, the *Peterson-Weil* metric[23], which is a Kähler metric, hence locally much nicer. Moreover, it has strictly negative Ricci curvature and holomorphic sectional curvatures. In particular, holomorphic maps $f : \Delta \to \mathfrak{I}_{g,n}$ will also be distance decreasing for ρ_{P-W} (suitably normalized) by the Ahlfors-Pick lemma. All the spaces $\mathfrak{M}_{g,n}$ inherit both metrics (with possible singularities where $\alpha : \mathfrak{I}_{g,n} \to \mathfrak{M}^{\Gamma}_{g,n}$ is ramified), and, esp. with ρ_{P-W}, this makes them rather hyperbolic. A closely related hyperbolic property of $\mathfrak{M}_{g,n}$ is:

[22] If this holds, one can assume $n = 3g - 3$ by restricting f_n to $U \cap L$, L a sufficiently general $3g - 3$-dimensional subspace of $\mathbb{A}^n$; hence $\mathbb{C}(\mathfrak{M}_g) \subset \mathbb{C}(t_1, \dots, t_{3g-3})$ with finite index too.

[23] If $n > 0$, more precisely, there is a family of $P - W$ metrics depending on assigning branch numbers σ_i, $2 \leq \sigma_i \leq \infty$, to the base points x_i.

The Rigidity Theorem of Arakelov-Paršin-Manin-Grauert. *(also called the "Šafarevitch-Mordell conjecture in the function field case"). Fix* $g \geq 2$ *and let* C *be any curve,* S *a finite set of points of* C*. Then there are only finitely many families of curves of genus* g *over* $C - S$*, i.e.,*

$$\pi : X \longrightarrow C - S$$

which are "non-constant" (i.e., the fibres $\pi^{-1}(s)$ *not all isomorphic), and if*

$$2(\text{genus } C) - 2 + \#S \leq 0$$

there are none at all; moreover, for each such family there are only finitely many sections, and even for "constant" families, there are only finitely many non-constant sections.

Corollary. *Fix* g, n, C, S *as above. Then there are only finitely many non-constant morphisms*

$$\phi : C - S \longrightarrow \mathfrak{M}_{g,n}$$

which are locally liftable to $\mathfrak{T}_{g,n}$*: i.e., if* $x \in C - S$*, and* $\phi(x)$ *is a ramification point for* $\mathfrak{T}_{g,n} \to \mathfrak{M}_{g,n}$*, one asks that in a small neighborhood of* x*,* ϕ *factor through* $\mathfrak{T}_{g,n}$*.*

A sketch of the proof is given in an appendix below. Finally I want to conclude by giving a conjecture which I am hopeful will very soon be a theorem!

Conjecture. For each g, n, there is a $\Gamma_0 \subset \Gamma_{g,n}$ of finite index such that for all $\Gamma \subset \Gamma_0$, $\mathfrak{M}^{\Gamma}_{g,n}$ is a variety *of general type* in Kodaira's sense.

Here "general type" for a variety X of dimension n means that you compactify X to $\bar{X}$, then blow-up $\bar{X}$ to X^* which is non-singular, and then you look for differential forms of type

$$\omega = a(x)\,(dx_1 \wedge \ldots \wedge dx_n)^k$$

on X^*, with no poles. It means that if k is large enough, you can find $n+2$ such forms whose ratios generate the field of rational functions $\mathbb{C}(X)$ on X. Since on unirational varieties, there are no non-zero differential forms of any type, the conjecture means that for Γ small, $\mathfrak{M}^{\Gamma}_{g,n}$ is more or less the opposite of being unirational.

[Added in 1997 edition] After these notes were written, a remarkable fact was discovered: $\mathfrak{M}_g$ *itself*, for $g \geq 24$, is of general type. In other words, one finds in the sequence of spaces $\mathfrak{M}_g$ this transition from elliptic to hyperbolic. For $g \leq 13$, $\mathfrak{M}_g$ is unirational, $\mathfrak{M}_{15}$ has Kodaira dimension $-\infty$ (no such ω's, any k), $\mathfrak{M}_{23}$ has positive Kodaira dimension (2 such ω's for some k) and general type thereafter.

Appendix: The idea of the proof of rigidity

The proof has 2 steps. The first consists in showing that the set of all families $\pi : X \to C - S$, and the set of all sections $s : C - S \to X$ of families π, *itself consists in a finite number of families.* The second consists in showing that given one $\pi : X \to C - S$ or one section $s : C - S \to X$ of such a π, then one cannot *deform* π or s, i.e., that the only families π or s lie in are 0-dimensional. Since a finite number of 0-dimensional families is just a finite set, we are done.

To carry out the first step, one can use an explicit projective embedding of $\overline{\mathfrak{M}}_{g,n}$, and for all $\phi : C - S \to \mathfrak{M}_{g,n}$ with $\phi(C - S) \neq$ point, extend ϕ to $\overline{\phi} : C \to \overline{\mathfrak{M}}_{g,n}$ and seek a bound on degree $\overline{\phi}(C)$. Then by general results, the set of morphisms $\phi : C - S \to \mathfrak{M}_{g,n}$ with degree $\overline{\phi}(C)$ bounded can be grouped into a finite number of nice families, the parameter space of each of which is some auxiliary variety. Equivalently, this means take a particular ample line bundle L on $\overline{\mathfrak{M}}_{g,n}$ and seek a bound on $c_1(\overline{\phi}^* L)$. (In fact, the nicest line bundle to pick is not quite ample, but near enough to make the proof go through: we will ignore details like this here.) Choosing a nice L, the next step is to identify $\overline{\phi}^* L$ from the geometry of the family $\pi : X \to C - S$ and the section s. One extends the family π of non-singular curves over $C - S$ to a larger family

$$\overline{\pi} : \overline{X} \longrightarrow C$$

over C of curves, some of which have double points (as in the definition of $\overline{\mathfrak{M}}_{g,n}$). Then it turns out that for the most natural L on $\overline{\mathfrak{M}}_{g,0}$,

$$\overline{\phi}^* L \cong \Lambda^g \overline{\pi}_* \left(\widetilde{\Omega}_{\overline{X}/C} \right)$$

where $\widetilde{\Omega}_{\overline{X}/C}$ denotes the line bundle whose sectinons are differential forms on the curves $\pi^{-1}(s)$, i.e., the cotangent bundle to the fibres of π, except that where $\pi^{-1}(s)$ has a double point, the forms may have simple poles with opposite residues at the 2 branches of $\pi^{-1}(s)$ at this double point. If one is dealing with n sections s_i too, hence a morphism $\phi : C - S \to \mathfrak{M}_{g,n}$ with $n > 0$, then $\overline{\phi}^* L$ is a tensor product of powers of this bundle and the line bundles

$$\overline{s}_i^* \widetilde{\Omega}_{\overline{X}/C}$$

where $\overline{s}_i : C \to \overline{X}$ is the extension of s_i. Now, in fact, by using the theory of algebraic surfaces, one gets a very good bound:

$$c_1 \left(\Lambda^g \overline{\pi}_* \left(\widetilde{\Omega}_{\overline{X}/C} \right) \right) \leq \left(q - 1 + \frac{s}{2} \right) (g - g_0)$$

where

$$\begin{aligned} q &= \text{genus } C \\ s &= \# S \\ g_0 &= \text{dimension of biggest abelian variety} \\ &\quad\ \text{which appears in the Jacobian of every} \\ &\quad\ \text{curve } \pi^{-1}(s) \text{ of the family.} \end{aligned}$$

$c_1\left(\overline{s}^*\widetilde{\Omega}_{\overline{X}/C}\right)$ seems harder to bound: I don't know a nice small explicit bound. However, following Grauert one can show that one exists by showing first that the cotangent bundle $\Omega^1_{\overline{X}}$ (of rank 2) is *ample* on almost all fibres $\pi^{-1}(s)$ of $\overline{X}$ over C and then applying general results on ample vector bundles. A good explicit bound here would be very interesting.

To carry out the second step, one applies Kodaira-Spencer-Grothendieck deformation theory to calculate the vector space of *infinitesimal* deformations of $\pi : X \to C - S$ and of $s : C - S \to X$. More precisely, one looks at deformations of $\overline{X}$ such that the map $\pi : \overline{X} \to C$ extends to this deformation and all singular fibres remain concentrated in $\pi^{-1}(S)$. It turns out that:

$$\left\{\begin{array}{l}\text{Space of}\\ \text{infinitesimal}\\ \text{deformations of}\\ \pi : X \longrightarrow C - S\end{array}\right\} \cong H^1\left(\overline{X}, \widetilde{\Omega}^{-1}_{\overline{X}/S}\right)$$

and

$$\left\{\begin{array}{l}\text{Space of}\\ \text{infinitesimal}\\ \text{deformations of}\\ s : C - S \longrightarrow X\end{array}\right\} \cong H^0\left(C, \overline{s}^*\widetilde{\Omega}^{-1}_{\overline{X}/C}\right).$$

To show these vector spaces are (0), one shows – and this is Arakelov's deepest contribution – that $\widetilde{\Omega}_{\overline{X}/C}$ is an ample line bundle on $\overline{X}$. Then the first space is (0) by Kodaira's Vanishing Theorem, and the second space is (0) because the line bundle has negative degree. Amazingly, Arakelov's proof here involves studying the curve $D \subset \overline{X}$ such that

$$D \cap \pi^{-1}(s) \doteq \text{ the Weierstrass points of } \pi^{-1}(s)$$

and identifying via differentials the line bundle on $\overline{X}$ of which D is a section.

Lecture III: How Jacobians and Theta Functions Arise

I would like to begin by introducing Jacobians in the way that they actually were discovered historically. Unfortunately, my knowledge of 19th-century literature is very scant so this should not be taken too literally. You know the story began with Abel and Jacobi investigating general algebraic integrals

$$I = \int f(x)dx$$

where f was a multi-valued algebraic function of X, i.e., the solution to

$$g\big(x, f(x)\big) \equiv 0, \qquad g \text{ polynomial in 2 variables.}$$

So we can write I as

$$I = \int_\gamma y\ dx$$

where γ is a path in plane curve $g(x,y) = 0$; or we may reformulate this as the study of integrals

$$I(a) = \int_{a_0}^{a} \overbrace{\frac{P(x,y)}{Q(x,y)}\ dx}^{\omega}\ , \qquad \begin{array}{l} P, Q \text{ polynomials} \\ a, a_0 \in \text{ plane curve } C : g(x,y) = 0 \end{array}$$

of *rational* differentials ω on plane curves C. The main result is that such integrals always admit an addition theorem: i.e., there is an integer g such that if a_0 is a base point, and $a_1, \ldots, a_{g+1}$ are any points of C, then one can determine up to permutation $b_1, \ldots, b_g \in C$ rationally in terms of the a's[24] such that

$$\int_{a_0}^{a_1} \omega + \ldots + \int_{a_0}^{a_{g+1}} \omega \equiv \int_{a_0}^{b_1} \omega + \ldots + \int_{a_0}^{b_g} \omega, \text{ mod periods of } \int \omega.$$

For instance, if $C = \mathbb{P}^1$, $\omega = dx/x$, then $g = 1$ and:

$$\int_1^{a_1} \frac{dx}{x} + \int_1^{a_2} \frac{dx}{x} = \int_1^{a_1 a_2} \frac{dx}{x}.$$

Iterating, this implies that for all $a_1, \ldots, a_g, b_1, \ldots, b_g \in C$, there are $c_1, \ldots, c_g \in C$ depending up to permutation rationally on the a's and b's such that

[24] E.g., one can find polynomials $g_i(x, y; a)$ in x, y and the coordinates of the a's such that the b_i's are the set of all $b \in c$ such that $g_i(b; a) = 0$.

$$\sum_{i=1}^{g} \int_{a_0}^{a_i} \omega + \sum_{i=1}^{g} \int_{a_0}^{b_i} \omega \equiv \sum_{i=1}^{g} \int_{a_0}^{c_i} \omega \qquad \text{(mod periods)}.$$

Now this looks like a group law! Only a very slight strengthening will lead us to a reformulation in which this most classical of all theorems will suddenly sound very modern. We introduce the concept of an algebraic group G: succinctly, this is a "group object in the category of varieties," i.e., it is simultaneously a variety and a group where the group law $m : G \times G \to G$ and the inverse $i : G \to G$ are morphisms of varieties. Such a G is, of course, automatically a complex analytic Lie group too, hence it has a Lie algebra $\mathrm{Lie}(G)$, and an exponential map $\exp : \mathrm{Lie}(G) \to G$. Now I wish to rephrase Abel's theorem as asserting that if C is a curve, and ω is any rational differential on C, then the multi-valued function

$$a \longmapsto \int_{a_0}^{a} \omega$$

can be factored into a composition of 3 functions:

$$C - (\text{poles of } \omega) \xrightarrow{\phi} J \xleftarrow{\exp}\!\!\!\!\leftarrow \mathrm{Lie}\, J \xrightarrow{\ell} \mathbb{C}$$

where:

i) J is a commutative algebraic group,
ii) ℓ is a *linear* map from Lie J to $\mathbb{C}$
iii) ϕ is a morphism of varieties; and, in fact, if $g = \dim J$, then if we use addition on J to extend ϕ to

$$\phi^{(g)} : [(C\text{-poles } \omega) \times \ldots \times (C\text{-poles } \omega) / \underset{S_g}{\text{permutations}}] \longrightarrow J$$

then $\phi^{(g)}$ is birational, i.e., is bijective on a Zariski-open set.

In our example

$$C = \mathbb{P}^1, \qquad \omega = dx/x,$$

then $J = \mathbb{P}^1 - (0, \infty)$ which is an algebraic group where the group law is multiplication, and ϕ is the identity. The point is that J is the object that realizes the rule by which 2 g-tuples $(a_1, \ldots, a_g)$, $(b_1, \ldots, b_g)$ are "added" to form a third $(c_1, \ldots, c_g)$, and so that the integral $\sum_{i=1}^{g} I(x_i)$ becomes a homomorphism from J to $\mathbb{C}$. A slightly less fancy way to put it is that there is a $\phi : C\text{-(poles } \omega) \to J$ and a *translation-invariant* differential η on J such that

$$\phi^* \eta = \omega,$$

hence

$$\int_{\phi(a_0)}^{\phi(a)} \eta \equiv \int_{a_0}^{a_0} \omega \qquad \text{(mod periods)}.$$

Among the ω's, the most important are those of 1st kind, i.e., without poles, and if we integrate all of them at once, we are led to the most important J of all: the *Jacobian*, which we call Jac. From property (iii), we find that Jac must be a *compact* commutative algebraic group, i.e., a complex torus, and we want that

$$\phi : C \longrightarrow \text{Jac}\ ,$$

should set up a bijection:

iv) $$\phi^* : \begin{bmatrix} \text{translation}- \\ \text{invariant 1-forms} \\ \eta \ \text{ on Jac} \end{bmatrix} \longrightarrow \begin{bmatrix} \text{rational differentials} \\ \omega \text{ on } C \ w/o \text{ poles} \end{bmatrix} = R_1(C).$$

Thus

$$\begin{aligned} \dim\ \text{Jac} &= \dim R_1(C) \\ &= \text{genus } g \text{ of } C. \end{aligned}$$

To construct Jac explicitly, there are 2 simple ways:

v) *Analytically:* write Jac $= V/L$, V complex vector space, L a lattice. Define:

$$\begin{aligned} V &= \text{dual of } R_1(C) \\ L &= \left\{ \begin{array}{l} \text{set of } \ell \in V \ \text{ obtained as periods, i.e.,} \\ \ell(\omega) = \int_\gamma \omega \ \text{ for some 1-cycle } \gamma \text{ on } C \end{array} \right. . \end{aligned}$$

Fixing a base point $a_0 \in C$, define for all $a \in C$

$$\phi(a) = \left\{ \begin{array}{c} \text{image in } V/L \text{ of any } \ell \in V \ \text{ defined by} \\ \ell(\omega) = \int_{a_0}^{a} \omega, \\ \text{where we fix a path from } a_0 \text{ to } a \end{array} \right. .$$

Note that since Jac is a group,

$$V^* \cong \begin{pmatrix} \text{translation} - \text{invariant} \\ \text{1-forms on Jac} \end{pmatrix} \cong \begin{pmatrix} \text{cotangent sp. to Jac at } \alpha \\ \text{any } \alpha \in \ \text{Jac} \end{pmatrix} \cong R_1(C).$$

vi) *Algebraically:* following Weil's original idea, introduce $S^gC = C \times \ldots \times C/S_g$ and construct by the Riemann-Roch theorem, a "group-chunk" structure on S^gC, i.e., a partial group law:

$$\begin{aligned} &m : U_1 \times U_2 \longrightarrow U_3 \\ &U_i \subset S^gC \ \text{ Zariski-open.} \end{aligned}$$

He then showed that any such algebraic group-chunk prolonged automatically into an algebraic group J with $S^gC \supset U_4 \subset J$ (some Zariski-open U_4).

An important point is that ϕ is an integrated form of the canonical map Φ : $C \to \mathbb{P}^{g-1}$ discussed at length above –

vii) Φ is the Gauss map of ϕ, i.e., for all $x \in C$, $d\phi(T_{x,C})$ is a 1-dimensional subspace of $T_{\phi(x),\mathrm{Jac}}$, and by translation this is isomorphic to Lie(Jac). If $\mathbb{P}^{g-1}$ = [space of 1-dimensional subsp. of Lie(Jac)], then $d\phi : C \to \mathbb{P}^{g-1}$ is just Φ.
(Proof: this is really just a rephrasing of (iv).)

The Jacobian has always been the corner-stone in the analysis of algebraic curves and compact Riemann surfaces. Its power lies in the fact that it *abelianizes* the curve and is a *reification* of H_1, e.g.,

viii) Via $\phi : C \to \mathrm{Jac}$, every abelian covering $\pi : C_1 \to C$ is the "pull-back" of a unique covering $p : G_1 \to \mathrm{Jac}$ (i.e., $C_1 \cong C \underset{\mathrm{Jac}}{\times} G_1$).

Weil's construction in vi) above was the basis of his epoch-making proof of the Riemann Hypothesis for curves over finite fields, which really put characteristic p algebraic geometry on its feet.

There are very close connections between the geometry of the curve C (e.g., whether or not C is hyperelliptic) and Jac. We want to describe these next in order to tie in Jac with the special cases studied in Lecture I, and in order to "see" Jac very concretely in low genus. The main tool we want to use is:

Abel's Theorem. *Given $x_1, \ldots, x_k, y_1, \ldots, y_k \in C$, then*

$$\left\{\begin{array}{l} \exists \text{ rational function } f \\ \text{on } C \text{ with} \\ (f) \underset{\text{def}}{=} (\text{zeroes of } f) - (\text{poles of } f) \\ \quad = \Sigma x_i - \Sigma y_i \end{array}\right\} \Longleftrightarrow \sum_{i=1}^{k} \phi(x_i) = \sum_{i=1}^{k} \phi(y_i).$$

When this holds, we say $\Sigma x_i \equiv \Sigma y_i$, or $\Sigma x_i, \Sigma y_i$ are *linearly equivalent.* For instance, when $C = \mathbb{P}^1$, any 2 points a, b are linearly equivalent via the function

$$f(x) = \frac{x-a}{x-b}.$$

For every k, we consider the map:

$$\begin{array}{ccc} \overbrace{C \times \ldots \times C}^{k \text{ times}} & \longrightarrow & \mathrm{Jac} \\ (x_1, \ldots, x_k) & \longmapsto & \sum_{i=1}^{k} \phi(x_i). \end{array}$$

If $S^k C$ denotes C^k divided by permutations, i.e., the k^{th} symmetric power of C, then this map factors via

$$\phi^{(k)} : S^k C \longrightarrow \mathrm{Jac}\ .$$

Define

$$W_k = \mathrm{Im}\ \phi^{(k)}, \qquad 1 \le k \le g-1$$

($\phi^{(k)}$ surjective if $k \geq g$). The fibres of this map[25] are called the *linear systems* on C of degree k, and by Abel's theorem they are the equivalence classes under linear equivalence and can be constructed as follows:

a) Pick one point $\mathfrak{A} = \sum_{i=1}^{k} x_i \in S^k C$.
b) Let

$$L(\mathfrak{A}) = \left\{ \begin{array}{l} \text{v. sp. of fcns. } f \text{ on } C \text{ with } (f) + \mathfrak{A} \geq 0, \text{ i.e.,} \\ \text{poles only at } x_i, \text{ order bounded by mult. of } x_i \\ \text{in } \mathfrak{A} \end{array} \right\}.$$

c) Let $|\mathfrak{A}| = \{\text{set of divisors } \Sigma y_i = (f) + \Sigma x_i, \ \ f \in L(\mathfrak{A}), \ \ f \neq 0\} \subset S^k C$.
d) Then $|\mathfrak{A}| = \phi^{(k)-1}\left(\phi^{(k)}(\mathfrak{A})\right)$. Note that it follows
$|\mathfrak{A}| \cong$ *projective space* of 1-dimensional subspaces of $L(\mathfrak{A})$.
e) We also want to use the Riemann-Roch theorem that tells us that

$$\dim |\mathfrak{A}| = k - g + i$$

where

$$i = \left\{ \begin{array}{ll} \dim \ \text{ of v.sp. } R_1(-\mathfrak{A}) & \text{of differentials} \\ \omega \in R_1(C), & \text{with } \textit{zeroes} \text{ on } \mathfrak{A} \end{array} \right\}.$$

Now let's look at low genus cases:

$\boxed{g = 0 :}$ Jac $= (0)$

$\boxed{g = 1 :}$ (a) $\phi : C \to \text{Jac}$ is an isomorphism, i.e., $C = \text{Jac}$. In fact, for any genus $g \geq 1$,

$$\phi^{(1)} : C \to \text{ Jac}$$

is an *embedding*, hence an isomorphism of C with W_1. (*Proof*: the fibres of $\phi^{(k)}$ being $\mathbb{P}^n$'s, $\phi^{(1)}$ would be either an embedding or C itself would be $\mathbb{P}^1$.)

(b) If $k \geq 2$,

$$\phi^{(k)} : S^k C \longrightarrow \text{ Jac}$$

makes $S^k C$ into a $\mathbb{P}^{k-1}$-bundle over Jac, whose fibres are the linear systems of degree k. In general, if $k > 2g - 2$,

$$\phi^{(k)} : S^k C \longrightarrow \text{ Jac}$$

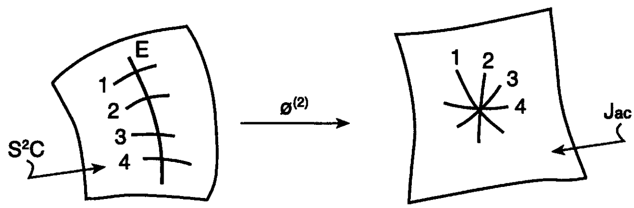
makes $S^k C$ into a $\mathbb{P}^{k-g}$-bundle over Jac. (*Proof*: This is a consequence of the Riemann-Roch theorem since no differential can have more than $2g - 2$ zeroes.)

[25] A technical aside: the complete ideal of functions on $S^k C$ vanishing on $\phi^{(k)-1}(\alpha)$ is generated by the functions on Jac vanishing at α – this is needed to make rigorous some of the points made below.

$\boxed{g = 2:}$ The interesting case is $1 < k \leq 2g - 2$, i.e., $k = 2$: the map

$$\phi^{(2)} : S^2C \longrightarrow \text{Jac}.$$

Recall that there is a degree 2 map $\pi : C \to \mathbb{P}^1$. Since the points of $\mathbb{P}^1$ are all linearly equivalent to each other, the degree 2 cycles $\pi^{-1}(x)$ are also linearly equivalent. This gives us a copy E of $\mathbb{P}^1$ inside S^2C. The result is that Jac is isomorphic to the quotient of S^2C after identifying all points of E; i.e., that Jac is obtained by "blowing down" $E \subset S^2C$. Here is a picture:



where, as is customary in the theory of algebraic surfaces, we draw real 2-dimensional manifolds in place of manifolds of 2 complex dimensions, which are 4 real-dimensional, hence undrawable! Going backwards, we may say that S^2C is obtained from Jac by "blowing up" $e = \phi^{(2)}(E)$: this is a process applicable to any variety X that replaces one of its points x by the set of tangent lines to X at x, giving you a new variety $B_x(X)$ birational to the first. We see here clearly that if we take the group law $m:$ Jac $\times$ Jac $\to$ Jac and try to transfer it to S^2C, we get merely a group chunk as in Weil's treatment because of E.

$\boxed{g = 3:}$ Consider first $k = 3$:

$$\phi^{(3)} : S^3C \longrightarrow \text{Jac}.$$

For any $x \in C$, consider the differentials ω on C zero at x: they form a 2-dimensional vector space and have 3 zeroes besides x. These zeroes form a degree 3 cycle, and as ω varies all these are linearly equivalent (use the functions ω_1/ω_2): this gives us a copy E_x of $\mathbb{P}^1$ in S^3C. It turns out:

$$\text{Jac} \cong (S^3C \text{ modulo collapsing each } E_x \text{ to a point }),$$

or putting it backwards if $\gamma =$ locus of points $\phi^{(3)}(E_x)$, then

$$S^3C \cong (\text{ Jac, with a curve } \gamma \subset \text{ Jac, isom. to } C\text{, blown up}).$$

Most interesting is the case $k = 2$:

$$\phi^{(2)} : S^2C \twoheadrightarrow W_2 \subset \text{Jac}.$$

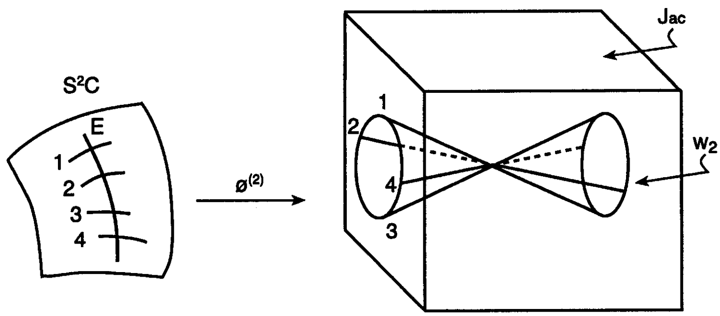
Then if C is *not hyperelliptic*, there are no non-trivial degree 2 linear systems, so

$$S^2C \xrightarrow{\approx} W_2.$$

But if C is *hyperelliptic*, you get one degree 2 linear system as in the $g = 2$ case, so

$$W_2 \cong (S^2C \text{ with a copy } E \text{ of } \mathbb{P}^1 \text{ blown down}).$$

The image $e \in W_2$ of E is a now double point and it looks like this:



$\boxed{g = 4 :}$ In this case, I want to consider because of its importance in Lecture IV only the case $k = 3$:

$$\phi^{(3)} : S^3C \twoheadrightarrow W_3 \subset \text{ Jac}.$$

We mentioned briefly in Lecture I that either a) C is hyperelliptic, or b) C was an intersection in $\mathbb{P}^3$ of a quadric F and a cubic G. Now we also distinguish b_1) F singular, hence a quadric cone, and b_2) F non-singular. b_2) is the most common case. Using the 2 rulings on a non-singular quadric, it is a standard fact that such a quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Thus $C \cong \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \cap G$, and since G is a *cubic*, C meets the curves $\left(\mathbb{P}^1 \times \text{ pt.}\right)$ or $\left(\text{pt. } \times \mathbb{P}^1\right)$ in 3 points. Thus the 2 projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ induce 2 maps π_1, π_2 from C to $\mathbb{P}^1$ of degree 3. The 2 families of degree 3 cycles $\left\{\pi_1^{-1}(x)\right\}$ and $\left\{\pi_2^{-1}(x)\right\}$ form 2 linear systems $E_1, E_2 \subset S^3C$, with $E_1 \cong E_2 \cong \mathbb{P}^1$. Then:

case b_2: $$W_3 \cong \begin{pmatrix} S^3C \text{ with } E_1, E_2 \text{ blown down to} \\ 2 \text{ points } e_1, e_2 \end{pmatrix}$$

and

$$e_1, e_2 = \text{ ordinary double points of } W_3.$$

In case b_1, the 2 rulings "come together"; in fact, S^3C contains only one non-trivial linear system E, and *case* b_1: $\qquad W_3 \cong \left(S^3C \text{ with } E \text{ blown down to } e\right)$

and

$e =$ higher double point of W_3.

In the hyperelliptic case a, it turns out that there is a whole curve of linear systems $E_x \subset S^3C$ depending on a point $x \in C$: in fact, take the degree 2 linear system, and just add x to each of its members. Thus

case a: $$W_3 \cong \begin{pmatrix} S^3C \text{ with the surface } \cup E_x \text{ blown} \\ \text{down to a curve } \gamma \text{ isomorphic to } C \end{pmatrix}$$

and

$\gamma =$ double curve of W_3.

Enough examples: the moral is that W_k's and their singularities display like an illustrated book the vagaries of the curve C from which they arise. The general result is the following:

Theorem (Riemann-Kempf). *Let $\alpha \in W_k \subset$ Jac, let $L = \phi^{(k)-1}(\alpha) \subset S^kC$ and suppose $L \cong \mathbb{P}^\ell$. Then W_k has a singularity at α of multiplicity $\binom{g-k+\ell}{\ell}$, and the tangent cone to W_k inside $T_{\alpha,\,\mathrm{Jac}}$ (= tangent sp. to* Jac *at α) is equal to:*

$$\bigcup_{\mathfrak{A}\in L} D\phi^{(k)}\left(T_{\mathfrak{A},S^kC}\right).$$

Here $D\phi^{(k)}$ is the differential of $\phi^{(k)}$ and it gives rise to an exact sequence:

$$(*) \qquad 0 \longrightarrow T_{\mathfrak{A},L} \longrightarrow T_{\mathfrak{A},S^kC} \xrightarrow{D\phi^{(k)}} T_{\alpha,\,\mathrm{Jac}}.$$

In fact, this sequence actually "displays" the Riemann-Roch formula in a beautiful way: using the fact that

$$\begin{aligned} R_1(C) &\cong \text{ translation-invariant differentials on Jac} \\ &\cong T^*_{\alpha,\,\mathrm{Jac}} \ (= \text{ cotangent sp. to Jac at } \alpha), \text{ for all } \alpha, \end{aligned}$$

it is not hard to check that if $\omega \in R_1(C)$ corresponds to $[\omega] \in T^*_{\alpha,\,\mathrm{Jac}}$, then:

$$\left(D\phi^{(k)}\right)^*[\omega] = 0 \text{ in } T_{\mathfrak{A},S^kC} \Longleftrightarrow \omega \text{ is zero on } \mathfrak{A}.$$

Therefore

$$\text{Coker } D\phi^{(k)} \cong \text{ dual of } R_1(-\mathfrak{A}), \text{ the space of differentials zero on } \mathfrak{A}.$$

Therefore counting the dimensions of the vector spaces in $(*)$:

$$\begin{aligned} \dim L &= \dim S^kC - \dim \mathrm{Jac} + \dim \mathrm{Coker}\, D\phi^{(k)} \\ &= k - g + \dim R_1(-\mathfrak{A}), \end{aligned}$$

which is the Riemann-Roch theorem! What comes next is going to be harder to follow, but we can go much further: let $\{\omega_i\}$ be a basis of $R_1(-\mathfrak{A})$ and let $\{f_j\}$ be a basis of $L(\mathfrak{A})$. Then a general member of L is given by

$$\mathfrak{A}_t = \mathfrak{A} + \left(\sum_{j=0}^{\ell} t_j f_j\right)$$

and a basis of $R_1(-\mathfrak{A}_t)$ is given by $\left(\sum_{j=0}^{\ell} t_j f_j\right) \cdot \omega_i$. Therefore $\sum_{j=0}^{\ell} t_j [f_j \omega_i]$ span the dual of the cokernel of

$$T_{\mathfrak{A}_t, S^k C} \xrightarrow{D\phi^{(k)}} T_{\alpha,\mathrm{Jac}},$$

or $\sum_{j=0}^{\ell} t_j [f_j \omega_i] = 0$ are linear equations on $T_{\alpha,\mathrm{Jac}}$ which define the subspace $D\phi^{(k)}\left(T_{\mathfrak{A}_t,S^kC}\right)$. It follows that if we put together a big $(\ell+1)\times(g-k+\ell)$-matrix of linear functions on $T_{\alpha,\mathrm{Jac}}$ out of $[f_j\omega_i]$, then all its $(\ell+1)\times(\ell+1)$ minors vanish on each $D\phi^{(k)}\left(T_{\mathfrak{A}_t,S^kC}\right)$, hence vanish on the whole tangent cone to W_k. Kempf proved that these equations suffice, and that W_k itself has equations of this type:

Theorem (Kempf). *There is a $(\ell+1)\times(g-k+\ell)$-matrix of holomorphic functions (f_{ij}) on* Jac *near α such that W_k is the set of zeroes of all its $(\ell+1)\times(\ell+1)$ minors: i.e., W_k is a* determinantal variety. *Moreover, $[f_j\omega_i]$ = linear term of f_{ij} and the tangent cone to W_k is the set of zeroes of the $(\ell+1)\times(\ell+1)$-minors of the matrix $[f_j\omega_i]$ of linear functions.*

The feature of the Jacobian, however, which really gives it its punch is the theta function. There are 3 very good reasons to look next at the function theory of Jac –

a) to define projective embeddings of Jac, hence understand better the algebraic structure, moduli, etc.
b) because Jac is a group, one hopes that its function theory will reflect this in interesting ways,
c) by pull-back, functions on Jac will define functions on S^gC, hence on C, and may give a good way to expand functions on C, prove the Riemann-Roch theorem, etc.

So write

$$\mathrm{Jac} = \mathbb{C}^g/L.$$

Instead of constructing L-periodic meromorphic functions f on $\mathbb{C}^g$, one seeks L-automorphic entire functions f, i.e.,

$$f(x+\alpha) = e_\alpha(x) \cdot f(x), \qquad \alpha \in L, \ x \in \mathbb{C}^g$$
$$\{e_\alpha\} = \text{“automorphy factor”}^{26}$$

Equivalently, such f are holomorphic sections of a line bundle $L_{\{e_\alpha\}}$ on Jac and clearly the quotient of 2 such f is always L-periodic. The simplest choice of $\{e_\alpha\}$ is something in the general form:

$$e_\alpha(x) = e^{\beta B(x,\alpha)+c(\alpha)}, \qquad \beta \text{ bilinear.}$$

(e_α = (constant) is too simple, because no f's will exist.)

Now if $g \geq 2$, most complex tori $\mathbb{C}^g/L$ have no non-constant meromorphic functions on them at all, and are not algebraic varieties, and do not carry any but "trivial" $\{e_\alpha\}$'s[27]. In the case of a curve C, however, special things happen; let's look for bilinear forms as candidates for B. We saw above that on $R_1(C)$ one has a positive definite Hermitian form:

$$(\omega_1, \omega_2) = \int_C \omega_1 \wedge \overline{\omega}_2$$

hence its dual, which is the universal covering space $\mathbb{C}^g$ of Jac, gets a Hermitian form that we will write H. But also $H_1(C,\mathbb{Z})$ carries an integral skew-symmetric form

$$E : H_1(C,\mathbb{Z}) \times H_1(C,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

given by intersection pairing. As we saw in (v) above, there is an isomorphism $H_1(C,\mathbb{Z}) \cong L$, hence L carries such an E. It is not hard to show that H and E are connected by:

$$(*) \qquad E(x_1,x_2) = \operatorname{Im} H(x_1,x_2), \qquad \text{all } x_1, x_2 \in L,$$

and that when $(*)$ holds there is a (nearly canonical[28]) choice of $\{e_\alpha\}$, viz.

$$e_\alpha(x) = \pm e^{\pi[H(x,\alpha)+\frac{1}{2}H(\alpha,\alpha)]}.$$

Moreover, one has the beautiful theorem:

---

[26] i.e., entire functions on $\mathbb{C}^g$, nowhere zero, such that

$$e_{\alpha+\beta}(x) \equiv e_\alpha(x+\beta) \cdot e_\beta(x).$$

[27] $\{e_\alpha\}$ is trivial if $e_\alpha(x) \equiv e(x+\alpha)/e(x)$ for some e.

[28] The sign $\pm$ is not canonical; it satisfies some funny identities that I don't want to discuss; any 2 choices, however, are related by a transformation

$$e'_\alpha(x) = \ell(\alpha)e_\alpha(x), \qquad \ell \in \operatorname{Hom}(L/2L, \pm 1).$$

Theorem. *The existence of a positive definite Hermitian H on $\mathbb{C}^g$ and an integral skew-symmetric E on L satisfying $E = \operatorname{Im} H$ is necessary and sufficient for a complex torus $\mathbb{C}^g/L$ to carry g algebraically independent meromorphic functions and if it has such functions, it admits an embedding into $\mathbb{P}^n$, some n, hence is a projective variety*[29].

Here we see the principle emerging that a complex torus does not fit easily in $\mathbb{P}^n$: non-trivial identities $(*)$ are required before it will fit at all. Now define a *theta-function*[30] *of order n* to be an entire function f on $\mathbb{C}^g$ such that

$$f(x+\alpha) = \left(\pm e^{\pi[H(x,\alpha)+\frac{1}{2}H(\alpha,\alpha)]}\right)^n \cdot f(x),$$

and let S_n be the space of such f. Then $S = \sum S_n$ is a graded ring. Elementary Fourier analysis combined with the fact that E is a unimodular pairing leads to

$$(**) \qquad \dim S_n = n^g, \qquad (n \geq 1).$$

In particular, there is exactly one first order theta-function, up to scalars. This important function, written $\vartheta(x)$, is called *Riemann's theta function*[31]. If, instead, we take any $n \geq 3$, and let $\psi_1, \ldots, \psi_{n^g}$ be a basis of S_n, we get:

Lefschetz's embedding theorem. *$\mathbb{C}^g/L$ is embedded in $\mathbb{P}^{n^g-1}$ by*

$$x \longmapsto (\psi_1(x), \ldots, \psi_{n^g}(x)) = \Psi_n(x).$$

This makes sense because ψ_i/ψ_j are single-valued functions on $\mathbb{C}^g/L$. This solves problem (a) raised above. (b) however is even more remarkable. In fact, to introduce the group structure into the picture, for all $\beta \in \mathbb{C}^g$, define

$$(T_\beta f)(x) = f(x+\beta).$$

For all nowhere zero holomorphic functions e on $\mathbb{C}^g$, define

---

29 By Chow's theorem, if you embed it in projective space at all, the image is projective a variety; and if you embed it in 2 ways, the 2 projective varieties are isomorphic algebraically as well as analytically.

30 Since this is not *exactly* the classical definition, let me indicate the connection. Classically, one splits $L = L_1 + L_2$, when $L_i \cong \mathbb{Z}^g$ and $E(x_1, x_2) = 0$ if x_1, x_2 are *both* in L_1 or *both* in L_2. For all $\alpha \in L_2$, define a *complex* linear $\ell_\alpha : \mathbb{C}^g \to \mathbb{C}$ by $\ell_\alpha(x) = E(x, \alpha)$ if $x \in L_1$. Require instead

$$\begin{aligned} f(x+\alpha) &= f(x), \qquad \alpha \in L_1 \\ f(x+\alpha) &= e^{2\pi i n(\ell_\alpha(x)+\frac{1}{2}\ell_\alpha(\alpha))} f(x), \qquad \forall \alpha \in L_2. \end{aligned}$$

Then these f's differ from the other f's by an elementary factor.

31 In the classical normalization, it is

$$\vartheta(x) = \sum_{\alpha \in L_2} e^{-2\pi i(\ell_\alpha(x)+\frac{1}{2}\ell_\alpha(\alpha))}.$$

$$(U_e f)(x) = e(x) f(x).$$

Then refining the analysis leading to (**), one finds

Lemma. *i) $\forall \beta \in \mathbb{C}^g$, there exists e such that $U_e T_\beta S_n = S_n$ if and only if $\beta \in \frac{1}{n} L$.*

ii) Choosing such an $e(\beta)$ for each $\beta \in \frac{1}{n} L$, $\beta \mapsto U_{e(\beta)} \cdot T_\beta$ defines a projective representation[32] *of $\frac{1}{n} L/L$ on S_n: this representation is* irreducible.

It seems to me remarkable that although $\mathbb{C}^g/L$ is an abelian group, its function-theory is full of irreducible representations of dimensions bigger than one: in fact, these are ordinary representations of a finite 2-step nilpotent group $\mathfrak{G}_n$:

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathfrak{G}_n \longrightarrow \frac{1}{n} L/L \longrightarrow 1$$

analogous to the nilpotent Lie group:

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{D} \longrightarrow V \oplus \hat{V} \longrightarrow 1 \qquad (V = \text{ real vector space})$$

whose Lie algebra is the Heisenberg commutation relations[33]. This rather easy lemma has lots of consequences.

Corollary. *i) In the embedding Ψ_n, translation by β on $\mathbb{C}^g/L$ extends to a linear transformation $\mathbb{P}^{n^g-1} \to \mathbb{P}^{n^g-1}$ if and only if $\beta \in \frac{1}{n} L/L$.*

ii) Modulo the choice of distinguished generators for the associated finite group $\mathfrak{G}_n$, we get, up to scalars, a distinguished basis of S_n, hence a normalization of

$$\Psi_n : \mathbb{C}^g/L \longrightarrow \mathbb{P}^{n^g-1}$$

under projective transformations. In this normalization, translations by $\beta \in \frac{1}{n} L/L$ acts on $\Psi_n(\mathbb{C}^g/L)$ by a simple set of explicit $n^g \times n^g$-matrices.

To be more explicit, start with $\vartheta \in S_1$. Choose $\phi : \mathbb{Z}^g \times \mathbb{Z}^g \xrightarrow{\approx} L$ such that

$$E(\phi(n,m), \phi(n',m')) = \langle n, m' \rangle - \langle m, n' \rangle,$$

and extend ϕ to $\mathbb{Q}^g \times \mathbb{Q}^g \xrightarrow{\approx} L \otimes \mathbb{Q}$. Then if $n = m^2$, a typical distinguished basis of S_{m^2} is of the form

---

[32] I.e., $g \mapsto U_g$ is a projective representation of G if

$$U_{g_1 g_2} = \text{ const. } U_{g_1} \cdot U_{g_2},$$

all $g_1, g_2 \in G$.

[33] $\mathfrak{D}$ = set of triples (α, x, ξ), $\alpha \in \mathbb{R}$, $x \in V$, $\xi \in \hat{V}$, with group law:

$$(\alpha, x, \xi) \cdot (\alpha', x', \xi') = (\alpha + \alpha' + \langle x, \xi' \rangle, x + x', \xi + \xi').$$

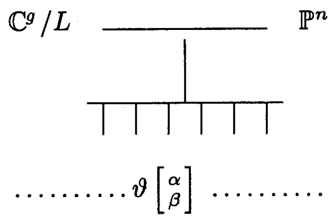
$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(x)=\begin{bmatrix}\text{exponential}\\ \text{of suitable}\\ \text{linear fcn.}\end{bmatrix}\cdot\vartheta(mx+\phi(\alpha,\beta))$$

where α, β range over coset representatives of $\frac{1}{m}\mathbb{Z}^g$ modulo $\mathbb{Z}^g$. Thus

$$x\longmapsto\left(\ldots\ldots,\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(x),\ldots\ldots\right)$$

is the normalized projective embedding of $\mathbb{C}^g/L$. The most important point here is that while translations by $\beta\in\frac{1}{n}L/L$ are normalized, $\Psi_n(0)$ is *not normalized.* Hence $\Psi_n(0)=\left(\ldots,\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0),\ldots\right)$ is an *invariant* of the torus $\mathbb{C}^g/L$ and the distinguished generators of $\mathfrak{G}_n$. These $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0)$ are classically called the *theta-null werte.* We will discuss their role as moduli at more length in Lecture IV.

Summarizing this discussion, you can say that you can take a) $\mathbb{C}^g/L$, and b) $\mathbb{P}^n$: both innocent *homogeneous* complex manifolds. You marry them via Ψ_n and the children they produce are these highly unsymmetric and intricate functions $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$:



We pass on now to problem (c): when $\mathbb{C}^g/L$ is the Jacobian of C, pull back functions on $\mathbb{C}^g/L$ to C and see what you get. We follow Riemann and consider the basic functions:

$$E_e(x,y)=\vartheta\left(\int_y^x\omega-\mathbf{e}\right)$$

where $\mathbf{e}\in\mathbb{C}^g$, and $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_g)$, ω_i the basis of $R_1(C)$ (recall that ϑ is naturally a function on the dual space to $R_1(C)$, and we have identified this space with $\mathbb{C}^g$; so $\{\omega_i\}$ is the dual basis of $R_1(C)$). For fixed y and $\mathbf{e}$, this is a multi-valued function on C that changes by a multiplicative factor

$$\mathrm{e}^{\left(\int_y^x\omega+\text{ const.}\right)}$$

when analytically continued around a cycle. Riemann showed that when not identically zero it had exactly g zeroes $z_1,\ldots,z_g$ and that there was a point $\Delta\in$ Jac (depending on the choice of sign $\pm$ in the definition of the automorphy factor $\{e_\alpha\}$ for ϑ) such that in Jac:

$$\sum_{i=1}^g\phi(z_i)=\phi(y)+\mathbf{e}+\Delta.$$

In fact, we saw that

$$\phi^{(g)} : S^g C \longrightarrow \text{Jac}$$

was birational, and this shows that if $\mathbf{e}_0 = \phi(y) + \Delta$ we get an *inverse* to $\phi^{(g)}$ almost everywhere by

$$\mathbf{e} \longmapsto g\text{-tuple of zeroes of } \vartheta\left(\int_y^x \omega - \mathbf{e} + \mathbf{e}_0\right).$$

Moreover, we find:

$$\vartheta(\mathbf{e}) = 0 \Longleftrightarrow E_e(y) = 0 \Longleftrightarrow \begin{array}{l}\text{some } z_i\\ \text{equals } y\end{array} \Longleftrightarrow \begin{array}{l}\exists z_1, \ldots, z_{g-1} \in C\\ \text{such that } \sum_{i=1}^{g-1} \phi(z_i) = \Delta + \mathbf{e}\end{array}.$$

This means that if we define the codimension 1 subset $\Theta \subset$ Jac by

$$\Theta = \{x \in \text{ Jac} \mid \vartheta(x) = 0\},$$

then, up to a translation, Θ is just W_{g-1}! This is the basic link between theta-functions and the geometry discussed earlier. Moreover, if we fix $\mathbf{e}$ such that $\vartheta(\mathbf{e}) = 0$, and fix $z_1, \ldots, z_{g-1} \in C$ such that $\sum \phi(z_i) = \Delta + \mathbf{e}$, then consider the function $E_e(x, y)$ for variables x and y. It follows that so long as $x \mapsto E_e(x, y)$ is not identically 0, its zeroes are $x = y$ and $x = z_1, \ldots, z_{g-1}$, i.e., ignoring certain bad points z_i independent of y, $E_e(x, y)$ is a *Prime Form* as a function of x: has a unique variable zero at y. Using this, we can show that every rational function f on C has a *unique factorization*:

If

$$\begin{aligned} a_i &= \text{zeroes of } f \\ b_i &= \text{poles of } f \end{aligned}$$

then

$$f(x) = e^{\int^x \omega} \cdot \frac{\prod_i E_e(x, a_i)}{\prod_i E_e(x, b_i)}$$

(for some $\omega \in R_1(C)$).

This beautiful decomposition is the higher genus analog of the factorization:

$$f(x) = C \cdot \frac{\prod(x - a_i)}{\prod(x - b_i)}$$

of rational functions on $\mathbb{P}^1$. Nor do these factorizations depend much on $\mathbf{e}$, because if $\vartheta(\mathbf{e}_1) = \vartheta(\mathbf{e}_2) = 0$, then

$$E_{e_1}(x, y) = (\text{fcn of } x \text{ alone})(\text{fcn of } y \text{ alone}) E_{e_2}(x, y).$$

Using the E_e's, we also get beautiful expressions for differentials on C with various poles too, e.g.,

$$\left(\frac{\partial}{\partial x} \log \frac{E_e(x, a)}{E_e(x, b)}\right) dx$$

is a rational 1-form on C, with simple poles at a, b only, residues ± 1 respectively; and

$$\left(\frac{\partial^2}{\partial x \partial y} \log E_e(x, y)\right)\bigg|_{y=a} dx$$

is a rational 1-form on C, with a double pole at $x = a$ and no others.

Lecture IV: The Torelli Theorem and the Schottky Problem

The purpose of this lecture is to consider the map carrying C to its Jacobian Jac from a moduli point of view. Jac is a particular kind of complex torus and the Schottky problem is simply the problem of characterizing the complex tori that arise as Jacobians. The Torelli theorem says that Jac, plus the form H on its universal covering space, determine the curve C up to isomorphism.

First of all, we saw that if $g \geq 2$, not all complex tori $X = \mathbb{C}^g/L$ are even projective varieties: in fact, necessary and sufficient for X to be a projective variety is that there exists a positive definite Hermitian form H on $\mathbb{C}^g$, such that $E \underset{\text{def}}{=} \operatorname{Im} H$ is integral on $L \times L$. The varieties that arise this way are called *abelian varieties*. The forms H are called *polarizations* of X. Since $rk\ L = 2g$, and E is skew-symmetric and integral on L, $\det E = (-1)^g d^2$, for some $d \in \mathbb{Z}$, $d \geq 1$: d is called the *degree* of the polarization. A polarization of degree 1 is called a *principal polarization.* Jacobians come with a natural polarization in which E is just the intersection form on $L \cong H_1(C, \mathbb{Z})$: this form is unimodular, so this is in fact a principal polarization. In general, if $(\mathbb{C}^g/L, H)$ is any polarized abelian variety, one can find $L_1 \subset L$ of finite index and $n \geq 1$ such that $\left(\mathbb{C}^g/L_1, \frac{1}{n} H\right)$ is a principally polarized abelian variety – so in studying all abelian varieties, the principally polarized ones play a central role.

Secondly, we saw in Lecture III that starting with any principal polarized abelian variety $(\mathbb{C}^g/L, H)$, we get Riemann's theta function $\vartheta : \mathbb{C}^g \to \mathbb{C}$, hence $\Theta = (\text{zeroes of } \vartheta) \subset \mathbb{C}^g/L$. A more succinct way to describe how $\mathbb{C}^g/L$ and H canonically determine the codimension 1 subvariety Θ up to translation[34] is the following:

$$\begin{aligned} \Theta \;=\;& \text{any codim. 1 subvariety } D \text{ of } \mathbb{C}^g/L \text{ whose fundamental} \\ & \text{class } [D] \in H^2(\mathbb{C}^g/L, \mathbb{Z}) \text{ is represented by } E, \text{ under the} \\ & \text{canonical identification:} \\ & H^2(\mathbb{C}^g/L, \mathbb{Z}) \cong (\text{skew-symmetric, integral forms on } L). \end{aligned}$$

Up to a translation, the only such Θ is the set of zeroes of ϑ. This shows that H, or Θ (up to translation) are equivalent data. Moreover it is also possible to describe which codimension 1 subvarieties $D \subset \mathbb{C}^g/L$ arise from H and a ϑ: for

[34] From Lecture III, Θ looks like it is unique even without a possible translation: however, remember the annoying ambiguity of sign in $\{e_\alpha\}$ – this means we actually only found Θ up to translation by a point $x \in \frac{1}{2}\, L/L$.

any $a \in \mathbb{C}^g/L$, let D_a = translate of D by a. For any D, choose $a_1, \ldots, a_g$ so that $D_{a_1}, \ldots, D_{a_g}$ meet transversely and consider the number of intersections:

$$D_{a_1} \bigcap \cdots \bigcap D_{a_g}.$$

This is denoted (D^g) and is always divisible by $g!$ Then

$$D = \text{ some } \Theta \qquad \text{iff} \qquad (D^g)/g! = 1.$$

We say such D's are of degree one. Therefore, instead of describing principal polarizations on $\mathbb{C}^g/L$ as forms H with Im E integral and unimodular on L, we can describe them as codimension 1 subvarieties $\Theta \subset \mathbb{C}^g/L$ with $(\Theta^g) = g!$ given up to translation.

This gives a completely algebraic way to describe such polarizations. There are also quite simple ways to describe algebraically polarizations of higher degree, but we do not need to know these[35]. We can now introduce the moduli space of principally polarized abelian varieties:

$$\mathcal{A}_g = \left\{ \begin{array}{l} \text{set of pairs } (X, \Theta), \text{ where } X \\ \text{is an abelian variety and} \\ \Theta \subset X \text{ is a codimension 1} \\ \text{subvariety such that} \\ (\Theta^g) = g! \end{array} \right\} \Bigg/ \left\{ \begin{array}{l} \text{isomorphisms} \\ f : X_1 \to X_2 \\ \text{such that} \\ f\Theta_1 = \Theta_2 \ - \\ \text{but } f \text{ need not} \\ \text{take the identity} \\ 0 \in X_1 \text{ to } 0 \in X_2 \end{array} \right\}.$$

As in Lecture II, it turns out that $\mathcal{A}_g$ has a natural structure of normal quasi-projective variety. Moreover, we obtain a morphism:

$$t : \mathfrak{M}_g \longrightarrow \mathcal{A}_g$$

by defining $t(C) = (\text{Jac},\ W_{g-1})$. The Torelli theorem simply says that t is injective and the Schottky problem can be rephrased as asking for a characterization of the image $t(\mathfrak{M}_g)$. Before studying these in more detail, I would like, in parallel with the treatment in Lecture II, to i) indicate the analytic description of $\mathcal{A}_g$ via an infinite covering, ii) indicate how to explicitly coordinatize $\mathcal{A}_g$, and iii) describe the closure of $t(\mathfrak{M}_g)$ in $\mathcal{A}_g$.

In regards to (i), we consider set-theoretically:

$$\mathfrak{H}_g = \left\{ \begin{array}{l} \text{set of 4-tuples } (V, L, H, \alpha), \text{ where} \\ V = \text{ a complex vector space} \\ L = \text{ a lattice in } V \\ H = \text{ a positive definite Hermitian form on } V \\ \alpha = \text{ an isomorphism } \mathbb{Z}^g \times \mathbb{Z}^g \to L \\ \text{with} \\ \operatorname{Im} H\big(\alpha(n,m), \alpha(n',m')\big) = \langle n, m' \rangle - \langle n', m \rangle \end{array} \right\} \Bigg/ \text{isomorphism}$$

[35] The 2 methods are i) by a suitable line bundle $\mathfrak{L}$ on $\mathbb{C}^g/L$ given up to translation, or ii) by a suitable homomorphism $\phi : \mathbb{C}^g/L \to (\mathbb{C}^g/L)^\wedge$ where $(\mathbb{C}^g/L)^\wedge$ is the "dual" abelian variety.

$$\cong \left\{ \begin{array}{l} \text{set of 3-tuples } (X, \Theta, \alpha), \text{ where} \\ X = \text{ an abelian variety} \\ \Theta = \text{ codim 1 subvariety with } (\Theta^g) = g! \\ \alpha = \text{ an isomorphism } \mathbb{Z}^g \times \mathbb{Z}^g \to H_1(X, \mathbb{Z}) \\ \text{where if } [\Theta] = \text{ fundamental class of } \Theta, \\ \text{then} \\ [\Theta]\big(\alpha(u,m), \alpha(u,m')\big) = \langle n, m' \rangle - \langle n', m \rangle \end{array} \right\} \Big/ \text{isomorphism}$$

(The connection being given by $X = V/L$, $\Theta \rightleftarrows H$ as above). Clearly, forgetting α defines a map

$$\mathfrak{H}_g \longrightarrow \mathcal{A}_g$$

and for all $\sigma \in Sp(2g, \mathbb{Z}) =$ [group of $2g \times 2g$ integral symplectic matrices],

$$(X, \Theta, \alpha) \longmapsto (X, \Theta, \alpha \cdot \sigma)$$

defines an action of $Sp(2g, \mathbb{Z})$ on $\mathfrak{H}_g$ such that

$$\mathcal{A}_g \cong \mathfrak{H}_g / Sp(2g, \mathbb{Z}).$$

On the other hand, given (V, L, H, α), there is a unique isomorphism $\Psi : V \cong \mathbb{C}^g$ such that $\Psi\big(\alpha(n, 0)\big) = n$, i.e., such that the first g generators of L are just the unit vectors in $\mathbb{C}^g$. Define the $g \times g$ complex matrix Ω by $\Psi\big(\alpha, (0, m)\big) = \Omega \cdot m$, i.e., the second g generators of L are just the g columns of Ω. Write H via a $g \times g$ Hermitian matrix h via

$$H(x, y) = {}^t\Psi(x) \cdot h \cdot \overline{\Psi(y)}.$$

Then the condition on Im H written out is:

$$\left. \begin{array}{l} \operatorname{Im} {}^t n \cdot h \cdot m = 0 \\ \operatorname{Im} {}^t n {}^t \Omega \cdot h \cdot \overline{\Omega} \cdot m = 0 \\ \operatorname{Im} {}^t n {}^t \Omega \cdot h \cdot m = {}^t n \cdot m \end{array} \right\} \forall n, m \in \mathbb{Z}^g$$

which works out as simply:

$$\Omega = {}^t\Omega, \qquad h = (\operatorname{Im} \Omega)^{-1}.$$

On the other hand, if Ω is any symmetric $g \times g$ complex matrix with Im Ω positive definite, then

$$\begin{aligned} V &= \mathbb{C}^g \\ L &= \mathbb{Z}^g + \Omega \cdot \mathbb{Z}^g \\ H(x, y) &= {}^t x \cdot (\operatorname{Im} \Omega)^{-1} \cdot y \\ \alpha(n, m) &= n + \Omega \cdot m \end{aligned}$$

is an element of $\mathfrak{H}_g$. This proves that

$$\mathfrak{H}_g \cong \left\{ \begin{array}{l} \text{open subset of } \mathbb{C}^{g(g+1)/2} \text{ of } g \times g \text{ complex symmetric} \\ \text{matrices } \Omega, \text{ with Im } \Omega \text{ positive definite} \end{array} \right\}$$

which is called the "Siegel upper half-space". Bringing in Teichmüller space again, it is not hard to see that we get the big diagram:

$$\begin{tikzcd}
\mathbb{C}^{3g-3} & & \mathbb{C}^{g(g+1)/2} & \\
\cup & & \cup & \\
\mathfrak{I}_g \arrow[rr, "\tilde{t}"] \arrow[d] & & \mathfrak{H}_g \arrow[d] & \text{Bounded Domains on this line} \\
\mathfrak{M}_g \arrow[rr, "t"] & & \mathcal{A}_g & \text{Quasi-Projective Varieties on this line}
\end{tikzcd}$$

where $\tilde{t}$ is even an equivariant holomorphic map for a homomorphism

$$\begin{aligned} \tau : \Gamma_g &\longrightarrow Sp(2g, \mathbb{Z}) \\ \Gamma_g &= \text{ Teichmüller modular group.} \end{aligned}$$

In regards to (ii), we want to mention how to use theta-null werte to explicitly embed $\mathcal{A}_g$ in a big projective space $\mathbb{P}^N$. To be precise, the ideas of Lecture III lead to the following: there is a subgroup $\Gamma_m \subset Sp(2g, \mathbb{Z})$ of finite index such that for all $m \geq 2$, the functions $\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0)$, $\alpha, \beta \in \frac{1}{m}\mathbb{Z}^g$ running over cosets mod $\mathbb{Z}^g$, which are called the *theta-nulls* of the abelian variety X, are global homogeneous coordinates not on $\mathcal{A}_g$ but on the covering space

$$\mathcal{A}_g^{\Gamma_m} \underset{\text{def}}{=} \mathfrak{H}_g / \Gamma_m,$$

i.e., define an embedding

$$\mathcal{A}_g^{\Gamma_m} \hookrightarrow \mathbb{P}^{(m^{2g}-1)}.$$

Suitable polynomials in the theta-nulls $\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0)$, invariant under the finite group $Sp(2g, \mathbb{Z})/\Gamma_m$, will then be coordinates on $\mathcal{A}_g$ itself. Assuming the injectivity of t, this gives by composition coordinates once more on $\mathfrak{M}_g$: this is method I alluded to in Lecture II. Other ways to get coordinates on $\mathcal{A}_g$ are to use other modular functions on $\mathcal{A}_g$, i.e., holomorphic functions automorphic with respect to $Sp(2g, \mathbb{Z})$, such as Poincaré series or Eisenstein series. The coordinates gotten in this way seem harder to interpret algebraically: in particular, finding an algebraic interpretation of the Eisenstein series in terms of the definition of $\mathcal{A}_g$ via moduli seems to be a very interesting problem.

In regards to (iii), although unfortunately $t(\mathfrak{M}_g)$ is not closed in $\mathcal{A}_g$, it is very nearly so. We can look at the compactification $\overline{\mathfrak{M}}_g$ of $\mathfrak{M}_g$ mentioned in Lecture II and study the "limit" of the Jacobian of a non-singular curve C as C approaches a singular curve C_0 representing a point of $\overline{\mathfrak{M}}_g - \mathfrak{M}_g$. It turns out these Jacobians have limits which are still abelian varieties if and only if C_0 is made up of a set of non-singular components $\{D_i\}$ connected together like a tree, and that in this case the limit of the Jacobian of C is the product of the Jacobians of the D_i. From this one proves:

$$\overline{t(\mathfrak{M}_g)} = \left\{ \begin{array}{l} \text{set of pairs } (X,\Theta) \text{ of the following type:} \\ X = \mathrm{Jac}(D_1) \times \ldots \times \ \mathrm{Jac}(D_k), \\ \quad D_i \ \text{ non-singular curves of genus } g_i \\ \quad \sum g_i = g \\ \Theta = \bigcup_{i=1}^k \mathrm{Jac}\ (D_1) \times \ldots \times \Theta_i \times \ldots \times \mathrm{Jac}(D_k) \end{array} \right\}.$$

Note that inside $\overline{t(\mathfrak{M}_g)}$, $t(\mathfrak{M}_g)$ is readily characterized as the set of (X,Θ) with *irreducible* Θ.

We now come to the final and most fascinating point (for me): exploration of the special properties that Jacobians have and that general principally polarized abelian varieties do not have. I would like to thank Harry Rauch, as well as Alan Mayer and John Fay, who introduced me to these questions and helped me see what a subtle thing was going on. The first step is to reconstruct C from Jac, i.e., prove t is injective (Torelli's theorem). Once this is proven, it follows that

$$\begin{aligned} \dim \overline{t(\mathfrak{M}_g)} &= 3g-3 \\ \dim \mathcal{A}_g &= g(g+1)/2, \end{aligned}$$

hence:

$$g \geq 4 \Longrightarrow \overline{t(\mathfrak{M}_g)} \overset{\subset}{\neq} \mathcal{A}_g.$$

The second point is to try to characterize $\overline{t(\mathfrak{M}_g)}$ by some special properties (Schottky's problem). I know of $\underline{\underline{4}}$ essentially different approaches to these closely related questions. At the outset, however, let me say that none of them seems to me to be a definitive solution to the second question and that I strongly suspect that although many special things about Jac are known, there is much more to be discovered in this direction.

Approach I: Reducibility of $\Theta \cap \Theta_a$

Recall that Θ_a denotes the translate of Θ by a. Almost all classical work on Torelli's theorem is closely related in some way to the lemma:

Lemma. *Let* Jac *be a Jacobian, Θ its theta-divisor. Then given $a \in$* Jac*, $a \neq 0$*

$$\left\{ \begin{array}{l} \Theta \cap \Theta_a \subset \Theta_b \cup \Theta_c \\ \text{for some } b,c \in \ \mathrm{Jac} \\ \text{distinct from } 0,a \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \text{for some } x,y \in C, \\ a = \phi(x) - \phi(y) \end{array} \right\}.$$

In fact, what this means is that if $a = \phi(x) - \phi(y)$, then $\Theta \cap \Theta_a$ breaks up into 2 components W_1, W_2 of dimension $g-2$, and $W_1 \subset \Theta_b$, $W_2 \subset \Theta_c$. This lemma is fairly elementary: let's check the easiest implication "$\Longleftarrow$". Using lecture 3, we recall that $\Theta = W_{g-1}$, esp. Θ is the image of:

$$\phi^{(g-1)} : S^{g-1}C \longrightarrow \text{Jac}\,.$$

Then $\Theta \cap \Theta_a$ is the image under $\phi^{(g-1)}$ of:

$$W = \left\{ \mathfrak{A} \in S^{g-1}C \;\middle|\; \begin{array}{l} \exists \mathfrak{A}' \in S^{g-1}C \text{ with} \\ \phi^{(g-1)}(\mathfrak{A}') = \phi^{(g-1)}(\mathfrak{A}) - a \end{array} \right\}$$

hence if $a = \phi(x) - \phi(y)$, by Abel's theorem:

$$W = \left\{ \mathfrak{A} \in S^{g-1}C \;\middle|\; \begin{array}{l} \exists \mathfrak{A}' \in S^{g-1}C \text{ with} \\ \mathfrak{A}' \equiv \mathfrak{A} - x + y \end{array} \right\}$$

Clearly one way for $\mathfrak{A}'$ to exist is if x is one of the points in the divisor $\mathfrak{A}$: i.e., $\mathfrak{A} = \mathfrak{A}_0 + x$, $\mathfrak{A}' = \mathfrak{A}_0 + y$: thus

$$W \supset W_y = \{ \text{ set of divisors } \mathfrak{A}_0 + x,\ \mathfrak{A}_0 \in S^{g-2}C \}\,.$$

The only other way is if $\mathfrak{A} + y$, $\mathfrak{A}' + x$ are distinct linearly equivalent divisors of degree g; but by Riemann-Roch, $\dim |\mathfrak{A} + y| \geq 1$ if and only if there is a 1-form ω, zero on $\mathfrak{A} + y$. Such an ω must have $g - 2$ more zeroes: call these $\mathfrak{A}_0$. Then

$$W \supset W'_y = \left\{ \begin{array}{l} \text{set of divisors } \mathfrak{A}, \text{ where } \mathfrak{A} + \mathfrak{A}_0 + y = \text{ zeroes} \\ \text{of some } \omega \in R_1(C),\ \ \mathfrak{A}_0 \in S^{g-2}C \end{array} \right\}$$

and

$$W = W_x \cup W'_y.$$

Therefore

$$\Theta \cap \Theta_a = \left(\phi^{(g-1)} W_x\right) \cup \left(\phi^{(g-1)} W'_y\right)$$

and it is not hard to see that:

$$\begin{aligned} \phi^{(g-1)}(W_x) &= (W_{g-2})_{\phi(x)} \\ \phi^{(g-1)}(W'_y) &= (-W_{g-2})_{k-\phi(y)} \end{aligned}$$

(where $-W_{g-2}$ is the set of points $-x$, $x \in W_{g-2}$; and $k = \sum_{i=1}^{2g-2} \phi(x_i)$, $\{x_i\}$ the zeroes of some $\omega \in R_1(C)$). Finally, if $b = \phi(x) - \phi(z)$, $c = \phi(w) - \phi(y)$, then the same argument shows:

$$\begin{aligned} \Theta \cap \Theta_b &= (W_{g-2})_{\phi(x)} \cup (-W_{g-2})_{k-\phi(z)} \\ \Theta \cap \Theta_c &= (W_{g-2})_{\phi(w)} \cup (-W_{g-2})_{k-\phi(y)} \end{aligned}$$

hence

$$\Theta \cap \Theta_a \subset \Theta_b \cup \Theta_c.$$

Weil investigated the deeper problem of classifying *all* cases where $\Theta \cap \Theta_a$ was reducible: it appears that for most curves, this only happens if $a = \phi(x) - \phi(y)$ again. But for some curves of genus 3 or 4 or for curves C which are double coverings of elliptic curves, there are other a's for which $\Theta \cap \Theta_a$ is reducible.

However, for *general principally polarized abelian varieties X, it seems very likely that $\Theta \cap \Theta_a$ is irreducible for all $a \in X$.*

There are various ways to use variants of the lemma to prove Torelli's theorem: one can stick to the implication "$\Longleftarrow$", and generalize it substantially, playing an elaborate Boolean algebra game with all the translates of all the $W_r \subset$ Jac. This leads eventually to the conclusion that there are only two possible ways to set up this whole Boolean configuration inside Jac, given the divisor Θ: one being obtained from the other by reflection in the origin. Or using the full strength of the lemma, one sees that Θ determines the surface

$$V = \{\phi(x) - \phi(y) \big| x, y \in C\} \subset \text{ Jac} .$$

If C is not hyperelliptic, it turns out that the tangent cone $T_{V,0}$ to V at $0 \in$ Jac is just a cone over the curve C itself: more precisely, if P is the projective space of 1-dimensional subspaces of $T_{\text{Jac},0}$, then the canonical curve $\Phi(X)$ sits in P, and

$$T_{V,0} = \begin{pmatrix} \text{union of lines } \ell, \quad \ell \text{ corresponding to} \\ \text{points of } \Phi(C) \end{pmatrix} .$$

If C is hyperelliptic, other arguments are needed. Or one may use variants of the lemma where a is infinitesimal. The geometric meaning of $\Theta \cap \Theta_a$, a infinitesimal, is the following: let $\check{P}$ be the projective space of $(g-1)$-dimensional subspaces of $T_{\text{Jac},0}$. Then we get the so-called Gauss map:

$$\pi : \Theta\text{-(singular pts. of } \Theta) \longrightarrow \check{P}$$

defined by

$$\pi(x) = \begin{pmatrix} \text{tangent space } T_{\Theta,x}, \quad \text{translated} \\ \text{to a subspace of } T_{\text{Jac},0} \end{pmatrix} .$$

The divisors $\pi^{-1}(H)$, $H \subset \check{P}$ a hyperplane, are the limits of the intersections $\Theta \cap \Theta_a$ as $a \to 0$. Thus the lemma says that at least a 1-dimensional family of divisors $\pi^{-1}(H)$ is reducible: in fact, note that hyperplanes in $\check{P}$ are points in P, and if we let $H_x \subset \check{P}$ denote the hyperplane corresponding to $x \in P$, the lemma says that $\pi^{-1}(H_x)$ is reducible for $x \in \Phi(C)$. Andreotti showed that one could say more: let $B \subset \check{P}$ be the branch locus of π, then

$$B = \text{"envelope" of the family of hyperplanes } \{H_x\},$$

i.e.,

$$\begin{aligned} B &= \bigcup_{x \in \Phi(C)} (H_x \cap H_{x+\delta x}) \quad (\delta x = \text{ infinitesimal change of } x) \\ &= \bigcup_{x \in C} \begin{pmatrix} \text{linear } (g-2)\text{-space of } \check{P} \text{ dual to the line} \\ \text{in } P \text{ tangent to } \Phi(C) \text{ at } \Phi(x) \end{pmatrix}. \end{aligned}$$

Again for non-hyperelliptic C's, this enables one to reconstruct $\Phi(C)$ immediately from (Jac, Θ).

Fay has given an analytic form of the lemma: if Jac $= \mathbb{C}^g / L$ and $\vartheta : \mathbb{C}^g \longrightarrow \mathbb{C}$ is the theta function whose zeroes are Θ, then he shows:

$$\begin{aligned}
& E(x,v)\cdot E(u,y)\cdot \vartheta\left(\mathbf{z}+\int_u^x \omega\right)\cdot \vartheta\left(\mathbf{z}+\int_v^y \omega\right) \\
+ \quad & E(x,u)\cdot E(v,y)\cdot \vartheta\left(\mathbf{z}+\int_v^x \omega\right)\cdot \vartheta\left(\mathbf{z}+\int_u^y \omega\right) \\
= \quad & E(x,y)\cdot E(u,v)\cdot \vartheta\left(\mathbf{z}+\int_{u+v}^{x+y} \omega\right)\cdot \vartheta(\mathbf{z})
\end{aligned}$$

where $E(x,y)$ is a certain "Prime form" on $C \times C$. In particular, it follows that if $x \neq u$:

$$\vartheta(\mathbf{z}) = \vartheta\left(\mathbf{z}+\int_u^x \omega\right) = 0 \Longrightarrow \vartheta\left(\mathbf{z}+\int_v^x \omega\right) = 0 \text{ or } \vartheta\left(\mathbf{z}+\int_u^y \omega\right) = 0$$

which is the "$\Longleftarrow$" of the lemma. Another pretty way to interpret this half of the lemma is via the Kummer variety: one uses the set of theta-functions of order 2 to map Jac to a projective space. All these functions are *even*, so the map factors through $K =$ Jac$/(\pm 1)$ (here $-1 =$ inverse of group law on Jac), and defines:

$$\Psi : K \hookrightarrow \mathbb{P}^k, \qquad k = 2^g - 1.$$

Then we find that $\Psi(K)$ has *trisecants*; more precisely, for any $x, y, u, v \in C$, fix a point $a \in$ Jac such that:

$$2a = \phi(x) + \phi(y) - \phi(u) - \phi(v).$$

Write $a = \frac{1}{2}(x+y-u-v)$ for clarity. Define $\frac{1}{2}(x-y+u-v)$ and $\frac{1}{2}(x-y-u+v)$ as $a - \phi(y) + \phi(u)$ and as $a - \phi(y) + \phi(v)$. Then:

$$\Psi\left(\frac{1}{2}(x+y-u-v)\right), \qquad \Psi\left(\frac{1}{2}(x-y+u-v)\right), \qquad \Psi\left(\frac{1}{2}(x-y-u+v)\right)$$

are collinear. Contrast this with the situation for generic principally polarized abelian varieties: because $\dim K \ll \dim \mathbb{P}^k$ for g large, it seems very likely that $\Psi(K)$ has no trisecants.

In connection with the Schottky problem, I would like to raise the following questions: given a principally polarized abelian variety (X, Θ), suppose there is a 2-dimensional set of points $a \in X$ such that $\Theta \cap \Theta_a$ is contained in $\Theta_b \cup \Theta_c(\{b,c\} \cap \{0,a\} = \phi)$. Then is X a Jacobian? Or if not, are there some small extra conditions that suffice to characterize Jacobians?

Approach II: Θ of translation type

Since $\Theta = W_{g-1}$, Θ is just the sum, using the group law of Jac, of the curve $\phi(C) = W_1$ with itself $(g-1)$-times. One can localize this property and come up with the following:

– Let H be a germ of a hypersurface at $0 \in \mathbb{C}^n$.

– Then H is of *translation type* if there are $(n-1)$ germs of analytic curves γ_i at $0 \in \mathbb{C}^n$ such that $H = \gamma_1 + \ldots + \gamma_{n-1}$. ($+$ represents vector addition pointwise of these subsets of $\mathbb{C}^n$.)

In fact, since up to translation Θ is symmetric, i.e., $-\Theta + k = \Theta$ for some $k \in$ Jac, whereas $\phi(C)$ is symmetric only when C is hyperelliptic, we find that for non-hyperelliptic C, Θ is *doubly of translation type*: i.e., for all $x \in \Theta$, represent Θ near x as a sum:

$$\begin{aligned} (\text{germ of } \Theta \text{ at } x) &= \gamma_1^{(x)} + \ldots + \gamma_{g-1}^{(x)} + x \\ \gamma_i^{(x)} &= \text{a 1-dimensional germ at } 0. \end{aligned}$$

Then if $-\Theta + k = \Theta$, so that $k - x \in \Theta$

$$(\text{germ of } \Theta \text{ at } x) = -\gamma_1^{(k-x)} - \ldots - \gamma_{g-1}^{(k-x)} + x$$

gives a 2[nd] representation of Θ as a hypersurface of translation type. The beautiful fact, which was conjectured by Sophus Lie and proven by W. Wirtinger, is that the only hypersurfaces doubly of translation type are the theta divisors in Jacobians and certain degenerate limits. Moreover, the theta divisor is never of translation type in a *third* way, which then proves Torelli's theorem! The following sort of answer to the Schottky problem is presumably a consequence – although the details have not been written down: given a principally polarized abelian variety (X, Θ),

$$\left(\begin{array}{l} \Theta \text{ is doubly of translation} \\ \text{type at some point } x \in \Theta \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} (X, \Theta) \text{ is the Jacobian of a} \\ \text{non-hyperelliptic curve} \end{array} \right).$$

The only thing lacking here is a nice differential-geometric criterion for a hypersurface to be singly or doubly of translation type. Also, since Θ is symmetric, one would hope that "usually" being *simply* of translation type by analytic prolongation would force it to be *doubly* so: but this is not clear.

Recently, Saint-Donat discovered a very beautiful proof of the Lie-Wirtinger results that I want to sketch. It is based on the following beautiful criterion:

Theorem. *Let $H \subset \mathbb{P}^n$ be a hyperplane, let $x_1, \ldots, x_d \in H$ and let $\gamma_1, \ldots, \gamma_d$ be germs of analytic curves at $x_1, \ldots, x_d$ which cross H transversely. Suppose $t_1, \ldots, t_d$ are coordinate functions on $\gamma_1, \ldots, \gamma_d$ such that for all hyperplanes H' near H:*

$$\sum_{i=1}^{d} t_i\,(H' \cap \gamma_i) = 0.$$

Then by analytic continuation, the $\gamma_1, \ldots, \gamma_d$ are part of an algebraic curve $\Gamma \subset \mathbb{P}^n$ of degree d (possibly reducible), so that in some neighbourhood U of H, $\Gamma \cap U = \bigcup_{i=1}^{d} \gamma_i \cap U$.

This can be proven, e.g., by reducing to $n = 2$ and in this case showing that in some neighbourhood U of H, there is a meromorphic 2-form ω on U, with simple poles on $\cup\gamma_i \cap U$, such that

$$t_i(z) = \int_{x_i}^{z} \mathrm{Res}_{\gamma_i}\omega,$$

and finally using the pseudoconcavity of U to extend ω to a rational 2-form on $\mathbb{P}^2$, whose poles will be Γ. To apply the theorem, let

$$H \subset \mathbb{C}^n$$

be a germ of a hypersurface such that

$$H = \sum_{i=1}^{n-1} \gamma_i = \sum_{i=1}^{n-1} \delta_i, \qquad \gamma_i, \delta_i \text{ germs of analytic curves.}$$

Let P denote the projective space of lines in $\mathbb{C}^n$ through 0. Associating to each point x of γ_i or δ_i the tangent line to γ_i or δ_i at x and translating to the origin, we get germs of analytic curves $\dot\gamma_i, \dot\delta_i \subset P$. For each $z \in H$, write

$$(*) \qquad z = \sum_{i=1}^{n-1} \gamma_i(x_i) = \sum_{i=1}^{n-1} \delta_i(y_i).$$

Then $T_{H,z}$ defines a hyperplane $H(z) \subset P$, and since $T_{H,z} \supset T_{\gamma_i,x_i} \cup T_{\delta_i,y_i}$, it follows that $\dot\gamma_i(x_i), \dot\delta_i(x_i) \in H(z)$. Now parametrize the branches γ_i and δ_i by any linear function L on $\mathbb{C}^n$, i.e.,

$$L(\gamma_i(x)) = x, \qquad L(\delta_i(y)) = y.$$

Then it follows from $(*)$ that:

$$\sum_{i=1}^{n-1} x_i = L\left(\sum \gamma_i(x_i)\right) = L(z) = L\left(\sum \delta_i(y_i)\right) = \sum_{i=1}^{n-1} y_i.$$

Assuming that all hyperplanes H' near $H(0)$ are of the form $H(z)$, this proves that the $2n-2$ branches $\dot\gamma_i, \dot\delta_i$ with coordinate functions $x_i, -y_i$ satisfy the condition of the theorem! Analytically prolonging, the $\dot\gamma_i, \dot\delta_i$ therefore are part of a curve $C \subset P$ and one goes on to prove that C is a canonically embedded curve of genus n (or a singular limit of such) and H is its theta divisor.

Approach III: Singularities of Θ

We use the fact that $\Theta = W_{g-1}$ and apply the results of Lecture III: it follows that every $\alpha \in \Theta$ is equal to $\phi^{(g-1)}(\mathfrak{A})$ for some divisor $\mathfrak{A} = \sum_{i=1}^{g-1} x_i$ in C. If $\ell = \dim|\mathfrak{A}|$, i.e.,

$$\left(\phi^{(g-1)}\right)^{-1}(\alpha) \cong \mathbb{P}^\ell,$$

then Kempf's results show in this case:

1) Θ is defined near α by an equation $\det(f_{ij}) = 0$, where f_{ij} is an $(\ell+1)\times(\ell+1)$ matrix of holomorphic functions at α,
2) the multiplicity of Θ at α is $\ell+1$ and in fact the tangent cone to Θ is defined by the polynomial equation $\det(df_{ij}) = 0$ on $T_{\alpha,\mathrm{Jac}}$.

Let Sing Θ denote the set of singular points on Θ. Then it is not hard to see that $\mathrm{Sing}_2\Theta$, the set of double points, is dense in Sing Θ; also, by the results quoted in Lecture I,

$$\begin{aligned} g \geq 4 \quad &\Longrightarrow \quad \exists \text{ at least one map } \pi : C \longrightarrow \mathbb{P}^1 \text{ of degree } d \leq g-1 \\ &\Longrightarrow \quad \exists \text{ at least one } \mathfrak{A} \in S^{g-1}C \text{ with } \dim|\mathfrak{A}| \geq 1 \\ &\Longrightarrow \quad \mathrm{Sing}\,\Theta \neq \emptyset. \end{aligned}$$

But if α is a double point of Θ, Θ is defined near α by an equation

$$f_{11}f_{22} - f_{12}f_{21} = 0.$$

Therefore Θ is also singular at any point x where $f_{11}(x) = f_{12}(x) = f_{21}(x) = f_{22}(x) = 0$. But 4 equations define a set of points of codimension at most 4, hence

3) Sing $\Theta \neq \emptyset$ and all components have dim $\geq g-4$[36].

In fact, a closer analysis shows that:

3′) (C hyperelliptic) $\Longrightarrow$ (Sing Θ irreducible of dim. exactly $g-3$)
(C not hyperelliptic) $\Longrightarrow \left(\begin{array}{l}\text{all comp. of Sing } \Theta \text{ have dim.} \\ \text{exactly } g-4\end{array}\right)$.

Notice, for instance, that this was exactly what we found in Lecture III if $g = 3$ or $g = 4$. This immediately distinguishes Jacobians from generic abelian varieties when $g \geq 4$, because for almost all $(X, \Theta) \in a_g$, Θ will be non-singular! In fact, Andreotti and Mayer prove:

[36] A heuristic argument for "proving" this is to count the dimension of the space S of coverings C of $\mathbb{P}^1$, of degree $g-1$, simple branching, and genus g. Simple topology shows that there must be $4g-4$ branch points, so we get $\dim S = 4g-4$. Looking at the curve C, we get a morphism $p : S \to \mathfrak{M}_g$, hence almost all fibres of p have $\dim g-1$, i.e., almost all curves C admit of $g-1$-dimensional family of maps $\pi : C \to \mathbb{P}^1$. Allowing for the 3-dimensional automorphism group of $\mathbb{P}^1$, this gives a $g-4$-dimensional family of 1-dimensional linear systems $L \subset S^{g-1}C$, hence dim Sing $\Theta \geq g-4$.

Theorem. *Let*

$$\mathcal{A}_g^{(n)} = \{(X,\Theta) \in \mathcal{A}_g \mid \dim \operatorname{Sing} \Theta \geq n\}.$$

Then $\overline{t(\mathfrak{M}_g)}$ *is a component of* $\mathcal{A}_g^{(g-4)}$.

The proof of the former fact and one ingredient of the proof of the theorem is the *heat equation* that ϑ satisfies: if we describe a principally polarized abelian variety X as above

$$X = \mathbb{C}^g/\mathbb{Z}^g + \Omega \cdot \mathbb{Z}^g$$

then in the classical normalization:

$$\vartheta(z) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i ({}^t n z + \frac{1}{2} {}^t n \Omega n)}$$

and it is immediate that considering ϑ as a function of z *and* Ω:

$$\frac{1}{2\pi i} \cdot \frac{\partial^2 \vartheta}{\partial z_i \partial z_j} = (1 + \delta_{ij}) \cdot \frac{\partial \vartheta}{\partial \Omega_{ij}}.$$

If $\alpha \in \Theta$ is a double point, and H_α is the hyperplane $\sum_{i,j=1}^g \frac{\partial^2 \vartheta}{\partial z_i \partial z_j}(\alpha) \cdot d\Omega_{ij} = 0$ in the tangent space to $\mathcal{A}_g$ at (X, Θ), then this shows that the singularity α "disappears" if you move (X, Θ) in a direction transversal to H_α. The idea of Andreotti and Mayer's proof is to show that for almost all curves C, corresponding to points $\gamma \in \mathfrak{I}_g$, $\tilde{t}(\mathfrak{I}_g)$ is non-singular at $\tilde{t}(\gamma)$ and its tangent space is the intersection of these H_α's, hence upstairs in $\mathfrak{H}_g$, $t(\mathfrak{I}_g)$ and the inverse image of $\mathcal{A}_g^{(g-4)}$ are *both* non-singular with the *same* tangent space at $\tilde{t}(\gamma)$. This will prove their theorem.

Now if

$$X = \mathbb{C}^g/\mathbb{Z}^g + \Omega \cdot \mathbb{Z}^g \cong \operatorname{Jac}(C),$$

let ω_i be the differential on C gotten by restricting dz_i to C; then one shows that if C is not hyperelliptic, $\tilde{t}(\mathfrak{I}_g)$ is non-singular at $\tilde{t}(\gamma)$ and its tangent space is defined by equations $\sum \lambda_{ij} d\Omega_{ij} = 0$ for all $\{\lambda_{ij}\}$ such that the quadratic differential $\sum \lambda_{ij}\omega_i\omega_j$ vanishes identically on C. More canonically, the point here is that the cotangent spaces to $\mathfrak{I}_g$ and $\mathfrak{H}_g$ can be identified as follows:

$$\begin{aligned} T^*_{\mathfrak{I}_g,\gamma} &\cong R_2(C), \qquad \text{quadratic differentials on } C \\ T^*_{\mathfrak{H}_g,\Omega} &\cong \operatorname{Symm}^2 \left(\begin{array}{l} \text{space of transl.-inv.} \\ \text{1-forms on } X \end{array} \right) \end{aligned}$$

and when $X = \operatorname{Jac}$, $\tilde{t}^*$ is multiplication taking quadratic expressions in the $\omega \in R_1(C)$ to the corresponding quadratic differentials; the kernel is thus the quadratic forms in $R_1(C)$ which vanish identically on C: call this $\operatorname{Ker}(\operatorname{Symm}^2 R_1 C \to R_2 C)$ or I_2. So what Adreotti and Mayer need is that for almost all curves C, I_2 is spanned by the forms:

$$q_\alpha = \sum \frac{\partial^2 \vartheta}{\partial z_i \partial z_j}(\alpha)\cdot[\omega_i]\cdot[\omega_j], \qquad \alpha = \text{ double pt. of } \Theta.$$

Looking back at Lecture III, we can see what these special quadratic forms are: we take $\mathfrak{A}$ of degree $g-1$ such that $\dim|\mathfrak{A}| = 1$, i.e., $L(\mathfrak{A})$ has a basis $\{1, f\}$. By Riemann-Roch, $R_1(-\mathfrak{A})$ is 2-dimensional: let ω_1, ω_2 be a basis. It follows that $\eta_i = f\omega_i$, $i = 1, 2$, have no poles, hence are in $R_1(C)$. Then the 2 quadratic differentials $\eta_1\omega_2$ and $\eta_2\omega_1$ are equal, i.e.,

$$q_{\mathfrak{A}} = [\eta_1]\cdot[\omega_2] - [\eta_2]\cdot[\omega_1]$$

is a quadratic form in $R_1(C)$ which vanishes on C (equivalently represents a quadric in $\mathbb{P}^{g-1}$ vanishing on $\Phi(C)$). According to the results of Lecture III, if $\alpha = \phi^{(g-1)}(\mathfrak{A})$, then $q_\alpha = \text{ constant } \cdot q_{\mathfrak{A}}$. It appears to be an open question whether or not for *every* non-hyperelliptic C, these $q_{\mathfrak{A}}$'s span $I_2 = \operatorname{Ker}\left(\operatorname{Symm}^2 R_1 C \to R_2 C\right)$. However, Andreotti and Mayer were able to check this for C which were triple covers of $\mathbb{P}^1$, hence it does hold for *almost all* C and their theorem is proven.

This approach does not establish Torelli's theorem for all curves C, but it does show that for *almost all* C's, $t^{-1}\big(t(C)\big) = \{C\}$. In fact, for almost all C, 2 good things happen – i) in the canonical embedding $\Phi : C \to \mathbb{P}^{g-1}$, $\phi(C)$ is the intersection of the quadrics containing it, and ii) the space I_2 of quadrics through $\Phi(C)$ is generated by the tangent cones q_α to Θ at its double points. Thus we have a simple prescription for recovering C from (Jac, Θ) when C is "good": i) take the tangent cone to Θ at all its double points, ii) translate to the origin of Jac and projectivize to get a quadric in $\mathbb{P}^{g-1}$, iii) intersect all these quadrics: this is generally the C you started with!

These ideas, though very elegant, do not seem to work without exception – e.g., $\mathcal{A}_g^{(g-4)}$ has other components besides $\overline{t(\mathfrak{M}_g)}$.

Approach IV: Prym varieties

The final approach to the Schottky problem is due to Schottky himself, in collaboration with Jung. One may start like this: since the curve C has a non-abelian π_1, can one use the *non-abelian coverings* of C to derive additional invariants of C which will be related by certain identities to the natural invariants of the "abelian part of C", i.e., to the theta-nulls of the Jacobian? And then, perhaps, use this whole set of identities to show that the theta-nulls of the Jacobian alone satisfy non-trivial identities? Now the simplest non-abelian groups are the dihedral groups, and this leads us to consider unramified covering spaces:

$$\begin{array}{c} C_2 \\ \downarrow \text{ degree } n\text{, abelian covering, group } A \\ C_1 \\ \downarrow \text{ degree 2} \\ C \end{array}$$

where the involution $\iota : C_1 \to C_1$ of C_1 over C lifts to an involution on C_2 and $\iota\alpha\iota^{-1} = -\alpha$, all $\alpha \in A$. These, in turn, may be constructed by starting with the degree 2 covering C_1, taking its Jacobian Jac_1, and taking the "odd" part of Jac, when it is decomposed into a product of even and odd pieces under ι. More precisely, we define:

$$\begin{aligned} \text{Prym } (C_1/C) &= \begin{pmatrix} \text{subabelian variety of } \mathrm{Jac}_1 \text{ of all} \\ \text{points } x - \iota(X), \quad x \in \mathrm{Jac}_1 \end{pmatrix} \\ &= \begin{pmatrix} \text{connected component of the set of} \\ x \in \mathrm{Jac}_1 \quad \text{such that } \iota(x) = -x. \end{pmatrix} \end{aligned}$$

There is a natural map

$$\phi_- : C_1 \longrightarrow \text{Prym}$$

given by

$$\begin{aligned} &\phi_-(x) = \phi_1(x) - \iota(\phi_1(x)), \quad \text{with} \\ &\phi_1 : C_1 \longrightarrow \mathrm{Jac}_1 \quad \text{the canonical map.} \end{aligned}$$

Then the coverings C_2 in question are pull-backs of abelian coverings of Prym via ϕ_-.

Now Jac_1 is very nearly the product of Jac and Prym: in fact there are homomorphisms

$$\text{Jac} \times \text{Prym} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Jac}_1$$

such that $\alpha \circ \beta$, $\beta \circ \alpha$ are multiplication by 2, and $\ker \alpha, \ker \beta$ are finite abelian groups isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g-1}$ (g = genus of C). The genus of C_1 will be $2g - 1$, hence

$$\begin{aligned} \dim \mathrm{Jac}_1 &= 2g - 1 \\ \dim \mathrm{Jac} &= g \\ \dim \mathrm{Prym} &= g - 1. \end{aligned}$$

The beautiful and surprising fact is that the new abelian variety Prym carries a canonical principal polarization too. In fact, if $\Theta \subset \mathrm{Jac}$, $\Theta_1 \subset \mathrm{Jac}_1$ and $\Xi \subset$ Prym are the 3 theta-divisors, Ξ is characterized by either of the properties:

$$\alpha^{-1}(\Theta_1) \sim 2\Theta + 2\Xi$$
$$\beta^{-1}(\Theta + \Xi) \sim 2\Theta_1$$

(where $\sim$ means the fundamental classes of the divisors are cohomologous; or equivalently that suitable translates of the divisors are linearly equivalent). So far, these facts tie Jac, Jac_1 and Prym into a tight but quite elementary configuration of abelian varieties, but one that does not impose any restriction on Jac itself. Thus if ϑ, ϑ_1 and ξ are the theta functions of these three abelian varieties, one can calculate ϑ_1 from ϑ and ξ and vice-versa, but ϑ and ξ can be arbitrary theta-functions of g and $g-1$ variables respectively. But now the underlying configuration of curves comes in and tells us:

$$(*) \qquad \big(\mathrm{Jac} \times (0)\big) \cap \alpha^{-1}\Theta_1 = \Theta + \Theta_\eta$$

where $\eta \in \mathrm{Jac}$ is the one non-trivial point of order 2 such that $\alpha(\eta) = 0$ (i.e., the original double cover C_1/C "corresponds" to η). This follows in fact directly from the interpretations $\Theta = W_{g-1} \subset \mathrm{Jac}$ and $\Theta_1 = W_{2g} \subset \mathrm{Jac}_1$. Now ϑ and ξ cannot be arbitrary any longer: $(*)$ turns out to be equivalent to asserting that the squares of the theta-nulls of Prym:

$$\xi^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0), \qquad \alpha, \beta \in \frac{1}{2}\mathbb{Z}^{g-1}$$

are *proportional* to certain monomials in the theta-nulls of Jac:

$$\vartheta \begin{bmatrix} \alpha\, 0 \\ \beta\, 0 \end{bmatrix} (0) \cdot \vartheta \begin{bmatrix} \alpha\, 0 \\ \beta\, 1 \end{bmatrix} (0), \qquad \alpha, \beta \in \frac{1}{2}\mathbb{Z}^{g-1}$$

(after one makes the correct simultaneous coordinatization of Prym and Jac). A third way to interpret $(*)$ is via the Kummer variety: embed $\mathrm{Jac}/\pm 1$ and $\mathrm{Prym}/\pm 1$ in projective spaces.

$$\Phi: \quad \mathrm{Jac}\,/\pm 1 \hookrightarrow \mathbb{P}^{2^g-1} = \mathbb{P}_j$$
$$\Psi: \quad \mathrm{Prym}\,/\pm 1 \hookrightarrow \mathbb{P}^{2^{g-1}-1} = \mathbb{P}_p.$$

Via suitable normalized coordinate systems as in Lecture III, there is a canonical way to identify $\mathbb{P}_p$ with a linear subspace of $\mathbb{P}_j$. Then $(*)$ says:

$$(**) \qquad \Psi(0) = \Phi\left(\frac{\eta}{2}\right).$$

Since Im Φ, Im Ψ have such large codimension, one certainly expects that for most g and $(g-1)$-dimensional principally polarized abelian varieties X, Y, $\Psi(Y)$ and $\Phi(X)$ would be disjoint.

The case when $g = 4$ is the first one where $\overline{t(\mathfrak{M}_g)} \overset{\subset}{\neq} \mathcal{A}_g$ and in this case:

$$\dim \mathcal{A}_4 = 10$$
$$\dim \overline{t(\mathfrak{M}_4)} = 9.$$

Schottky was able to show that the above identities on ϑ and ξ implied one identity on ϑ alone, of degree 8, and Igusa has asserted that this identity holds only on $\overline{t(\mathfrak{M}_4)}$! Moreover, when $g > 4$, no efficient method of eliminating ξ from the above identities is known and the ultimate problem of characterizing $\overline{t(\mathfrak{M}_g)}$ by simple identities in the theta-nulls remains open. I am confident that Schottky's approach has not been exhausted, however, and a full theta-function theoretic analysis of the dihedral (or even higher non-abelian) coverings of C remains to be carried out.

I hope these lectures have perhaps convinced the patient reader that nature's secrets in this corner of existence are fascinating and subtle and worthy of his time!

Survey of Work on the Schottky Problem up to 1996

Enrico Arbarello

Approach I.

The reducibility condition for the theta divisor on a p.p.a.v. X can be written as

$$\Theta_a \cap \Theta_b \subset \Theta_c \cup \Theta_{-c}$$

(here $\pm c \neq a, b$). If $a \neq -b$, this condition expresses the fact that $\psi(a), \psi(b), \psi(c)$ belong to a trisecant of the Kummer variety $K = K(X)$. Gunning [Gu] defines the locus

$$V_{a,b,c} = \{2x \mid \Theta_{x+a} \cap \Theta_{x+b} \subset \Theta_{x+c} \cup \Theta_{-x-c}\} \subset X$$

($V_{a,b,c}$ has a natural scheme structure) and shows that if $V_{a,b,c}$ is positive dimensional, and if some mild condition is satisfied, then $V_{a,b,c}$ is a smooth curve and X is its Jacobian. So, in this sense,

"infinitely many trisecants" $\Rightarrow$ "X is a Jacobian".

In [W3], Welters defines a subscheme $V_Y \subset X$, for any length 3 artinian subscheme $Y \subset X$ (except for the case $Y = \operatorname{Spec} \mathbb{C}[\epsilon_1, \epsilon_2] / (\epsilon_1, \epsilon_2)^2$) by letting

$$V_Y = \{2x \mid \exists \text{ a line } \mathrm{l} \subset \mathbb{P}^{2^g-1}, \text{ such that } \psi^{-1}(\mathrm{l}) \supset \mathrm{Y} + 2\mathrm{x}\},$$

and proves, without extra conditions, that

"V_Y is positive dimensional" $\Rightarrow$ "V_Y is a smooth curve and X is its Jacobian".

It is easy to see that V_Y always contains $Y - s$, where $\mathrm{Supp Y} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, and $s = a + b + c$ (with a, b, c possibly coincident). Welters asks the question: is the condition $V_Y \not\supseteq Y - s$ sufficient to insure that V_Y is a smooth curve and X is its Jacobian? This is the so-called trisecant conjecture.

Particular cases of this question are: is the existence of a trisecant or of an inflectionary tangent to the Kummer variety sufficient to conclude that X is a Jacobian? These questions are analyzed and partially answered in [D2] and [M2], but they remain open.

Another particular case of this question is the one in which $Y = \{a + \epsilon D_1 + \epsilon^2 D_2\}$, $2x \in Y - s$, $2x + \epsilon D_1 + \epsilon^2 D_2 + \epsilon^3 D_3 \in V_Y$. This translates into the K.P. equation

$$D_1^4\theta \cdot \theta - 4D_1^3\theta \cdot D_1\theta + 3(D_1^2\theta)^2$$
$$-3(D_2\theta)^2 + 3D_2^2\theta \cdot \theta + 3D_1\theta \cdot D_3\theta - 3D_1D_3\theta \cdot \theta + d\theta \cdot \theta = 0,$$

where d is a suitable constant. This case is Novikov's conjecture solved by T. Shiota [S].

The relationship between the K.P. hierarchy of differential equations and the scheme V_Y is established in [A-D2]. Following the point of view of [A], [A-D2], a geometrical proof of Shiota's Theorem is given in [M1]. There, to overcome the difficulties coming from the singularities of the theta divisor, an important role is played by the following fundamental result of Ein and Lazarsfeld [E-L]:

If X is a g-dimensional, indecomposable p.p.a.v. and Θ its theta divisor, then

$$\dim(\Theta_{\text{sing}}) < g - 2.$$

Approach II.

One can read a complete survey of the classical point of view in the papers by J. Little [Li1], [Li2]. There, one can find the developments of the ideas in Saint-Donat's paper [S-D]. A discussion of Darboux's criterion [D] for the algebraicity of local analytic arcs can be found in [Gr].

In [Li3] it is proved that the existence of a curve of flexes of the Kummer variety is equivalent to the double translation property for the Θ-divisor, thus relating Approaches I and II.

Approach III.

The loci $\mathcal{A}_g^{(k)}$ (often denoted by $\mathcal{N}_k$) have been studied especially in the case $k \le g - 4$.

In [B 1], Beauville proves that $\mathcal{A}_4^{(0)}$ is the union of $\mathrm{t}(\mathcal{M}_4)$ and the locus θ_{null} of 4-dimensional p.p.a.v. with a vanishing theta-null.

$\mathcal{A}_5^{(1)}$ has five components that are described in [Del] and [Do4]. In [Del] a list of components for $\mathcal{A}_g^{(g-4)}$ is described.

In [B-D1], a bridge between Approaches I and III is established by showing that the locus of g-dimensional p.p.a.v. whose associated Kummer has a trisecant line is contained in $\mathcal{A}_g^{(g-4)}$.

As it was mentioned above, the case $k = g - 2$ has been settled by Ein and Lazarsfeld:

$$\mathcal{A}_g^{(g-2)} = \text{decomposable p.p.a.v.}$$

The study of tangent cones at singular points of the theta-divisor has been carried out by M. Green in [G] where he proves the following theorem: *For* *any non-hyperelliptic curve C of genusg ≥ 4, the space of quadrics through the canonical curve is spanned by tangent cones at double points of its theta-divisor.*

When the curve in question is non-trigonal, and not a smooth plane quintic, this gives a new proof of Torelli's theorem. Green's proof relies on a subtle theorem by Kempf [K] which asserts that, when C is not hyperrelliptic, the Kodaira-Spencer map:

$$H^1(C, \mathcal{T}_C) \to H^1(C^{(g-1)}, \mathcal{T}_{C^{(g-1)}})$$

is an isomorphism.

A detailed analysis of the local Torelli theorem including the case of hyperelliptic curves is contained in [O-S].

The heat equation satisfied by the theta function of an abelian variety, which plays a central role in the paper of Andreotti and Mayer, has been studied by Welters in [W1]. This paper was instrumental in Hitchin's laying the foundation of a parallel theory in the case of higher rank vector bundles on curves [H].

Approach IV.

Using Kummer varieties and suitable normalized coordinate systems one can define the Schottky loci $\mathcal{S}_g$ and $\mathcal{S}_g^{(\mathrm{big})}$ to be

$$\begin{aligned}\mathcal{S}_g^{(\mathrm{big})} &= \{X \in \mathcal{A}_g \mid \exists\, \eta \in X\backslash\{0\} \text{ of order } 2\text{, and } P \in \mathcal{A}_{g-1},\ \text{s.t. } \Psi(0) = \Phi(\tfrac{\eta}{2})\}\\ \mathcal{S} &= \{X \in \mathcal{A}_g \mid \forall\, \eta \in X\backslash\{0\} \text{ of order 2, } \exists\, P \in \mathcal{A}_{g-1},\ \text{s.t. } \Psi(0) = \Phi(\tfrac{\eta}{2})\end{aligned}$$

By virtue of Prym varieties one has

$$t(\mathcal{M}_g) \subseteq \mathcal{S}_g \subseteq \mathcal{S}_g^{(\mathrm{big})}\,.$$

In [vG1], van Geemen proves that $t(\mathcal{M}_g)$ is an irreducible component of $\mathcal{S}_g$ and Donagi [Do1] proves that $t(\mathcal{M}_g)$ is an irreducible component of $\mathcal{S}_g^{(\mathrm{big})}$.

An important tool in the proofs is Welter's characterization (for $g \geq 5$) of the difference $C - C \subset J(C)$ as the support of the base locus of the linear system $\Gamma_{00} = |\{s \in \mathcal{O}_X(2\Theta) \mid \mathrm{mult}_0(\mathrm{s}) \geq 4\}|$. This result was conjectured in [vG-vG]. There it is also conjectured that this equality is yet another way to characterize Jacobians, to the point that what should happen is: *if X is not a Jacobian, the support of Γ_{00} is $\{0\} \in X$.* In the same paper an infinitesimal version of this conjecture is also discussed.

In [B-D3] it is proved that for a generic p.p.a.v. of genus $g \geq 4$ the base locus of Γ_{00} is finite, and that the infinitesimal version of the conjecture holds. Moreover, the conjecture itself is related with the trisecant conjecture. In [Iz] the conjecture is proved for $g = 4$.

Analogous to the Torelli map t is the Prym map

$$p : \mathcal{R}_{g+1} \to \mathcal{A}_g\ ,$$

where $\mathcal{R}_{g+1}$ is the moduli space of unramified 2-sheeted coverings of genus $g+1$ curves, or equivalently, of pairs (C,η) where C is a smooth curve and $\eta \in H^1(C,\mathbb{Z}_2)$ a non-zero half period.

The Prym map is generically injective for $g \geq 6$ [F-S] but, unlike the Torelli map, it is never injective because of Donagi's tetragonal construction [Do 4].

Turning the attention to low genera, Igusa [I] and Freitag [Fr] prove that $t(\mathcal{M}_4) = \mathcal{S}_4 = \mathcal{S}_4^{(\mathrm{big})}$.

Also, the Prym map is generically surjective for $g \leq 5$ and its structure has been extensively analyzed in [D-S] and [Do5].

Donagi, in an unpublished work, establishes the equality $t(\mathcal{M}_5) = \mathcal{S}_5$. On the other hand, $\mathcal{S}_5 \neq \mathcal{S}_5^{(\mathrm{big})}$, as the latter contains intermediate Jacobians of cubic threefolds [Do2].

The conjecture $t(\mathcal{M}_g) = \mathcal{S}_g$ is still open.

The Schottky problem for Prym varieties is studied in [B-D2] where it is proved that the moduli space of Prym varieties of dimension ≥ 7 is an irreducible components of the locus of p.p.a.v. whose associated Kummer varieties admit a *quadrisecant* plane.

Finally, two words about new ideas. Let $(X = \mathbb{C}^g/\Lambda, \Theta)$ be p.p.a.v.; Buser and Sarnak defined a metric invariant $m(X) = m(X,\Theta)$ by setting it equal to the minimum square length, with respect to the Hermitian form, of a non-zero lattice vector. Then, on the one hand, they show that for a general p.p.a.v. X ,

$$m(X) \geq \frac{1}{\pi}(2g!)^{1/g} \approx \frac{g}{\pi e},$$

while on the other hand, they show that for a Jacobian $J(C)$,

$$m(J(C)) \leq \frac{3}{\pi}\log(4g+3) .$$

As a consequence, for $g \gg 0$, the Jacobian locus $t(\mathcal{M}_g)$ lies in a small neighbourhood of the boundary of $\mathcal{A}_g$.

Lazarsfeld [L] takes a different point of view. He considers the Seshadri constant

$$\epsilon(X) = \epsilon(X,\Theta) = \sup\{\epsilon \geq 0 \mid f^*c_1(\Theta) - \epsilon[E] \text{ is nef}\},$$

where f is the blow-up of X at one of its points x, and E is the exceptional divisor of f (by homogeneity the definition does not depend on x). He proves that

$$\epsilon(X) \geq \frac{\pi}{4}m(X).$$

In particular, for a general p.p.a.v. X ,

$$\epsilon(X) \geq \frac{1}{4}(2g!)^{1/g} \approx \frac{g}{4e}.$$

He then gets upper bounds for Jacobians of curves

$$\epsilon(J(C)) \leq g^{1/g},$$

and, even more precisely, for Jacobians of d-gonal curves

$$\epsilon(J(C)) \leq \frac{4d}{\pi}.$$

In conclusion, the minimal length of lattice vectors and the Seshadri constant behave in a similar way and both tend to distinguish Jacobians of curves among all principally polarized abelian varieties.