PROBABILISTIC METHODS IN NUMBER THEORY

By A. RÉNYI

1. Introduction

Probability theory was created to describe random mass-phenomena. Since the appearance in 1933 of the fundamental book[1] of Kolmogoroff, however, probability theory has become an abstract, axiomatic theory, and as such is capable of other interpretations too. Thus methods and results of probability theory may be applied as tools in any other branch of mathematics. Many important applications of probabilistic methods have been made in number theory. There exist excellent previous surveys of these results (see [2, 3, 4]); these surveys contain also many references to the literature. In the present paper I should like to give an account of some recent results, obtained since the appearance of the surveys mentioned.

I do not aim at completeness, and shall mention mainly such results as have some connection with my own work, done partly in collaboration with others, especially with Erdős, to whom I am indebted for kindly agreeing to include in the present paper some yet unpublished results of our collaboration.

Erdős and the author of the present paper are, following a suggestion of Doob, preparing a monograph on 'Probabilistic methods in number theory' to appear in the series '*Ergebnisse der Mathematik*' published by the Springer Verlag. This monograph will contain a full bibliography of the subject.

2. Additive number theoretical functions

A real valued function $f(n)$ defined for all natural numbers $n = 1, 2, \ldots$ is called *additive* if

$$f(nm) = f(n) + f(m), \tag{1}$$

provided that $(n, m) = 1$, where (n, m) denotes the greatest common divisor of n and m. Typical additive functions are: the function $V(n)$ denoting the number of all prime factors of n; the function $U(n)$ denoting the number of different prime factors of n; the function $\log d(n)$ where $d(n)$ denotes the number of divisors of n. If $f(n)$ has besides (1) the property that $f(p^{\alpha}) = f(p)$ if p is a prime and $\alpha = 2, 3, \ldots, f(n)$ is called *strongly additive*. If $f(n)$ is such that (1) holds for all n and $m, f(n)$ is called *absolutely additive.* Clearly $U(n)$ is strongly additive and $V(n)$ absolutely additive, but $\log d(n)$ has neither of these properties.

The distribution of values of additive number theoretic functions has been investigated in detail. To express the results we may make use of the terminology of conditional probability spaces (see [5,6,7]). Let Ω be the set of all natural numbers, Ω_N the set of the first N natural numbers. Let $\mathscr{A}$ denote the set of all subsets of Ω and $\mathscr{B}$ the set of all finite and non-empty subsets of Ω; we denote by $v(A)$ the number of elements of $A \in \mathscr{A}$. We denote by AB the intersection of the sets A and B, and put for $A \in \mathscr{A}$, $B \in \mathscr{B}$

$$P(A \mid B) = \frac{v(AB)}{v(B)}.$$

Then clearly $[\Omega, \mathscr{A}, \mathscr{B}, P]$ is a conditional probability space in the sense of [5] and [6]. All results concerning the distribution of values of additive number theoretical functions can be conveniently expressed in terms of this conditional probability space.

The first fundamental result concerning additive number theoretical functions was the theorem of Erdős and Kac[8]. For the sake of brevity we shall formulate their result only for the function $V(n)$. If $S(n)$ is a proposition concerning the natural number n, we denote the set of those n for which this proposition is valid also by $S(n)$. The theorem of Erdős and Kac contains as a special case the assertion that

$$\lim_{N \to +\infty} P(V(n) - \log\log n < x\sqrt{(\log\log n)} \mid \Omega_N) = \Phi(x), \tag{2}$$

where

$$\Phi(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du \tag{3}$$

is the distribution function of the normal probability law. Thus the distribution of values of $V(n)$ on Ω_N tends for $N \to +\infty$ to the normal distribution. A similar statement holds for a broad class of additive number theoretical functions; the most general results have been obtained by Kubilius (see [4]). The method of proof used by Erdős and Kac, as well as by Kubilius, was a combination of probabilistic methods (theorems on the limiting distribution of sums of independent random variables) with elementary number theoretical results (the sieve of Viggo Brun). LeVeque[9] and Kubilius[4] proved certain improvements of (2) too, by estimating the remainder term. LeVeque conjectured that the best result is

$$P(V(n) - \log\log n < x\sqrt{(\log\log n)} \mid \Omega_N) = \Phi(x) + O\left(\frac{1}{\sqrt{(\log\log N)}}\right) \tag{4}$$

uniformly for $-\infty < x < +\infty$. Other proofs of (3) have been given by Delange[10] and Halberstam[11] using the method of moments.

Recently we succeeded with Turán[12] in proving the conjecture (4) of LeVeque. The method used in [12] was a combination of an analytical approach,† working with standard tools of analytical number theory (such as the zeta-function of Riemann and contour-integration) with the method of characteristic functions. This method can be applied for other functions than $V(n)$ too, and to other related problems.

3. The probabilistic generalization of the large sieve of Linnik

Linnik[14] discovered in 1941 an ingenious new method, called by him 'the large sieve'. I generalized this method in 1947, and used it in the proof of the following theorem (see [15]), being a step towards the unsolved hypothesis of Goldbach: there exists a constant K such that every natural number n can be represented in the form $n = p+P$ where p is a prime and $V(P) \leqslant K$. For another application of the large sieve see [16]. Later on, in 1948–9, I realized that the large sieve is a special case of a general theorem of probability theory (see [17,18,19]). This theorem, in a recently obtained improved form (see [40]), can be stated as follows:

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be random variables each having a distribution of the discrete type. Let x_{nk} $(k = 1, 2, \ldots)$ denote the values taken on by ξ_n with positive probability; let us denote by A_{nk} the event $\xi_n = x_{nk}$ and by $P(A_{nk})$ the probability of this event. Let us denote by $\phi(\xi_n, \xi_m)$ the mean square contingency of ξ_n and ξ_m $(n \neq m)$ as defined by Pearson (see [20]), i.e. put

$$\phi(\xi_n, \xi_m) = \left(\sum_k \sum_l \frac{(P(A_{nk}A_{ml}) - P(A_{nk})\,P(A_{ml}))^2}{P(A_{nk})\,P(A_{ml})}\right)^{\frac{1}{2}}. \tag{5}$$

We call the sequence ξ_n *weakly dependent with bound* B if for any sequence x_n of real numbers such that $\sum_n x_n^2 < +\infty$ we have

$$\Big|\sum_{\substack{n\ m\\ n\neq m}} \sum \phi(\xi_n, \xi_m)\, x_n x_m\Big| \leqslant B \sum_n x_n^2. \tag{6}$$

Let $M(\eta)$ and $D^2(\eta)$ denote respectively the mean value and variance of the random variable η, $M(\eta \mid \xi)$ the conditional mean value of η with respect to a fixed value of ξ, and $D_\xi^2(\eta)$ the variance of the random variable $M(\eta \mid \xi)$. Denote by $\Theta_\xi(\eta)$ the correlation ratio of η on ξ, as defined by Pearson (see [20]), i.e. put

$$\Theta_\xi^2(\eta) = \frac{D_\xi^2(\eta)}{D^2(\eta)}. \tag{7}$$

† This approach has already been applied to the investigation of the distribution of values of additive number theoretical functions in 1934 in the paper[13] of Turán.

Our theorem asserts that if η is an arbitrary random variable having finite second moment $M(\eta^2)$, and if $\{\xi_n\}$ is a sequence of weakly dependent random variables with bound B, then

$$\sum_n \Theta^2_{\xi_n}(\eta) \leqslant (1+B). \tag{8}$$

The application of this theorem to number theory makes it possible to prove that an arbitrary sufficiently dense sequence of integers $\leqslant N$ is 'almost' uniformly distributed among 'almost' all residue classes with respect to 'almost' all primes $p \leqslant N^{\alpha}$, where $\alpha < \frac{1}{2}$. For the exact meaning of the term 'almost' (occurring three times in the above sentence, and having a different meaning on each occasion) we refer to [18] and [19].

4. Statistical properties of the digits in various representations of real numbers

The history of the subject of this paragraph goes back to Gauss, whose conjecture on continued fractions was proved in 1928 by Kuzmin[21]. The starting-point of recent investigations is the classical result of Borel[22], according to which the limiting frequencies of all possible digits in the q-adic representation of a real number x are equal to one another $\left(\text{and thus equal to } \frac{1}{q}\right)$ for almost all x. Some years ago I generalized this theorem by considering, instead of q-adic expansions, general *Cantor's* series[23]. Let $q_n \geqslant 2$ $(n = 1, 2, \ldots)$ be an arbitrary sequence of positive integers. Then every real number x $(0 \leqslant x \leqslant 1)$ can be represented in the form

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n(x)}{q_1 q_2 \cdots q_n}, \tag{9}$$

where the 'digit' $\epsilon_n(x)$ may take on the values $0, 1, 2, \ldots, q_n - 1$ $(n = 1, 2, \ldots)$. The digits $\epsilon_n(x)$, considered as random variables on the probability space $[\Omega, \mathscr{A}, P]$, where Ω is the interval $(0, 1)$, $\mathscr{A}$ the set of all measurable subsets of Ω and P the ordinary Lebesgue measure, are independent, and $\epsilon_n(x)$ takes on each of the values $0, 1, \ldots, q_n - 1$ with probability $1/q_n$. Now it was shown in [23] that if $N_n(k, x)$ denotes the frequency of the number k $(k = 0, 1, \ldots)$ among the first n digits in the representation (9) of x, then

$$\lim_{n\to\infty} \frac{N_n(k, x)}{\sum\limits_{\substack{j=1\\ q_j>k}}^{n} \frac{1}{q_j}} = 1 \tag{10}$$

for almost all x, provided that

$$\lim_{n\to\infty} \sum_{\substack{j=1\\ q_j>k}}^{n} \frac{1}{q_j} = +\infty.$$

Equation (10) implies that, if the series $\Sigma(1/q_n)$ is divergent and $q_n \to +\infty$, then for any pair k, l of non-negative integers we have

$$\lim_{n\to\infty} \frac{N_n(k,x)}{N_n(l,x)} = 1. \tag{11}$$

Thus in this case all digits $k = 0, 1, 2, \dots$ occur in the limit in a certain sense equally frequently among the digits in the representation (9) of almost every real number x.

Recently we have obtained, with Erdős, the following results. Let us put

$$M_n(x) = \max_{(k)} N_n(k,x), \tag{12}$$

i.e. let $M_n(x)$ denote the frequency of the most frequent number among the first n digits in (9). Then if, putting $Q_n = \sum_{k=1}^{n} 1/q_k$, we have $\lim_{n\to+\infty} \frac{Q_n}{\log n} \to +\infty$, then for almost all x

$$\lim_{n\to\infty} \frac{M_n(x)}{Q_n} = 1. \tag{13}$$

On the other hand, in case $q_n/n \to +\infty$, and $Q_n \to +\infty$ we have for almost all x

$$\lim_{n\to\infty} \frac{M_n(x)}{Q_n} = +\infty \tag{14}$$

Borel's theorem has been generalized by Raikov[24] as follows: If $g(x)$ is integrable in the interval $(0, 1)$ and periodic with period 1, then for almost all x $(0 \leqslant x \leqslant 1)$ and for all integers $q \geqslant 2$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(q^k x) = \int_0^1 g(t)\,dt. \tag{15}$$

This theorem, as has been remarked by Riesz[25], is a special case of the individual ergodic theorem. An analogous result for continued fractions has been obtained by Ryll-Nardzewski[26]. Let x denote a real number $(0 < x < 1)$ and consider its continued fraction

$$x = \cfrac{1}{\epsilon_1(x) + \cfrac{1}{\epsilon_2(x) + \cfrac{1}{\epsilon_3(x) + \dots}}}, \tag{16}$$

where the digits $\epsilon_n(x)$ are positive integers. Put $r_0(x) = x$ and

$$r_n(x) = \cfrac{1}{\epsilon_{n+1}(x) + \cfrac{1}{\epsilon_{n+2}(x) + \cfrac{1}{\epsilon_{n+3}(x) + \dots}}} \qquad (n = 1, 2, \dots), \tag{17}$$

i.e. $r_n(x)$ is the nth remainder of the continued fraction (16). Then if $g(x)$ is L-integrable in $[0, 1]$ the theorem of Ryll-Nardzewski asserts that, for almost every x,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(r_k(x)) = \frac{1}{\log 2} \int_0^1 \frac{g(t)}{1+t} dt. \tag{18}$$

The fundamental idea of Ryll-Nardzewski was the following: he introduced the measure

$$v(E) = \frac{1}{\log 2} \int_E \frac{dt}{1+t}, \tag{19}$$

where E is a measurable subset of $(0, 1)$. Now the transformation† $Tx = (1/x)$ of the interval $(0, 1)$ leaves the measure $v(E)$ invariant; on the other hand $r_n(x) = T^n x$ $(n = 1, 2, \dots)$. Thus (18) follows also from the individual ergodic theorem.

The q-adic expansion and the continued fraction expansion (16) are both special cases of a general class of representations of real numbers x $(0 < x < 1)$ in the form (called the 'f-expansion' of x)

$$x = f(\epsilon_1(x) + f(\epsilon_2(x) + f(\epsilon_3(x) + \dots))), \tag{20}$$

where the monotonic function $f(x)$ has to satisfy certain conditions and the digits $\epsilon_n(x)$ are non-negative integers, determined by the following algorithm: Let $x = \phi(y)$ denote the inverse function of $y = f(x)$, determine the sequence $r_n(x)$ by the recursion

$$r_0(x) = x, \quad r_{n+1}(x) = (\phi(r_n(x))) \tag{21}$$

and put

$$\epsilon_{n+1}(x) = [\phi(r_n(x))]. \tag{22}$$

(Here (z) denotes again the fractional part and $[z]$ the integral part of z.)

The q-adic expansion is obtained from (20) as a special case if $f(x) = x/q$ for $0 \leqslant x \leqslant q$ and the continued fraction expansion if $f(x) = 1/x$ for $x \geqslant 1$.

f-expansions with decreasing $f(x)$ have been considered previously by Bissinger[27] and with increasing $f(x)$ by Everett[28]. f-expansions (both with increasing and decreasing f) have been considered under more general conditions by the author of the present paper in [29], where a general theorem on the distribution of digits of a general f-expansion is

† Here and in what follows (z) denotes the fractional part of z.

given, which includes as special cases the theorems of Raikov and Ryll-Nardzewski. The proof of the above-mentioned theorem is based on the ergodic theorem of Dunford and Miller. I mention only the following special case: Let $\beta > 1$ be an arbitrary real number; then every real number x $(0 \leqslant x \leqslant 1)$ can be represented in the form

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n(x)}{\beta^n}, \tag{23}$$

where the digits $\epsilon_n(x)$ may take on the values $0, 1, \ldots, [\beta]$ and are determined by the algorithm (20)–(22) with $f(x) = x/\beta$ $(0 \leqslant x \leqslant \beta)$. There exists a measure $v_\beta(E)$, defined on the interval $(0, 1)$, which is equivalent to the Lebesgue measure, and which is invariant with respect to the transformation $Tx = (\beta x)$. Thus we have for any $g(x)$ which is L-integrable on $[0, 1]$, for almost every x,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(r_k(x)) = \int_0^1 g(x)\, dv_\beta, \tag{24}$$

where $r_k(x)$ is defined by (21). Clearly, if β is an integer, v_β is the ordinary Lebesgue measure and (24) reduces to the theorem of Raikov. I have not succeeded in determining explicitly the measure v_β except for some special (algebraic) values of β. E.g. if β is the only positive root of the equation

$$\beta^n = \beta^{n-1} + 1 \quad (n \geqslant 1 \text{ integral}) \tag{25}$$

then

$$v_\beta(E) = \int_E h_\beta(x)\, dx, \tag{26}$$

where

$$h_\beta(x) = \left\{ \begin{array}{lll} \lambda & \text{for} & 0 < x < \dfrac{1}{\beta^{n-1}}, \\ \dfrac{\lambda}{\beta^k} & \text{for} & \dfrac{1}{\beta^{n-k}} < x < \dfrac{1}{\beta^{n-k-1}} \quad (k = 1, 2, \ldots, n-1), \end{array} \right\} \tag{27}$$

where

$$\lambda = \frac{\beta}{(\beta-1)(n(\beta-1)+1)}. \tag{28}$$

It follows from (24) that, denoting by p_0 and p_1 the limits of the relative frequencies of the digits 0 and 1 in the expansion (23) for almost all x (as (25) implies $1 < \beta < 2$ these are the only possible values of the digits $\epsilon_n(x)$), then

$$p_0 = \frac{(n-1)(\beta-1)+1}{n(\beta-1)+1} \quad \text{and} \quad p_1 = \frac{\beta - 1}{n(\beta-1)+1}. \tag{29}$$

In particular, for $n = 1$, $\beta = 2$ we obtain the well-known special case of Borel's theorem that in the dyadic representation of almost all numbers both digits 0 and 1 have the limiting frequency $\frac{1}{2}$.

Let us consider now the representation of the real number x $(0 < x < 1)$ in the form of Engel's series

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \dots + \frac{1}{q_1 q_2 \cdots q_n} + \dots, \tag{30}$$

where the q_n are integers, $q_{n+1} \geqslant q_n$. The probabilistic theory of Engel's series has been considered by Borel[30] and Lévy[31].

Borel has announced, without proof, that for almost all x

$$\lim_{n \to \infty} \sqrt[n]{q_n} = e. \tag{31}$$

A proof of (31) has been given by Lévy, who also proved that if we define the probability $P(E)$ of a measurable subset E of the interval $[0, 1]$ to be equal to the Lebesgue measure of E, we have

$$\lim_{n \to \infty} P\left(\frac{\log q_n - n}{\sqrt{n}} < y\right) = \Phi(y). \tag{32}$$

In a recent paper[32] by Erdős, Szüsz and the author, new and simple proofs of these theorems and of some other results have been given, based on the fact that the sequence q_n, considered as a sequence of random variables in the above-mentioned probabilistic interpretation, forms a homogeneous Markov chain, with the transition probabilities

$$P(q_n = k \mid q_{n-1} = j) = \frac{j-1}{k(k-1)} \quad \text{for} \quad k \geqslant j. \tag{33}$$

By using a theorem of the author on mixing sets[33] it can be shown (see [34]) that (32) can be replaced by the more general relation

$$\lim_{n \to \infty} Q\left(\frac{\log q_n - n}{\sqrt{n}} < y\right) = \Phi(y), \tag{34}$$

which is valid whenever Q is a measure which is absolutely continuous with respect to the Lebesgue measure.

In [32] similar results have been obtained for Sylvester's series

$$x = \frac{1}{Q_1} + \frac{1}{Q_2} + \dots + \frac{1}{Q_n} + \dots \tag{35}$$

(where the Q_n are integers, $Q_{n+1} \geqslant Q_n(Q_n - 1) + 1$), e.g. it has been shown that $\lim\limits_{n \to \infty} Q_n^{1/2^n} = L(x)$ exists for almost all x, but we were unable to determine the limit $L(x)$ explicitly. In [35] similar results have been obtained for Cantor's products

$$x = \prod_{n=1}^{\infty} \left(1 - \frac{1}{Q_n}\right). \tag{36}$$

5. Problems of additive number theory

In this section I should like to give an account of some yet unpublished results on the additive properties of random sequences of integers, obtained by Erdős and the author of the present paper. Some results in this direction have been announced without proof in [36]. Random sequences of integers $v = (v_1, v_2, \ldots, v_n, \ldots)$ are defined as follows: Let A_n denote for each n $(n = 1, 2, \ldots)$ the event that n belongs to the random sequence v, and let us suppose that $P(A_n) = p_n$ is given. Let us suppose further that the events A_n are independent. By these hypotheses a probability measure is determined on a σ-algebra $\mathscr{A}$ of subsets of the space Ω consisting of all possible sequences of integers. $\mathscr{A}$ is defined as the least σ-algebra which contains all subsets A of Ω defined by fixing for a finite number of integers whether it belongs to the sequence v or not.

Now let $\psi_2(N)$ denote the number of representations of the natural number N in the form $N = v_i + v_j$ $(i \leqslant j)$, where v_i and v_j are elements of the random sequence v. Then $\psi_2(N)$ is a random variable.

We have obtained, for example, the following result. If $p_n = cn^{-\frac{1}{2}}$ then the sequence $S_k^{(2)}$ of integers N, for which $\psi_2(N) = k$, has for almost all sequences v the density $d_k = (\lambda^k e^{-\lambda})/k!$ $(k = 0, 1, \ldots)$, with $\lambda = \frac{1}{2}c^2\pi$, i.e. the distribution $\{d_k\}$ is of Poisson's type.

More generally, if $\psi_r(N)$ denotes the number of representations of the natural number N in the form $N = v_{i_1} + v_{i_2} + \ldots + v_{i_r}$ $(i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r)$, and $p_n = c/n^{1-(1/r)}$ $(n = 1, 2, \ldots)$, then denoting by $S_k^{(r)}$ the sequence of integers N for which $\psi_r(N) = k$, $S_k^{(r)}$ has for almost all sequences v the density $d_k = (\lambda^k e^{-\lambda})/k!$ $(k = 0, 1, \ldots)$, where

$$\lambda = \frac{1}{r!}\left[c\Gamma\left(\frac{1}{r}\right)\right]^r \quad (r = 2, 3, \ldots).$$

We considered with Erdős also the distribution of differences of elements of a random sequence of integers. Other applications of probabilistic methods to problems of additive number theory are discussed in [36].

6. Some further applications in number theory

In this section we mention, without going into details, some other lines of research. Linnik[37] (see also [4] for further literature) has obtained interesting results by applying the theory of Markov chains to the arithmetic of quaternions. His results are of great importance in the theory of representation of integers by means of ternary quadratic forms. By the same method Linnik has proved[38] that the points (x, y, z) with integral co-ordinates which lie on the spherical surface

$$x^2+y^2+z^2 = m$$

are in the limit for $m \to +\infty$ uniformly distributed on this surface (where $m \equiv 1$ or $2 \bmod 4$ or $m \equiv 3 \bmod 8$).

Probabilistic methods have also been applied in the theory of diophantine approximation; see, for example, the paper[39] of Erdős and the author.

I am convinced that in spite of the wide variety of problems to which results or methods of probability theory have already been applied with success, only a small part of the possibilities of such an approach has yet been exhausted, and there will be a rapid development in this direction in the years to come. This remark applies to chapters of mathematics other than number theory too.

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