Motivation for the Jacobian Variety

[+12] [3] Rdrr
[2020-07-06 15:56:40]
[ag.algebraic-geometry arithmetic-geometry]
[https://mathoverflow.net/questions/364967/motivation-for-the-jacobian-variety]

I've been to many talks in Number Theory and for some reason I've yet to fully grasp, we all seem to like Jacobian Varieties a lot. I know that they are Abelian varieties, which give information about their respective curve, but I'm not sure what information exactly. I know of the analytic description of the Jacobian, but I'm still not exactly sure why the Jacobian is so studied.

In his AMS article, What is a motive [1] Barry Mazur seems to suggest that Jacobians encapsulate all cohomology theories. Is this true? How can I see this?

[1] https://www.ams.org/notices/200410/what-is.pdf

(4) Kleiman's chapter in FGA explained (available separately on arXiv) contains a great and very detailed historical account (albeit possibly a bit tangential to your main question) of the development of the Picard scheme (as well as details for the construction in a very general setting). - **R. van Dobben de Bruyn**

[+15] [2020-07-06 15:59:43] David E Speyer

If you are a number theorist, you presumably like class groups? Let X be a curve defined over $\mathbb{F}_p$, let J be its Jacobian and let x be an $\mathbb{F}_p$ point of X. Let A be the coordinate ring of the affine curve $X \setminus \{x\}$. Then the class group of A is $J(\mathbb{F}_p)$. (And similar statements can be made for deleting more than one point, or deleting points defined over extensions of $\mathbb{F}_p$.)

Why do you need to delete a point? - **Rdrr**
(1) Answer 1 to get an affine variety. If someone is truly coming from classical number theory, they may only know class groups of rings, not Pic. Answer 2 if you take $Pic(X)$, you get $\mathbb{Z} \times J(\mathbb{F}_p)$, not $J(\mathbb{F}_p)$. - **David E Speyer**
Could you give a reference for the fact that the class group of A is $J(\mathbb{F}_p)$? Thanks. - **user141691**
(1) @Ang I don't have a reference off the top of my head. We have $\mathrm{Pic}(X \setminus \{x\}) \cong \mathrm{Pic}^0(X)$, since every divisor on $X \setminus \{x\}$ can be extended to a degree 0 divisor on X in a unique way. (Here it matters that X is an $\mathbb{F}_p$ point.) The fact that $J(\mathbb{F}_p) \cong \mathrm{Pic}^0(X)$ is the defining property of the Picard functor. - **David E Speyer**

[+10] [2020-07-06 20:36:32] Jef

Suppose $X/\mathbb{Q}$ is a (smooth, projective, geometrically integral) curve of genus $g \geq 2$ and $J/\mathbb{Q}$ its Jacobian variety. If one is interested in determining the (finite, by Faltings) set of rational points $X(\mathbb{Q})$, then it can be useful to compute $J(\mathbb{Q})$ first. The latter is easier because $J(\mathbb{Q})$ is a finitely generated abelian group, and descent theory analogous to elliptic curves allows us to often do this in practice. If we pick a point $P \in X(\mathbb{Q})$ then we have an associated embedding $i_P : X \hookrightarrow J$. In favorable situations studying this embedding allows us to determine $X(\mathbb{Q})$ from $J(\mathbb{Q})$. For example, the method of Chabauty-Coleman gives a very concrete instance of this when the rank of $J(\mathbb{Q})$ is less than g (for a friendly introduction to this method see the nice survey of McCallum-Poonen).

The moral is: by replacing X by J, we somehow have made the geometry harder but the arithmetic easier.

The relation with motives can be explained in relatively concrete terms. The ℓ-adic cohomology groups $H^i(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$ are zero if $i \neq 0, 1, 2$ and isomorphic to $\mathbb{Q}_l, \mathbb{Q}_l(-1)$ if $i = 0, 2$ respectively. (The minus -1 denotes the Tate twist.) So the only interesting degree is $i = 1$, and pulling back via i_P will induce an isomorphism $H^1(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l) \simeq H^1(J_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$. This last group (with its Galois action) is isomorphic to the dual of the ℓ-adic Tate module of J. So J and its torsion points encapsulate all the cohomological information of X. Similar statements will hold for other Weil cohomology theories: the only interesting degree is 1 and i_P will induce an isomorphism on H^1.

Edit: as pointed out in the comments, the geometry of J is arguably easier than that of X. A better moral is thus maybe that we have made the space we're considering larger but richer in structure.

(5) I must respectfully disagree that replacing C with J "makes the geometry harder." It does increase the dimension, which one could argues makes the geometry harder, but it introduces a group structure, and I'd sugestt that the geometry of a high dimensional group variety (especially one that's compact) is much less difficult than the geometry of lower dimensional varieties having less structure. Or even ignoring the group structure, J has Kodaira dimension 0, while C has Kodaira dimension 1, again suggesting that J's geometry is simpler than C's. - **Joe Silverman**
Thanks for the comment, I'll edit my vague moral to make it more accurate. - **Jef**
What are the favourable situations that allow us to determine $J(\mathbb{Q})$ from $X(\mathbb{Q})$? Also, why does i_P become an isomorphism on H^1? - **Rdrr**

2

[+4] [2020-07-07 13:41:05] Damien Robert

As outlined by the other answers, the Jacobian J_X of a curve X defined over $\mathbb{F}_q$ indeed encapsulates all cohomology information of X. In particular one can read the zeta function ζ_X directly on J_X: the numerator of ζ_X is simply the (reciprocal) polynomial of the Frobenius π_q acting on J_X.

In particular André Weil's original proof of the Hasse-Weil bound for curves used Jacobians (implicitely). That was a big motivation in his *Foundations of algebraic geometry*: the algebraic construction of Jacobians over any field.

By the way over $\mathbb{C}$ the Abel-Jacobi map shows that the Jacobian of X is intimately related to the study of abelian integrals. I think historically that was the prime motivation to study Jacobians. A fun fact is that modular functions coming from hyperelliptic integrals can be used to solve algebraic equations. Cf the appendix of Mumford's TATA2.

3
