

NATIONAL INSTITUTE OF SCIENCE EDUCATION AND RESEARCH  
SCHOOL OF MATHEMATICAL SCIENCES

M599: DISSERTATION

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# Cousin problem for analytic hypersurface in $\mathbb{C}^n$

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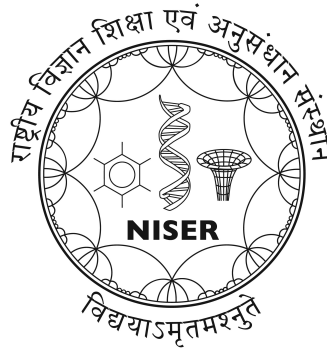
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### **Abstract**

The report begins with an introduction to the theory of complex manifolds. Then a proof of Dolbeault theorem, analogous to that of de Rham theorem, has been presented. Finally, the utility of Dolbeault-Čech isomorphism is illustrated by proving that every analytic hypersurface in  $\mathbb{C}^n$  can be described as the zero-locus of an entire function.

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# Contents

<b>Abstract</b>	<b>1</b>
<b>Introduction</b>	<b>3</b>
<b>1 Dolbeault cohomology</b>	<b>6</b>
1.1 Differential forms on $\mathbb{C}^n$	6
1.1.1 Tangent space	6
1.1.2 Cotangent space	8
1.1.3 Differential forms	9
1.1.4 Exterior derivative	10
1.2 $\bar{\partial}$ -closed and exact forms on $\mathbb{C}^n$	11
1.2.1 Cauchy integral formula	12
1.2.2 $\bar{\partial}$ -Poincaré lemma	17
1.3 Differential forms on complex manifolds	21
1.3.1 Complex differential forms	22
1.3.2 Holomorphic differential forms	26
1.4 $\bar{\partial}$ -closed and exact forms on complex manifolds	28
1.4.1 Dolbeault cohomology	28
1.4.2 $\bar{\partial}$ -Poincaré lemma for complex manifolds	29
<b>2 Cousin problems</b>	<b>30</b>
2.1 Cousin problems for $\mathbb{C}$	30
2.1.1 Mittag-Leffler theorem	30
2.1.2 Weierstrass theorem	32
2.2 Cousin problems for $\mathbb{C}^n$	34
2.2.1 Cousin I	34
2.2.2 Cousin II	35
2.3 Cousin problem for analytic hypersurface in $\mathbb{C}^n$	36
2.3.1 Analytic subvariety of a complex manifold	36
2.3.2 Sheaf theory and Čech cohomology	38
2.3.3 Dolbeault isomorphism	40
2.3.4 Solution of the problem	42
<b>Future work</b>	<b>44</b>
<b>A Algebra</b>	<b>47</b>
A.1 Complexification of vector space	47
A.2 Linear complex structure	50
A.3 Multilinear algebra	52

<b>B</b>	<b>Analysis</b>	<b>55</b>
B.1	Several variable holomorphic functions . . . . .	55
B.2	Algebraic properties of $\mathcal{O}_w$ . . . . .	59
B.3	Several variable holomorphic mappings . . . . .	62
<b>C</b>	<b>Topology</b>	<b>65</b>
	<b>Bibliography</b>	<b>69</b>

# Introduction

In 1876, Karl Weierstrass asked the following three questions regarding complex valued holomorphic and meromorphic functions defined on an open subset  $U$  of  $\mathbb{C}$  [20, Chapter 2]:

- W1. Does there exist a holomorphic function with prescribed zeros?
- W2. Is every meromorphic function on a quotient of two holomorphic functions?
- W3. Does there exist a meromorphic function with prescribed poles and their principal part?

The answer to all these questions is yes. The first two questions were answered by Weierstrass himself in 1876, and the third question was answered by Gösta Mittag-Leffler during 1876-1882. The answer to the first and second question follows from the *Weierstrass factorization theorem*. Moreover, the affirmative answer to the second question is a corollary to the first one [3, Theorem VII.5.15, Corollary VII.5.20]. The answer to the third question is known as the *Mittag-Leffler theorem*, and the Weierstrass factorization theorem can be deduced from it [3, Theorem VIII.3.2, Exercise VIII.3.3].

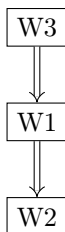


Figure 1: The relation between Weierstrass' questions identified by Mittag-Leffler.

The close bond between these three questions motivated other mathematicians to ask these question for complex valued holomorphic and meromorphic functions defined on open sets in  $\mathbb{C}^n$ . In 1883, Henri Poincaré generalized W2 by proving that every meromorphic function on  $\mathbb{C}^2$  is a quotient of two holomorphic functions on  $\mathbb{C}^2$  [15, Chapter 6] [2, §2]. However, there wasn't much progress made until 1895, when Pierre Cousin proved in his Ph.D. thesis that W1, W2 and W3 for product domains  $X = U_1 \times U_2 \times \dots \times U_n \subset \mathbb{C}^n$  are consequences of a single fundamental theorem [2, §3.1].

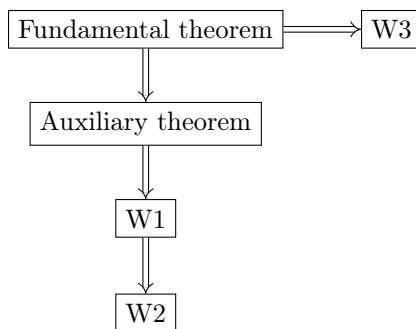


Figure 2: The relation between Weierstrass' questions for product domains identified by Cousin.

Therefore, Cousin was successful in bringing together the three problems of Weierstrass to make one coherent family. Moreover, the methods of Poincaré and Cousin exhibited what would later be called the “from local to global” problem form. However, in 1913, Thomas Hakon Grönwall and William Fogg Osgood found a counter example to W2, i.e. in the product of two ring-shaped domains there is a meromorphic function that cannot be written as the quotient of two holomorphic functions. Since W2 was an easy consequence of W1, they concluded that there was some flaw<sup>2</sup> in the proof of *auxiliary theorem*, which was the logarithmic variant of Cousin’s fundamental theorem [2, §3.3]. Later, in 1934, Henri Cartan published a three-page note to show that the three problems had not significantly changed since Cousin, and gave the following labels [2, §3.4]:

**Cousin I:** Name given to Cousin’s fundamental theorem. Also known as the additive problem.

**Cousin II:** Name given to Cousin’s auxiliary theorem. Also known as the multiplicative problem.

**Poincaré problem:** Name given to the problem about the quotient representation of meromorphic functions.

Kyoshi Oka made a breakthrough by first solving Cousin I for bounded domains of holomorphy in 1937 and then an year later establishing that Cousin II for domains of holomorphy is a problem of purely topological nature. That is, he proved that for domains of holomorphy, the solvability of Cousin II depends only on a topological property of the zero-locus [2, §3.4.2]. To illustrate the independence of Cousin II, he also gave an example of product domain (since every product domain is a domain of holomorphy), in which  $\boxed{\text{Cousin I}} \not\Rightarrow \boxed{\text{Cousin II}}$  [13, p. 250].

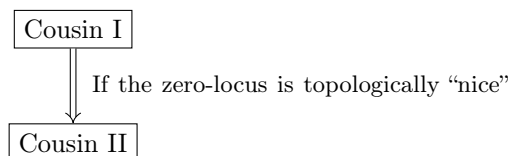


Figure 3: The relation between Cousin problems for domains of holomorphy identified by Oka.

In 1944, Cartan generalized the Cousin problems by recasting them in terms of ideals<sup>3</sup> [2, §4]. In particular, this theory in the new setting enabled the use of powerful abstract methods such as *Hilbert’s Nullstellensatz* available in algebraic geometry [10, Proposition 1.1.29].

We saw in previous report that during 1945-1951 the concept of sheaf and sheaf cohomology was developed [12, pp. 2-3]. Fortunately, during these developments, several important questions left pending in Cartan’s 1944 paper were also answered [2, §5]. From 1949 to 1953, Cartan organized various seminars which were devoted to the study of fibre-spaces, homotopy theory, cohomology theories and analytic functions in several variables. During the last three talks, the cohomology of coherent sheaves on Stein spaces was developed and Cartan proved two results concerning a coherent sheaf  $\mathcal{F}$  on a Stein manifold  $X$  which were analogous to Cousin problems (called Cartan A and Cartan B) [2, §5.5]. For more details, refer to the books by Gunning and Rossi [9], Kaup and Kaup [11], Fritzsche and Grauert [6], Maurin [15], Krantz [13] and Taylor [19].

In 1952, Cartan’s student Jean-Pierre Serre<sup>4</sup> gave the cohomological formulation of the conditions for solving the Cousin problems [2, p. 62]:

<sup>2</sup>Cousin thought he had established  $\boxed{\text{Fundamental theorem}} \implies \boxed{\text{Auxiliary theorem}}$  for any product domain, but he had proved it only for those product domains in which at most one of the components is not necessarily simply connected.

<sup>3</sup>This was actually the second half of a single work. The first half was published in 1940, but the *Second World War* caused the delay in the publication of the other half.

<sup>4</sup>In 1953, he also proved that Poincaré’s problem is solvable for Stein manifold, i.e. on a Stein manifold any meromorphic function is the quotient of two holomorphic functions.



Let  $X$  be a complex analytic variety<sup>5</sup>,  $\mathcal{O}$  be the sheaf of holomorphic complex valued functions and  $\mathcal{M}$  be the sheaf of meromorphic complex valued functions on  $X$ . Then Cousin I is solvable for  $X$  if and only if  $\check{H}^1(X, \mathcal{O}) \rightarrow \check{H}^1(X, \mathcal{M})$  is one to one and onto, and Cousin II is solvable for  $X$  if and only if  $\check{H}^1(X, \mathcal{O}^*) \rightarrow \check{H}^1(X, \mathcal{M}^*)$  is one to one and onto. In particular, for Cousin I to be solvable, it is sufficient that  $\check{H}^1(X, \mathcal{O}) = 0$  and for Cousin II to be solvable, it is sufficient that  $\check{H}^1(X, \mathcal{O}^*) = 0$ .

Pierre Dolbeault, another student of Cartan, in 1953 introduced the  $\bar{\partial}$ -cohomology<sup>6</sup> of the differential forms defined on complex analytic manifolds [7, §9.1.1]. He proved that this holomorphic analogue of de Rham cohomology defined on real manifolds is isomorphic to the sheaf cohomology of the sheaf of holomorphic differential forms [4]. Therefore, Dolbeault's theorem is a complex analogue of de Rham's theorem<sup>7</sup> [12, Theorem 3.1]. Using the Dolbeault-Čech isomorphism we get that  $\check{H}^1(\mathbb{C}^n, \mathcal{O}) = 0$  (Theorem 2.8). Combining this with the purely topological fact that  $\check{H}^1(\mathbb{C}^n, \mathbb{Z}) = 0$  (Corollary 2.2), and using the exponential sheaf sequence we can conclude that  $\check{H}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$  (Lemma 2.3). Hence proving that both the Cousin problems are solvable for  $\mathbb{C}^n$  [8, pp. 46-47].

This report consists of two chapters. In chapter 1 we will discuss various concepts related to complex differential forms and complex manifolds needed to define Dolbeault cohomology. We will also develop the tools like  $\bar{\partial}$ -Poincaré lemma, which will be used later to establish important sheaf theoretic results about the complex differential forms. In chapter 2 we will first illustrate the local to global principle by discussing the solution of Cousin problems for  $\mathbb{C}$ . Then we will prove Dolbeault theorem and use it to solve Cousin problem for analytic hypersurface in  $\mathbb{C}^n$ .

Apart from the two chapters, we have also included three appendices. In Appendix A, to supplement the discussions in the first chapter, we have stated a few facts from linear algebra. In Appendix B we have discussed the function theory of several complex variables, which will be used in both the chapters. In Appendix C some fundamental results about smooth partition of unity, which will play an important role in various arguments presented in the report, have been stated.

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<sup>5</sup>For example, complex manifold.

<sup>6</sup>Now called *Dolbeault cohomology*.

<sup>7</sup>However, note that, unlike de Rham cohomology, Dolbeault cohomology is not a topological invariant because it depends closely on the complex structure.

# Chapter 1

## Dolbeault cohomology

### 1.1 Differential forms on $\mathbb{C}^n$

This section generalizes the concepts discussed in previous report [12, §1.1, 1.3], following the discussion from [10, §1.3] and [24, §1.3].

#### 1.1.1 Tangent space

**Definition 1.1** (Real tangent space). Let  $U \subset \mathbb{C}^n$  be an open subset. In particular, we can consider  $U \subset \mathbb{R}^{2n}$ , to be a smooth manifold of dimension  $2n$ . Then for  $w \in U$  we define the *real tangent space* of  $U$  at the point  $w$  as the real vector space of  $\mathbb{R}$ -linear derivations on the ring of real-valued smooth functions in a neighborhood of  $w$ , i.e.

$$T_{w,\mathbb{R}}U = \{X_w : C_w^\infty(U) \rightarrow \mathbb{R} \mid X_w(fg) = X_w(f)g(w) + f(w)X_w(g)\}$$

**Remark 1.1.** If we write the standard coordinates on  $\mathbb{C}^n$  as  $z_j = x_j + iy_j$ , then a canonical basis of  $T_{w,\mathbb{R}}U$  is given by the tangent vectors

$$\left\{ \frac{\partial}{\partial x_1} \Big|_w, \dots, \frac{\partial}{\partial x_n} \Big|_w, \frac{\partial}{\partial y_1} \Big|_w, \dots, \frac{\partial}{\partial y_n} \Big|_w \right\}$$

Clearly,  $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}U) = 2n$  as seen in the case of smooth manifolds.

**Definition 1.2** (Complexified tangent space). Let  $U \subset \mathbb{C}^n$  be an open subset. Then we define the *complexified tangent space* of  $U$  at the point  $w$  to be the complexification<sup>1</sup> of real tangent space of  $U$  at  $w$

$$T_{w,\mathbb{C}}U = T_{w,\mathbb{R}}U \otimes_{\mathbb{R}} \mathbb{C}$$

**Remark 1.2.** We can also use the canonical basis of real tangent space to define its complexification [17, p. 379]. We can view  $T_{w,\mathbb{C}}U$  as the complex vector space of  $\mathbb{C}$ -linear derivations in the ring of complex-valued smooth functions<sup>2</sup> in a neighborhood of  $w$ , i.e.  $T_{w,\mathbb{C}}U$  also has the same basis

$$\left\{ \frac{\partial}{\partial x_1} \Big|_w, \dots, \frac{\partial}{\partial x_n} \Big|_w, \frac{\partial}{\partial y_1} \Big|_w, \dots, \frac{\partial}{\partial y_n} \Big|_w \right\}$$

Hence, as expected, we have  $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}U) = \dim_{\mathbb{C}}(T_{w,\mathbb{C}}U)$ .

---

<sup>1</sup>For the definition, see [Definition A.2](#).

<sup>2</sup>That is, they possess partial derivatives of all orders with respect to the  $2n$  real coordinates in  $\mathbb{C}^n$ .

**Definition 1.3** (Complex structure for  $T_{w,\mathbb{R}}U$ ). Each real tangent space  $T_{w,\mathbb{R}}U$  admits a natural complex structure<sup>3</sup> defined on the basis as

$$\begin{aligned} J : T_{w,\mathbb{R}}U &\rightarrow T_{w,\mathbb{R}}U \\ \frac{\partial}{\partial x_j} \Big|_w &\mapsto \frac{\partial}{\partial y_j} \Big|_w \\ \frac{\partial}{\partial y_j} \Big|_w &\mapsto -\frac{\partial}{\partial x_j} \Big|_w \end{aligned}$$

**Remark 1.3.** We will regard this  $J$  as a vector bundle endomorphism of the smooth vector bundle  $T_{\mathbb{R}}U$  over  $U$ .

**Proposition 1.1.** *The complexified tangent bundle  $T_{\mathbb{C}}U = T_{\mathbb{R}}U \otimes_{\mathbb{R}} \mathbb{C}$  decomposes as a direct sum of complex vector bundles*

$$T_{\mathbb{C}}U = (T_{\mathbb{R}}U)^{1,0} \oplus (T_{\mathbb{R}}U)^{0,1}$$

such that the  $\mathbb{C}$ -linear extension  $\tilde{J} = J \otimes \mathbf{1}_{\mathbb{C}}$  satisfies

$$\tilde{J}|_{(T_{\mathbb{R}}U)^{1,0}} = i \cdot \mathbf{1}_{T_{\mathbb{C}}U} \quad \text{and} \quad \tilde{J}|_{(T_{\mathbb{R}}U)^{0,1}} = -i \cdot \mathbf{1}_{T_{\mathbb{C}}U}$$

*Proof.* Fix a point  $w \in U$ , and substitute  $V = T_{w,\mathbb{R}}U$  and  $V_{\mathbb{C}} = T_{w,\mathbb{C}}U$  in the proof of [Proposition A.7](#). □

**Remark 1.4.** As seen in the proof of [Proposition A.7](#), we can write

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{1}{2} \left( \frac{\partial}{\partial x_j} - iJ \left( \frac{\partial}{\partial x_j} \right) \right) + \frac{1}{2} \left( \frac{\partial}{\partial x_j} + iJ \left( \frac{\partial}{\partial x_j} \right) \right) \\ \frac{\partial}{\partial y_j} &= \frac{1}{2} \left( \frac{\partial}{\partial y_j} - iJ \left( \frac{\partial}{\partial y_j} \right) \right) + \frac{1}{2} \left( \frac{\partial}{\partial y_j} + iJ \left( \frac{\partial}{\partial y_j} \right) \right) \end{aligned}$$

where

$$\frac{1}{2} \left( \frac{\partial}{\partial x_j} - iJ \left( \frac{\partial}{\partial x_j} \right) \right), \frac{1}{2} \left( \frac{\partial}{\partial y_j} - iJ \left( \frac{\partial}{\partial y_j} \right) \right) \in (T_{\mathbb{R}}U)^{1,0}$$

and

$$\frac{1}{2} \left( \frac{\partial}{\partial x_j} + iJ \left( \frac{\partial}{\partial x_j} \right) \right), \frac{1}{2} \left( \frac{\partial}{\partial y_j} + iJ \left( \frac{\partial}{\partial y_j} \right) \right) \in (T_{\mathbb{R}}U)^{0,1}$$

Next, use the definition of  $J$  to get:

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) + \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \\ \frac{\partial}{\partial y_j} &= \frac{i}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) - \frac{i}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \end{aligned}$$

**Definition 1.4** (Complex partial derivative). Based on the discussion above, we define the operators:

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

for  $j = 1, \dots, n$ .

---

<sup>3</sup>For definition, see [Definition A.3](#).

**Remark 1.5.** Hence we can say that  $\left\{ \frac{\partial}{\partial z_1} \Big|_w, \dots, \frac{\partial}{\partial z_n} \Big|_w \right\}$  is a basis for the complex vector space  $(T_{w,\mathbb{R}}U)^{1,0}$  and  $\left\{ \frac{\partial}{\partial \bar{z}_1} \Big|_w, \dots, \frac{\partial}{\partial \bar{z}_n} \Big|_w \right\}$  is a basis for the complex vector space  $(T_{w,\mathbb{R}}U)^{0,1}$ . Therefore, the following forms a basis of  $T_{w,\mathbb{C}}U$

$$\left\{ \frac{\partial}{\partial z_1} \Big|_w, \dots, \frac{\partial}{\partial z_n} \Big|_w, \frac{\partial}{\partial \bar{z}_1} \Big|_w, \dots, \frac{\partial}{\partial \bar{z}_n} \Big|_w \right\}$$

**Proposition 1.2.** Let  $f : U \rightarrow V$  be a holomorphic map between open subsets  $U \subset \mathbb{C}^n$  and  $V \subset \mathbb{C}^n$ . The  $\mathbb{C}$ -linear extension of the pushforward map<sup>4</sup>  $f_* : T_{w,\mathbb{R}}U \rightarrow T_{f(w),\mathbb{R}}V$  respects the above decomposition, i.e.  $\tilde{f}_* ((T_{w,\mathbb{R}}U)^{1,0}) \subset (T_{f(w),\mathbb{R}}V)^{1,0}$  and  $\tilde{f}_* ((T_{w,\mathbb{R}}U)^{0,1}) \subset (T_{f(w),\mathbb{R}}V)^{0,1}$ .

*Proof.* Follows directly from the [Remark B.5](#). □

### 1.1.2 Cotangent space

**Definition 1.5** (Real cotangent space). Let  $U \subset \mathbb{C}^n$  be an open subset. In particular, we can consider  $U \subset \mathbb{R}^{2n}$ , to be a smooth manifold of dimension  $2n$ . Then for  $w \in U$  we define the *real cotangent space* of  $U$  at the point  $w$  as dual space of the real vector space  $T_{w,\mathbb{R}}U$ , i.e.

$$T_{w,\mathbb{R}}^*U = \text{Hom}_{\mathbb{R}}(T_{w,\mathbb{R}}U, \mathbb{R})$$

**Remark 1.6.** If we write the standard coordinates on  $\mathbb{C}^n$  as  $z_j = x_j + iy_j$ , then a canonical basis of  $T_{w,\mathbb{R}}^*U$  is given by the cotangent vectors

$$\{dx_1|_w, \dots, dx_n|_w, dy_1|_w, \dots, dy_n|_w\}$$

Clearly,  $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}^*U) = 2n$  as seen in the case of smooth manifolds.

**Definition 1.6** (Complexified cotangent space). Let  $U \subset \mathbb{C}^n$  be an open subset. Then we defined the *complexified cotangent space* of  $U$  at the point  $w$  to be the complexification of real cotangent space

$$T_{w,\mathbb{C}}^*U = T_{w,\mathbb{R}}^*U \otimes_{\mathbb{R}} \mathbb{C}$$

**Remark 1.7.** We can also use the canonical basis of real cotangent space to define its complexification [17, p. 379]. We can view  $T_{w,\mathbb{C}}^*U$  as the complex vector space with the basis

$$\{dx_1|_w, \dots, dx_n|_w, dy_1|_w, \dots, dy_n|_w\}$$

Hence, as expected, we have  $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}^*U) = \dim_{\mathbb{C}}(T_{w,\mathbb{C}}^*U)$ .

**Remark 1.8.** As in [Proposition A.8](#), we get the complex structure  $\mathcal{J}$  on  $T_{w,\mathbb{R}}^*U$  from the complex structure  $J$  on  $T_{w,\mathbb{R}}U$ . We will regard this  $\mathcal{J}$  as a vector bundle endomorphism of the smooth vector bundle  $T_{\mathbb{R}}^*U$  over  $U$ .

**Proposition 1.3.** The complexified cotangent bundle  $T_{\mathbb{C}}^*U = T_{\mathbb{R}}^*U \otimes_{\mathbb{R}} \mathbb{C}$  decomposes as a direct sum of complex vector bundles

$$T_{\mathbb{C}}^*U = (T_{\mathbb{R}}^*U)^{1,0} \oplus (T_{\mathbb{R}}^*U)^{0,1}$$

such that the  $\mathbb{C}$ -linear extension  $\tilde{\mathcal{J}} = \mathcal{J} \otimes \mathbb{1}_{\mathbb{C}}$  satisfies

$$\tilde{\mathcal{J}}|_{(T_{\mathbb{R}}^*U)^{1,0}} = i \cdot \mathbb{1}_{T_{\mathbb{R}}^*U} \quad \text{and} \quad \tilde{\mathcal{J}}|_{(T_{\mathbb{R}}^*U)^{0,1}} = -i \cdot \mathbb{1}_{T_{\mathbb{R}}^*U}$$

*Proof.* Fix a point  $w \in U$ , and substitute  $V = T_{w,\mathbb{R}}U$  and  $V_{\mathbb{C}} = T_{w,\mathbb{C}}U$  in the proof of [Proposition A.8](#). □

<sup>4</sup>It was defined in the the previous report [12, Definition 1.5].

**Remark 1.9.** From [Corollary A.2](#), we have  $T_{w,\mathbb{C}}^*U = (T_{w,\mathbb{R}}^*U)_{\mathbb{C}} \cong (T_{w,\mathbb{C}}U)^*$ . Hence we can obtain another basis for  $T_{w,\mathbb{C}}^*U$  by defining the dual basis of  $(T_{w,\mathbb{R}}U)^{1,0}$  and  $(T_{w,\mathbb{R}}U)^{0,1}$ . Observe that:

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) (dx_k + i dy_k) &= \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \\ \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) (dx_k - i dy_k) &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) (dx_k + i dy_k) &= 0 \\ \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) (dx_k - i dy_k) &= \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \end{aligned}$$

**Definition 1.7** (Complex differential). Based on the discussion above, we define the differentials:

$$dz_j := dx_j + i dy_j \quad \text{and} \quad d\bar{z}_j := dx_j - i dy_j$$

for  $j = 1, \dots, n$ .

**Remark 1.10.** Hence we can say that  $\{dz_1|_w, \dots, dz_n|_w\}$  is a basis for the complex vector space  $(T_{w,\mathbb{R}}^*U)^{1,0}$  and  $\{d\bar{z}_1|_w, \dots, d\bar{z}_n|_w\}$  is a basis for the complex vector space  $(T_{w,\mathbb{R}}^*U)^{0,1}$ . Therefore, the following forms a basis of  $T_{w,\mathbb{C}}^*U$

$$\{dz_1|_w, \dots, dz_n|_w, d\bar{z}_1|_w, \dots, d\bar{z}_n|_w\}$$

### 1.1.3 Differential forms

**Definition 1.8** (Differential  $(p, q)$ -form). Let  $U \subset \mathbb{C}^n$  be an open subset. Over  $U$  one has the complex vector bundle<sup>5</sup> of rank  $\binom{n}{p} \binom{n}{q}$  defined as

$$\bigwedge^{p,q} T_{\mathbb{R}}^*U := \bigwedge^p ((T_{\mathbb{R}}^*U)^{1,0}) \otimes_{\mathbb{C}} \bigwedge^q ((T_{\mathbb{R}}^*U)^{0,1})$$

whose fiber is  $\bigwedge^{p,q} T_{w,\mathbb{R}}^*U$ . The smooth sections<sup>6</sup> of this vector bundle are called the *differential forms of type  $(p, q)$*  on  $U$ . The space of all smooth differential forms of type  $(p, q)$  on  $U$  is denoted by  $\Omega^{p,q}(U)$ .

**Remark 1.11.** Any  $(p, q)$ -form  $\omega \in \Omega^{p,q}(U)$  can be written uniquely as

$$\omega = \sum_{|\alpha|=p, |\beta|=q} f_{\alpha\beta} dz_{\alpha} \wedge d\bar{z}_{\beta}$$

where  $\alpha = (\alpha_1, \dots, \alpha_p)$  and  $\beta = (\beta_1, \dots, \beta_q)$  are multi-indices with  $1 \leq \alpha_j, \beta_k \leq n$ ;  $dz_{\alpha} = dz_{\alpha_1} \wedge \dots \wedge dz_{\alpha_p}$  and  $d\bar{z}_{\beta} = d\bar{z}_{\beta_1} \wedge \dots \wedge d\bar{z}_{\beta_q}$ ; and  $f_{\alpha\beta}$  is a complex-valued smooth function on  $U$ , i.e.  $f_{\alpha\beta} \in C^{\infty}(U)$ . In particular,  $\Omega^{0,0}(U) = C^{\infty}(U)$ .

<sup>5</sup>That is, in the definition of smooth vector bundle, replace  $\mathbb{R}$  by  $\mathbb{C}$ . This will be discussed in detail later, see [Remark 1.27](#).

<sup>6</sup>For definition, see [\[12, Definition 1.38\]](#).

**Remark 1.12.** Let  $\Omega_{\mathbb{C}}^k(U)$  be the space of sections of vector bundle  $\bigwedge^k T_{\mathbb{C}}^*U$ . Any element  $\omega \in \Omega_{\mathbb{C}}^1(U)$  can thus be written in a unique manner in the form

$$\omega = \sum_{j=1}^n f_j dz_j + \sum_{k=1}^n f_k d\bar{z}_k$$

Moreover, if  $\omega \in \Omega_{\mathbb{C}}^r(U)$  and  $\eta \in \Omega_{\mathbb{C}}^s(U)$  then  $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega \in \Omega_{\mathbb{C}}^{r+s}(U)$ .

**Remark 1.13.** By [Remark A.12](#) we have

$$\bigwedge^k T_{\mathbb{C}}^*U \cong \bigoplus_{p+q=k} \bigwedge^{p,q} T_{\mathbb{R}}^*U \implies \Omega_{\mathbb{C}}^k(U) \cong \bigoplus_{p+q=k} \Omega^{p,q}(U)$$

Thus we have natural projection operators  $\bigwedge^k T_{\mathbb{C}}^*U \rightarrow \bigwedge^{p,q} T_{\mathbb{R}}^*U$  and  $\Omega_{\mathbb{C}}^k(U) \rightarrow \Omega^{p,q}(U)$ , denoted by  $\Pi^{p,q}$  for  $p+q=k$ .

### 1.1.4 Exterior derivative

**Definition 1.9** (Differential of a  $(p, q)$ -form). Let  $U \subset \mathbb{C}^n$  be an open subset, and  $d : \Omega_{\mathbb{C}}^k(U) \rightarrow \Omega_{\mathbb{C}}^{k+1}(U)$  be the complex linear extension of the usual exterior differential<sup>7</sup>. Then

$$\partial : \Omega^{p,q}(U) \rightarrow \Omega^{p+1,q}(U) \quad \text{and} \quad \bar{\partial} : \Omega^{p,q}(U) \rightarrow \Omega^{p,q+1}(U)$$

are defined as  $\partial := \Pi^{p+1,q} \circ d$  and  $\bar{\partial} := \Pi^{p,q+1} \circ d$ .

**Remark 1.14.** For any  $f \in \Omega_{\mathbb{C}}^0(U) = C^\infty(U)$  one has

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j} dy_j = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j = \partial f + \bar{\partial} f$$

Since  $\{d\bar{z}_j\}$  are linearly independent, by [Theorem B.2](#),  $f$  is holomorphic if and only if  $\bar{\partial} f = 0$ .

**Lemma 1.1.** For the differential operators  $\partial$  and  $\bar{\partial}$  one has:

1.  $d = \partial + \bar{\partial}$
2.  $\partial^2 = \bar{\partial}^2 = 0$  and  $\partial \bar{\partial} = -\bar{\partial} \partial$
3. They satisfy the Leibniz's rule, i.e.

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta$$

$$\bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta$$

for  $\omega \in \Omega^{p,q}(U)$  and  $\eta \in \Omega^{r,s}(U)$ .

*Proof.* We will use the properties of  $d$  studied earlier<sup>8</sup>.

1. This follows from the local description of  $\partial$  and  $\bar{\partial}$ . Given  $\omega = \sum_{\alpha,\beta} f_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta \in \Omega^{p,q}(U)$ , we have

$$\partial\omega = \sum_{j=1}^n \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial z_j} dz_j \wedge dz_\alpha \wedge d\bar{z}_\beta$$

$$\bar{\partial}\omega = \sum_{j=1}^n \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_\alpha \wedge d\bar{z}_\beta$$

<sup>7</sup>For definition, see [12, Definition 1.17].

<sup>8</sup>In the previous report, see [12, Theorem IX, Theorem XXVI].

2. Recall that  $d^2 = 0$  since the second order partial derivatives commute. Since  $d = \partial + \bar{\partial}$ , we have

$$\begin{aligned} d^2 &= d \circ d \\ &= d \circ \partial + d \circ \bar{\partial} \\ &= \partial \circ \partial + \bar{\partial} \circ \partial + \partial \circ \bar{\partial} + \bar{\partial} \circ \bar{\partial} \\ &= \partial^2 + \bar{\partial}\partial + \partial\bar{\partial} + \bar{\partial}^2 \end{aligned}$$

Moreover, each operator projects to a different summand of  $\Omega_{\mathbb{C}}^{p+q+2}(U)$ , we obtain

$$\partial^2 = \bar{\partial}\partial + \partial\bar{\partial} = \bar{\partial}^2 = 0$$

Therefore,  $\partial^2 = \bar{\partial}^2 = 0$  and  $\partial\bar{\partial} = -\bar{\partial}\partial$ .

3. Recall that for  $\omega \in \Omega_{\mathbb{C}}^{p+q}(U)$  and  $\eta \in \Omega_{\mathbb{C}}^{r+s}(U)$  we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{p+q}\omega \wedge d\eta \in \Omega_{\mathbb{C}}^{p+q+r+s+1}(U)$$

Since  $\partial := \Pi^{p+r+1, q+s} \circ d$ , taking the  $(p+r+1, q+s)$ -parts on both sides one obtains

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$

Similarly, taking the  $(p+r, q+s+1)$ -parts one obtains

$$\bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \bar{\partial}\eta$$

□

**Remark 1.15.** As noted in [Remark A.13](#),  $\Omega_{\mathbb{C}}^k(U)$  does not reflect the complex structure  $J$ , whereas its decomposition into subspaces  $\Omega^{p,q}(U)$  does.

## 1.2 $\bar{\partial}$ -closed and exact forms on $\mathbb{C}^n$

In this section the proof of  $\bar{\partial}$ -Poincaré lemma will be discussed, following [9, §I.D] and [19, §1.4, 10.1].

**Definition 1.10** ( $\bar{\partial}$ -closed forms). Let  $U \subset \mathbb{C}^n$  be an open subset. Then a differential form  $\omega \in \Omega^{p,q}(U)$  is called  $\bar{\partial}$ -closed if  $\bar{\partial}\omega = 0$ .

**Remark 1.16.** If  $U$  is an open set in  $\mathbb{C}^n$ , let  $\mathcal{Z}^{p,q}(U)$  denote the set of all  $\bar{\partial}$ -closed  $(p, q)$ -forms on  $U$ . The sum of two such  $(p, q)$ -forms is another  $\bar{\partial}$ -closed  $(p, q)$ -form, and so is the product of a  $\bar{\partial}$ -closed  $(p, q)$ -form by a scalar. Hence  $\mathcal{Z}^{p,q}(U)$  is the vector sub-space of  $\Omega^{p,q}(U)$ . Also, from [Theorem B.2](#) it follows that  $\mathcal{Z}^{p,0}(U)$  is the space of  $(p, 0)$ -forms whose coefficients are complex-valued holomorphic functions in  $U$ . In particular, note that  $\mathcal{Z}^{0,0}(U) = \mathcal{O}(U)$ , the space of complex-valued functions holomorphic in  $U$ .

**Definition 1.11** ( $\bar{\partial}$ -exact forms). Let  $U \subset \mathbb{C}^n$  be an open subset. Then a differential form  $\omega \in \Omega^{p,q}(U)$ , for  $q > 0$ , is called  $\bar{\partial}$ -exact if  $\omega = \bar{\partial}\eta$  for some differential form  $\eta \in \Omega^{p,q-1}(U)$ .

**Remark 1.17.** If  $U$  is an open set in  $\mathbb{C}^n$ , let  $\mathcal{B}^{p,q}(U)$  denote the set of all  $\bar{\partial}$ -exact  $(p, q)$ -forms on  $U$ . The sum of two such  $(p, q)$ -forms is another  $\bar{\partial}$ -exact  $(p, q)$ -form, and so is the product of a  $\bar{\partial}$ -exact  $(p, q)$ -form by a scalar. Hence  $\mathcal{B}^{p,q}(U)$  is the vector sub-space of  $\Omega^{p,q}(U)$ . Moreover, the trivial form  $\omega \equiv 0$  is the only  $(p, 0)$ -form which is  $\bar{\partial}$ -exact for any value of  $p = 0, 1, \dots, n$ . That is,  $\mathcal{B}^{p,0}(U)$  consists only of zero.

**Theorem 1.1.** *Every  $\bar{\partial}$ -exact form is  $\bar{\partial}$ -closed.*

*Proof.* Let  $U$  be an open set in  $\mathbb{C}^n$  and  $\omega \in \mathcal{B}^{p,q}(U)$  such that  $\omega = \bar{\partial}\eta$  for some  $\eta \in \Omega^{p,q-1}(U)$ . From [Lemma 1.1](#) we know that  $\bar{\partial}\omega = \bar{\partial}(\bar{\partial}\eta) = 0$  hence  $\omega \in \mathcal{Z}^{p,q}(U)$  for all  $q \geq 1$ . For  $q = 0$ , the statement is trivially true.  $\square$

**Remark 1.18.** This theorem implies that  $\mathcal{B}^{p,q}(U) \subset \mathcal{Z}^{p,q}(U)$  for all  $q \geq 1$ . However, the converse doesn't always hold. For example, if  $U = \mathbb{C}^2 \setminus \{0\}$ , then the  $(0,1)$ -form

$$\omega = \begin{cases} \bar{\partial} \left( \frac{\bar{z}_2}{z_1 r^2} \right) & \text{when } z_1 \neq 0 \\ -\bar{\partial} \left( \frac{\bar{z}_1}{z_2 r^2} \right) & \text{when } z_2 \neq 0 \end{cases}$$

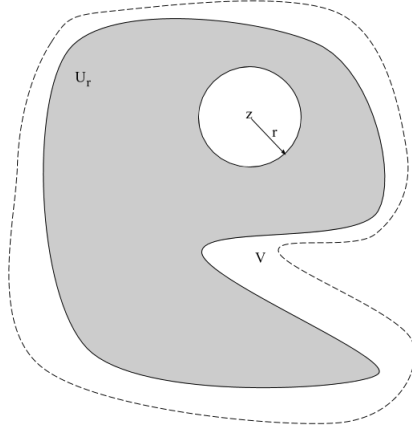
where  $(z_1, z_2) \in U$  and  $r^2 = |z_1|^2 + |z_2|^2$ , is  $\bar{\partial}$ -closed but not  $\bar{\partial}$ -exact [\[9, pp. 30–31\]](#).

### 1.2.1 Cauchy integral formula

**Proposition 1.4** (Generalized Cauchy integral formula). *Let  $U$  be a region<sup>9</sup> in  $\mathbb{C}$  bounded by a simple closed rectifiable curve<sup>10</sup>  $\gamma$ , and  $f$  be complex-valued smooth function in some open neighborhood  $V$  of  $\bar{U}$ . Then for any point  $z \in U$ ,*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} f(w) \frac{dw}{w-z} + \frac{1}{2\pi i} \iint_U \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}$$

*Proof.* For any point  $z \in U$  select a disc  $\Delta(z; r)$  with closure contained in  $U$ . Let  $\gamma_r$  be the boundary of the  $\Delta(z; r)$ , a circle of radius  $r$  centered at  $z$ . Furthermore, let  $U_r = U \setminus \bar{\Delta}(z; r)$  and observe that this is an open region bounded by  $\gamma - \gamma_r$ .



Now note that as a function of  $w$ , for a fixed  $z$ ,

$$\frac{\partial f(w)}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w-z} = \frac{\partial}{\partial \bar{w}} \left( \frac{f(w)}{w-z} \right) d\bar{w} \wedge dw = d \left( f(w) \frac{dw}{w-z} \right)$$

<sup>9</sup>A region is an open connected subset of the complex plane [\[3, p. 40\]](#).

<sup>10</sup>A rectifiable curve is a curve having finite length. In other words, the measure (for example, arc length or distance) between any two points of this curve is finite. For more details, see [\[3, p. 62\]](#).



whenever the functions involved are well defined<sup>11</sup>. Therefore, by the Stokes theorem<sup>12</sup> in the plane we get

$$\iint_{U_r} \frac{\partial f(w)}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w-z} = \iint_{U_r} d \left( f(w) \frac{dw}{w-z} \right) = \int_{\gamma} f(w) \frac{dw}{w-z} - \int_{\gamma_r} f(w) \frac{dw}{w-z} \quad (1.1)$$

Note that the integral of  $(w-z)^{-1} d\bar{w} \wedge dw$  exists on a bounded region, as seen by integrating it using polar coordinates centered at  $z$ . That is, substituting  $w = z + Re^{i\theta}$  and

$$\begin{aligned} d\bar{w} \wedge dw &= (dx + i dy) \wedge (dx - i dy) \\ &= -2i dx \wedge dy \\ &= -2i(\cos \theta dR - R \sin \theta d\theta) \wedge (\sin \theta dR + R \cos \theta d\theta) \\ &= 2iR d\theta \wedge dR \end{aligned}$$

for  $w = x + iy$ ,  $x = R \cos \theta$ , and  $y = R \sin \theta$ . We get

$$\iint_{U_r} \frac{d\bar{w} \wedge dw}{w-z} = 2i \iint_{U_r} e^{-i\theta} d\theta dR$$

Therefore, as  $r \rightarrow 0$ , the surface integral over  $U_r$  converges to the surface integral over  $U$

$$\lim_{r \rightarrow 0} \iint_{U_r} \frac{\partial f(w)}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w-z} = \iint_U \frac{\partial f(w)}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w-z} \quad (1.2)$$

Moreover, since  $\gamma_r$  is defined by  $w = z + re^{it}$  with  $0 \leq t \leq 2\pi$ , we have

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(w) \frac{dw}{w-z} = \lim_{r \rightarrow 0} \int_{t=0}^{2\pi} f(z + re^{it}) i dt = if(z) \int_{t=0}^{2\pi} dt = 2\pi if(z) \quad (1.3)$$

Letting  $r \rightarrow 0$  in (1.1), and using (1.2) and (1.3) we get

$$\begin{aligned} \iint_U \frac{\partial f(w)}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w-z} &= \int_{\gamma} f(w) \frac{dw}{w-z} - 2\pi if(z) \\ \implies f(z) &= \frac{1}{2\pi i} \int_{\gamma} f(w) \frac{dw}{w-z} + \frac{1}{2\pi i} \iint_U \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \end{aligned}$$

Hence completing the proof. □

**Remark 1.19.** If  $f$  is holomorphic then  $\frac{\partial f(w)}{\partial \bar{w}} = 0$  and we get the familiar Cauchy integral formula [3, Theorem IV.5.4]:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} f(w) \frac{dw}{w-z}$$

<sup>11</sup>Note the abuse of notations. Here  $f(w)$  is a function of  $w$  and  $\bar{w}$  which are linearly independent “variables”. The better notation would have been  $f(w, \bar{w})$  just like we have  $f(x, y)$  in  $\mathbb{R}^2$ . Hence  $\partial/\partial \bar{w}$  treats  $w$  as a constant. Moreover, the differential is well defined whenever  $w \neq z$ , which will hold when we apply the Stokes theorem.

<sup>12</sup>This is the standard Stokes theorem expressed in the complex notation [13, Theorem 1.1.1]: *Let  $U \subset \mathbb{C}^n$  be a bounded open set with rectifiable boundary and  $\omega \in \Omega^{p,q}(U)$  with  $p+q=2n$ . Then*

$$\int_{\partial U} \omega = \int_U d\omega = \int_U \partial\omega + \bar{\partial}\omega$$

**Corollary 1.1.** *Let  $U$  be a region in  $\mathbb{C}$  bounded by a simple closed rectifiable curve  $\gamma$ , and  $f$  be complex-valued smooth function in some open neighborhood  $V$  of  $\bar{U}$ . Then for any point  $z \in U$ ,*

$$f(z) = -\frac{1}{2\pi i} \int_{\gamma} f(w) \frac{d\bar{w}}{\bar{w} - \bar{z}} + \frac{1}{2\pi i} \iint_U \frac{\partial f(w)}{\partial w} \frac{dw \wedge d\bar{w}}{\bar{w} - \bar{z}}$$

*Proof.* Note that as a function of  $w$ , for a fixed  $z$ ,

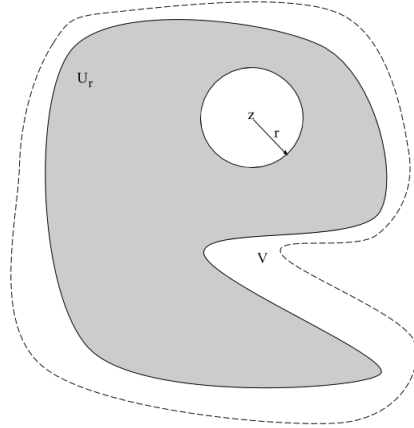
$$\frac{\partial f(w)}{\partial w} \frac{dw \wedge d\bar{w}}{\bar{w} - \bar{z}} = \frac{\partial}{\partial w} \left( \frac{f(w)}{\bar{w} - \bar{z}} \right) dw \wedge d\bar{w} = d \left( f(w) \frac{d\bar{w}}{\bar{w} - \bar{z}} \right)$$

whenever the functions involved are well defined. Now repeat the steps performed in the proof of previous result.  $\square$

**Proposition 1.5.** *Let  $U$  be an open subset of  $\mathbb{C}$  bounded by a simple closed rectifiable curve  $\gamma$ , and  $f$  be complex-valued smooth function in an open neighborhood  $V$  of  $\bar{U}$ . Then there exists a complex-valued smooth function  $g \in C^\infty(U)$  such that*

$$\frac{\partial g(z)}{\partial \bar{z}} = f(z)$$

*Proof.* For any point  $z \in U$  select a disc  $\Delta(z; r)$  with closure contained in  $U$ . Let  $\gamma_r$  be the boundary of the  $\Delta(z; r)$ , a circle of radius  $r$  centered at  $z$ . Furthermore, let  $U_r = U \setminus \bar{\Delta}(z; r)$  and observe that this is an open region bounded by  $\gamma - \gamma_r$ .



Now note that as a function of  $w$ , for a fixed  $z$ ,

$$d \log |w - z|^2 = d(\log(w - z) + \log(\bar{w} - \bar{z})) = \frac{dw}{w - z} + \frac{d\bar{w}}{\bar{w} - \bar{z}}$$

whenever the functions involved are well defined<sup>13</sup>. Therefore, by the Stokes theorem in the plane we get

$$\begin{aligned} \int_{\gamma} f(w) \log |w - z|^2 d\bar{w} - \int_{\gamma_r} f(w) \log |w - z|^2 d\bar{w} &= \iint_{U_r} d(f(w) \log |w - z|^2 d\bar{w}) \\ &= \iint_{U_r} \frac{\partial f(w)}{\partial w} \log |w - z|^2 dw \wedge d\bar{w} + \iint_{U_r} f(w) \frac{dw \wedge d\bar{w}}{w - z} \end{aligned} \quad (1.4)$$

<sup>13</sup>Note that  $\partial/\partial w$  and  $\partial/\partial \bar{w}$  treat  $\bar{w}$  and  $w$  as constants, respectively. Also recall that we can define the logarithm in every simply connected open set not containing 0 [3, Corollary IV.6.17]. In every of these open sets we can compute the differentials. It turns out that on the overlaps these differentials agree because different branches of the logarithm differ locally by a constant which is killed by taking a derivative [3, Corollary III.2.21]. Therefore, even though logarithm is not a globally defined function, its derivative is defined and smooth everywhere in  $\mathbb{C} \setminus \{0\}$ .

Observe that, as  $r \rightarrow 0$ , the surface integral over  $U_r$  converges to the surface integral over  $U$

$$\lim_{r \rightarrow 0} \iint_{U_r} \frac{\partial f(w)}{\partial w} \log |w - z|^2 dw \wedge d\bar{w} = \iint_U \frac{\partial f(w)}{\partial w} \log |w - z|^2 dw \wedge d\bar{w} \quad (1.5)$$

and

$$\lim_{r \rightarrow 0} \iint_{U_r} f(w) \frac{dw \wedge d\bar{w}}{w - z} = \iint_U f(w) \frac{dw \wedge d\bar{w}}{w - z} \quad (1.6)$$

Moreover, since  $\gamma_r$  is defined by  $w = z + re^{it}$  with  $0 \leq t \leq 2\pi$ , we have

$$\begin{aligned} \lim_{r \rightarrow 0} \int_{\gamma_r} f(w) \log |w - z|^2 d\bar{w} &= \lim_{r \rightarrow 0} \int_{t=0}^{2\pi} f(z + re^{it}) (-2r)(\log r) ie^{-it} dt \\ &\leq \lim_{r \rightarrow 0} \int_{t=0}^{2\pi} |f(z + re^{it})| (-2r)(\log r) |ie^{-it}| dt \\ &\leq \lim_{r \rightarrow 0} 2Mr(\log r) \int_{t=0}^{2\pi} dt \\ &= 4\pi M \lim_{r \rightarrow 0} r \log r = 0 \end{aligned} \quad (1.7)$$

where  $M = \sup_{z \in U} |f(z)|$  and  $|ie^{-it}| = 1$ . Letting  $r \rightarrow 0$  in (1.4), and using (1.5), (1.6) and (1.7) we get

$$\int_{\gamma} f(w) \log |w - z|^2 d\bar{w} = \iint_U \frac{\partial f(w)}{\partial w} \log |w - z|^2 dw \wedge d\bar{w} + \iint_U f(w) \frac{dw \wedge d\bar{w}}{w - z} \quad (1.8)$$

Next, we apply the operator  $\partial/\partial\bar{z}$  to each integral in (1.8). We can use Leibniz's differentiation under the integral sign<sup>14</sup> for the integrals where the integrand obtained after differentiation is still integrable. Hence we have

$$\begin{aligned} \frac{\partial}{\partial\bar{z}} \int_{\gamma} f(w) \log |w - z|^2 d\bar{w} &= \int_{\gamma} \frac{\partial \log |w - z|^2}{\partial\bar{z}} f(w) d\bar{w} = - \int_{\gamma} f(w) \frac{d\bar{w}}{\bar{w} - \bar{z}} \\ \frac{\partial}{\partial\bar{z}} \iint_U \frac{\partial f(w)}{\partial w} \log |w - z|^2 dw \wedge d\bar{w} &= \iint_U \frac{\partial \log |w - z|^2}{\partial\bar{z}} \frac{\partial f(w)}{\partial w} dw \wedge d\bar{w} = - \iint_U \frac{\partial f(w)}{\partial w} \frac{dw \wedge d\bar{w}}{\bar{w} - \bar{z}} \end{aligned}$$

Hence by applying  $\partial/\partial\bar{z}$  to (1.8), we get:

$$\begin{aligned} - \int_{\gamma} f(w) \frac{d\bar{w}}{\bar{w} - \bar{z}} &= - \iint_U \frac{\partial f(w)}{\partial w} \frac{dw \wedge d\bar{w}}{\bar{w} - \bar{z}} + \frac{\partial}{\partial\bar{z}} \iint_U f(w) \frac{dw \wedge d\bar{w}}{w - z} \\ \Rightarrow \frac{\partial}{\partial\bar{z}} \iint_U f(w) \frac{dw \wedge d\bar{w}}{w - z} &= - \int_{\gamma} f(w) \frac{d\bar{w}}{\bar{w} - \bar{z}} + \iint_U \frac{\partial f(w)}{\partial w} \frac{dw \wedge d\bar{w}}{\bar{w} - \bar{z}} = 2\pi i f(z) \quad (\text{Corollary 1.1}) \end{aligned}$$

Therefore, we have

$$\boxed{g(z) = \frac{1}{2\pi i} \iint_U f(w) \frac{dw \wedge d\bar{w}}{w - z}} \implies \frac{\partial g(z)}{\partial\bar{z}} = f(z)$$

<sup>14</sup>The proof of this result is an application of Dominated Convergence Theorem [5, Theorem 2.27].

Observe that from (1.8) it follows that  $g \in C^1(U)$  since

$$g(z) = \frac{1}{2\pi i} \int_{\gamma} f(w) \log |w - z|^2 d\bar{w} - \frac{1}{2\pi i} \iint_U \frac{\partial f(w)}{\partial w} \log |w - z|^2 dw \wedge d\bar{w}$$

and the differential equation shows that  $\partial g / \partial \bar{z} \in C^\infty(U)$ . In particular,  $g \in C^\infty(U)$ , as desired.  $\square$

**Corollary 1.2.** *Let  $V$  be an open neighborhood of the closure of a disc  $\Delta \subset \bar{\Delta} \subset V \subset \mathbb{C}$ . For  $f \in C^\infty(V)$ , the function*

$$g(z) := \frac{1}{2\pi i} \iint_{\Delta} \frac{f(w)}{w - z} dw \wedge d\bar{w}$$

*satisfies  $\partial g(z) / \partial \bar{z} = f(z)$  for  $z \in \Delta$ .*

**Corollary 1.3.** *Let  $f \in C^\infty(V)$  on an open set  $V$  of  $\mathbb{C}$ . Then, locally<sup>15</sup> on this open set, there exists a complex-valued smooth function  $g$  such that  $\partial g / \partial \bar{z} = f$ .*

**Corollary 1.4.** *If  $f \in C^\infty(V)$ , for an open set  $V \subset \mathbb{C}$  containing a compact set  $K$ , then there exists an open set  $U$ , with  $K \subset U \subset V$ , and a  $g \in C^\infty(U)$ , such that  $\partial g / \partial \bar{z} = f$  in  $U$ .*

**Remark 1.20.** We can prove the above three corollaries directly: Huybrechts [10, Proposition 1.3.7] and Kaup and Kaup [11, Lemma 61.6] prove Corollary 1.2 using Lemma C.2, Proposition B.4 and Stokes theorem; Voisin [21, Theorem 1.28] proves Corollary 1.3 by assuming that  $f$  has a compact support since we want to prove a local statement and using Stokes theorem; and Taylor [19, Proposition 1.4.2] proves Corollary 1.4 by using Lemma C.2 and the generalized Cauchy integral formula. The proof discussed here is by Gunning and Rossi [9, Lemma I.D.2].

**Theorem 1.2.** *If  $U$  is any open subset of  $\mathbb{C}$  and  $f \in C^\infty(U)$ , then there exists  $g \in C^\infty(U)$  such that  $\partial g / \partial \bar{z} = f$ .*

*Proof.* From Lemma C.1 we know that there exists a sequence  $\{K_n\}$  of compact subsets of  $U$  such that

1.  $K_n \subset \text{int}(K_{n+1})$  for each  $n$ ;
2.  $\bigcup_{n \in \mathbb{N}} \text{int}(K_n) = U$ ; and
3. each bounded component of the complement of  $K_n$  meets the complement of  $U$ .

First we will prove by induction that there exists a sequence of complex-valued smooth functions  $\{g_n\}$  satisfying  $\partial g_n / \partial \bar{z} = f$  on an open neighborhood of  $K_n$ , such that

$$|g_n(z) - g_{n-1}(z)| < \frac{1}{2^{n-1}} \quad \text{for all } z \in K_{n-1} \text{ if } n > 1$$

For the base case we get  $g_1$  by Corollary 1.4. Next, as the induction hypothesis, assume that there exist complex-valued smooth functions  $\{g_1, \dots, g_m\}$  satisfying the desired conditions. We again apply Corollary 1.4 to get a function  $h$  which is smooth in an open neighborhood of  $K_{m+1}$  and satisfies  $\partial h / \partial \bar{z} = f$  on this neighborhood. Since  $K_m \subset \text{int}(K_{m+1})$ , on an open neighborhood of  $K_m$  we have

$$\frac{\partial(h - g_m)}{\partial \bar{z}} = 0$$

---

<sup>15</sup>Here “locally” means that for any point  $z \in V$  there is some open neighborhood  $U$  of  $z$  where  $\partial g / \partial \bar{z} = f$ .

So, by [Theorem B.2](#),  $h - g_m$  is holomorphic on this neighborhood of  $K_m$ . By Runge's theorem [[3](#), Theorem VIII.1.7], we can choose a rational function  $r$ , with poles in  $\mathbb{C} \setminus U$ , such that

$$|h(z) - g_m(z) - r(z)| < \frac{1}{2^m} \quad \text{for all } z \in K_m$$

If we set  $\boxed{g_{m+1} = h - r}$ , then  $\partial g_{m+1}/\partial \bar{z} = f$  on an open neighborhood on  $K_{m+1}$  and

$$|g_{m+1}(z) - g_m(z)| < \frac{1}{2^m} \quad \text{for all } z \in K_m$$

By induction, a sequence  $\{g_n\}$  with the required properties exists.

Next, we note that<sup>16</sup> the sequence  $\{g_n\}$  of complex-valued smooth functions converges uniformly on each compact set  $K_n$  to a function  $g$  defined on  $U$ . Moreover,  $g_n - g_m$  is holomorphic on an open neighborhood of  $K_m$  for each  $n > m$ . Thus for each fixed  $m$ ,  $\{g_n - g_m\}$  is a sequence of complex-valued holomorphic functions on an open neighborhood of  $K_m$  which is uniformly convergent on  $K_m$ . Therefore, by Morera's theorem [[3](#), Exercise IV.5.8], the limit function  $g - g_m$  is holomorphic on  $\text{int}(K_m)$ . Hence,  $g$  is smooth on  $\text{int}(K_m)$ . Since this is true for each  $m$  and  $\bigcup_m \text{int}(K_m) = U$ , we conclude that  $g$  is a complex-valued smooth function on the whole of  $U$ . Clearly,  $\partial g/\partial \bar{z} = f$  in  $U$ .  $\square$

**Remark 1.21.** In particular, if  $U$  is simply connected and  $f : U \rightarrow \mathbb{C}$  is holomorphic, then  $f$  has a primitive in  $U$  [[3](#), Corollary IV.6.16].

### 1.2.2 $\bar{\partial}$ -Poincaré lemma

**Lemma 1.2.** Let  $\bar{\Delta} \subset \mathbb{C}^n$  be a compact polydisc<sup>17</sup>, and  $\omega \in \Omega^{p,q}(V)$  for some open neighborhood  $V$  of  $\bar{\Delta}$ . If  $q > 0$  and  $\bar{\partial}\omega = 0$ , then there is  $\eta \in \Omega^{p,q-1}(\bar{\Delta})$  such that  $\omega = \bar{\partial}\eta$ .

*Proof.* Consider the following explicit representation of  $\omega \in \Omega^{p,q}(V)$

$$\omega = \sum_{|\alpha|=p, |\beta|=q} f_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$$

Let  $\ell$  be the least integer such that the expression for  $\omega$  involves no conjugate differential  $d\bar{z}_j$  with  $j > \ell$ ; i.e.  $\omega$  can be written in terms of the conjugate differentials  $d\bar{z}_1, \dots, d\bar{z}_\ell$  and the differentials  $dz_1, \dots, dz_n$ . We will proceed by induction on  $\ell$ .

For the base case there is nothing to prove since for  $\ell = 0$  we have  $\omega = 0$  because by hypothesis  $q > 0$ . Next, as the induction hypothesis, assume that for  $0 < \ell < k$ , every  $\bar{\partial}$ -closed  $(p, q)$ -form in an open neighborhood of  $\bar{\Delta}$  is  $\bar{\partial}$ -exact on  $\bar{\Delta}$ . In general, for the induction step, we write

$$\omega = d\bar{z}_k \wedge \theta + \xi$$

where  $\theta$  and  $\xi$  involve only the conjugate differentials  $d\bar{z}_1, \dots, d\bar{z}_{k-1}$ . Since  $\omega$  is  $\bar{\partial}$ -closed, we have

$$\begin{aligned} 0 &= \bar{\partial}\omega = \bar{\partial}(d\bar{z}_k \wedge \theta) + \bar{\partial}\xi \\ &= (\bar{\partial}(d\bar{z}_k) \wedge \theta + (-1)^{0+1} d\bar{z}_k \wedge \bar{\partial}\theta) + \bar{\partial}\xi \\ &= (-d\bar{z}_k \wedge \bar{\partial}\theta) + \bar{\partial}\xi \end{aligned}$$

<sup>16</sup>Recall the following three facts from real analysis: (1). If a sequence  $(x_n)_{n=1}^\infty$  in  $\mathbb{R}^n$  satisfies  $\sum_{n \geq 1} |x_{n+1} - x_n| < \infty$ , then it is Cauchy.; (2). A sequence  $\{f_n\}$  converges uniformly if and only if  $\{f_n\}$  is uniformly Cauchy; (3). A sequence of functions  $\{f_n\}$  from a set  $A$  to a metric space  $X$  is said to be uniformly Cauchy if for all  $\varepsilon > 0$ , there exists  $N > 0$  such that for all  $a \in A$  we have  $|f_n(a) - f_m(a)| < \varepsilon$  whenever  $m, n > N$ .

<sup>17</sup>For its definition see [Definition B.1](#).

It follows, by [Theorem B.2](#), that the coefficients of the forms  $\theta$  and  $\xi$  are holomorphic in  $z_{k+1}, \dots, z_n$  since the partial derivatives  $\partial/\partial\bar{z}_{k+1}, \dots, \partial/\partial\bar{z}_n$  for any such coefficient are all zero. Consider the following explicit representation of  $\theta$

$$\theta = \sum_{\substack{|\alpha|=p \\ \beta_j \in \{1, \dots, k-1\}}} g_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$$

Observe that any coefficient  $g_{\alpha\beta}$  of  $\theta$  is a complex-valued smooth function of the variable  $z_k$  in an open neighborhood of  $\bar{\Delta}_k$ , where the original polydisc has the product decomposition<sup>18</sup>

$$\Delta = \Delta_1 \times \dots \times \Delta_n$$

where  $\Delta_j$  is a disc in  $\mathbb{C}$ . The function  $g_{\alpha\beta}$  is also a complex-valued smooth function of  $z_1, \dots, z_{k-1}$  and a holomorphic function of  $z_{k+1}, \dots, z_n$  in the corresponding domains. By [Corollary 1.2](#) there exists a function  $h_{\alpha\beta}$  which is smooth in  $z_k \in \Delta_k$ :

$$h_{\alpha\beta}(z) = h_{\alpha\beta}(z_1, \dots, z_n) = \frac{1}{2\pi i} \iint_{\Delta_k} \frac{g_{\alpha\beta}(z_1, \dots, z_{k-1}, w, z_{k+1}, \dots, z_n)}{w - z_k} dw \wedge d\bar{w}$$

such that

$$\frac{\partial h_{\alpha\beta}}{\partial \bar{z}_k} = g_{\alpha\beta}$$

Note that  $h_{\alpha\beta}$  is also<sup>19</sup> smooth in  $z_1, \dots, z_{k-1}$  and holomorphic in  $z_{k+1}, \dots, z_n$  in the same regions as  $g_{\alpha\beta}$  is. Replacing each coefficient  $g_{\alpha\beta}$  in the differential form  $\theta$  by such a function  $h_{\alpha\beta}$  yields a new  $(p, q-1)$ -form

$$\sigma = \sum_{\substack{|\alpha|=p \\ \beta_j \in \{1, \dots, k-1\}}} h_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$$

which by this construction satisfies the equation

$$\bar{\partial}\sigma = d\bar{z}_k \wedge \theta + \rho$$

for some differential form  $\rho$  involving only the conjugate differentials  $d\bar{z}_1, \dots, d\bar{z}_{k-1}$ . Now consider the differential form

$$\nu = \omega - \bar{\partial}\sigma = \xi - \rho$$

Note that  $\nu$  is a  $\bar{\partial}$ -closed form since

$$\bar{\partial}\nu = \bar{\partial}\omega - \bar{\partial}^2\sigma = 0$$

and it involves only the conjugate differentials  $d\bar{z}_1, \dots, d\bar{z}_{k-1}$  since  $\xi$  and  $\rho$  do. The induction hypothesis implies that  $\nu$  is  $\bar{\partial}$ -exact on  $\Delta$ , i.e.  $\nu = \bar{\partial}\lambda$  for some  $\lambda \in \Omega^{p, q-1}(\Delta)$ . Hence, for  $\eta = \sigma + \lambda$  we have  $\omega = \bar{\partial}\eta$ , completing the proof.  $\square$

**Corollary 1.5.** *Let  $\omega$  be a  $(p, q)$ -form such that  $\bar{\partial}\omega = 0$  and  $q > 0$ , then it is locally<sup>20</sup> expressible as  $\bar{\partial}\eta$  for some  $(p, q-1)$ -form  $\eta$ .*

<sup>18</sup>In this argument it is important that  $\bar{\Delta}$  is a Cartesian product of some compact sets  $\bar{\Delta}_1, \dots, \bar{\Delta}_n$  in  $\mathbb{C}$ , since it enables us to apply [Corollary 1.4](#) in each variable separately, while treating the other variables as parameters [[19](#), p. 241].

<sup>19</sup>From the proof of [Proposition 1.5](#) it is clear that the function  $g$  constructed is holomorphic or smooth in any additional parameters in which  $f$  is holomorphic or smooth [[9](#), Lemma I.D.2].

<sup>20</sup>Suppose  $V$  is an open set in  $\mathbb{C}^n$  and  $\omega \in \Omega^{p, q}(U)$  such that  $q > 0$  and  $\bar{\partial}\omega = 0$ , and for any point  $z \in U$ , then in some open neighborhood  $U$  of  $z$  such that  $\omega = \bar{\partial}\eta$  for some  $\eta \in \Omega^{p, q-1}(U)$ .

*Proof.* The open polydiscs form a basis for the product topology on  $\mathbb{C}^n$ . Therefore, this result follows from the previous one.  $\square$

**Theorem 1.3** ( $\bar{\partial}$ -Poincaré lemma). *Let  $\Delta$  be an open polydisc in the space  $\mathbb{C}^n$ , not necessarily having a compact closure, and  $\omega \in \Omega^{p,q}(\Delta)$ . If  $q > 0$  and  $\bar{\partial}\omega = 0$ , then there is  $\eta \in \Omega^{p,q-1}(\Delta)$  such that  $\omega = \bar{\partial}\eta$ .*

*Proof.* Let  $\{\Delta_j\}$  be a sequence of open polydiscs in  $\mathbb{C}^n$  which have same center as  $\Delta$  and satisfy the following conditions:

1.  $\bar{\Delta}_j \subset \Delta_{j+1}$ ; and
2.  $\Delta = \bigcup_j \Delta_j$

We will divide the proof into two cases:

Case 1. If  $q > 1$ .

We will inductively construct a sequence of  $(p, q - 1)$ -forms  $\{\eta_j\}$  such that

- (a)  $\eta_j \in \Omega^{p,q-1}(V_j)$  for some open neighborhood  $V_j$  of  $\bar{\Delta}_j$ ;
- (b)  $\bar{\partial}\eta_j = \omega$  on  $\Delta_j$ ; and
- (c)  $\eta_j|_{\Delta_{j-1}} = \eta_{j-1}$  if  $j > 1$ .

For the base case we get  $\eta_1 \in \Omega^{p,q-1}(V_1)$  by [Lemma 1.2](#). Next, as the induction hypothesis, assume that there exist  $(p, q - 1)$ -forms  $\{\eta_1, \dots, \eta_k\}$  satisfying the desired conditions. We again apply [Lemma 1.2](#) to get a  $(p, q - 1)$ -form  $\theta$  on an open neighborhood  $V$  of  $\bar{\Delta}_{k+1}$  such that  $\bar{\partial}\theta = \omega$  on this neighborhood. Since  $\bar{\Delta}_k \subset \Delta_{k+1}$ , on an open neighborhood of  $\bar{\Delta}_k$  we have

$$\bar{\partial}(\theta - \eta_k) = 0$$

So,  $\theta - \eta_k$  is a  $\bar{\partial}$ -closed  $(p, q - 1)$ -form with  $q - 1 > 0$ . By yet another application of [Lemma 1.2](#) there exists a  $(p, q - 2)$ -form  $\xi$  on an open neighborhood  $U$  of  $\bar{\Delta}_k$  such that  $\bar{\partial}\xi = \theta - \eta_k$  on this neighborhood. From [Lemma C.2](#) we know that there exists a real-valued smooth function  $F$  in  $\mathbb{C}^n$  such that

- (a)  $0 \leq F(z) \leq 1$  for all  $z \in \mathbb{C}^n$ ;
- (b)  $F(z) = 1$  for  $z \in \bar{\Delta}_k$ ; and
- (c)  $F(z) = 0$  for  $z \in \mathbb{C}^n \setminus U$ .

Hence we have  $F\xi \in \Omega^{p,q-2}(\mathbb{C}^n)$ . Then we get the  $(p, q - 1)$ -form  $\eta_{k+1} = \theta - \bar{\partial}(F\xi)$  defined on the open neighborhood  $V$  of  $\bar{\Delta}_{k+1}$ , which satisfies the desired conditions:

$$\bar{\partial}\eta_{k+1} = \omega \text{ on } \Delta_{k+1} \quad \text{and} \quad \eta_{k+1}|_{\Delta_k} = \theta - \bar{\partial}\xi = \eta_k$$

As a result of the above construction there is  $\eta \in \Omega^{p,q-1}(\Delta)$  such that  $\eta|_{\Delta_j} = \eta_j$  and  $\bar{\partial}\eta = \omega$ , which concludes the proof of this case.

Case 2. If  $q = 1$ .

First we will inductively construct a sequence of  $(p, 0)$ -forms  $\{\eta_j\}$  such that

- (a)  $\eta_j \in \Omega^{p,q-1}(V_j)$  for some open neighborhood  $V_j$  of  $\bar{\Delta}_j$ ;
- (b)  $\bar{\partial}\eta_j = \omega$  on  $\Delta_j$ ; and

(c) If  $\eta_j = \sum_{\alpha} f_{\alpha}^{(j)} dz_{\alpha}$  for  $\alpha = (\alpha_1, \dots, \alpha_p)$  and  $dz_{\alpha} = dz_{\alpha_1} \wedge \dots \wedge dz_{\alpha_p}$ , then

$$\left| f_{\alpha}^{(j)}(z) - f_{\alpha}^{(j-1)}(z) \right| < \frac{1}{2^{j-1}} \quad \text{for all } \alpha \text{ and } z \in \bar{\Delta}_{j-1} \text{ if } j > 1$$

For the base case we get  $\eta_1 \in \Omega^{p,q-1}(V_1)$  by [Lemma 1.2](#). Next, as the induction hypothesis, assume that there exist  $(p, 0)$ -forms  $\{\eta_1, \dots, \eta_k\}$  satisfying the desired conditions. We again apply [Lemma 1.2](#) to get a  $(p, 0)$ -form  $\theta$  on an open neighborhood  $V$  of  $\bar{\Delta}_{k+1}$  such that  $\bar{\partial}\theta = \omega$  on this neighborhood. Let the following be the explicit representation of  $\theta$

$$\theta = \sum_{\alpha} g_{\alpha} dz_{\alpha}$$

Then on an open neighborhood of  $\bar{\Delta}_k$  all the coefficients of the form  $\theta - \eta_k$  are holomorphic by [Remark 1.16](#) since  $\bar{\partial}(\theta - \eta_k) = 0$ . Observe that each coefficient has a power series expansion centered at the common center of all the polydiscs and converging uniformly in  $\bar{\Delta}_k$ . Hence choosing suitable partial sums, we find polynomial terms  $r_{\alpha}(z)$  such that

$$\left| g_{\alpha}(z) - f_{\alpha}^{(k)}(z) - r_{\alpha}(z) \right| < \frac{1}{2^k} \quad \text{for all } \alpha \text{ and } z \in \bar{\Delta}_k$$

Let  $\xi$  be the  $(p, 0)$ -form with the polynomials  $r_{\alpha}$  as coefficients

$$\xi = \sum_{\alpha} r_{\alpha} dz_{\alpha}$$

Note that  $\bar{\partial}\xi = 0$  since the coefficients are holomorphic. Then we get the  $(p, 0)$ -form  $\eta_{k+1} = \theta - \xi$  defined on the open neighborhood  $V$  of  $\bar{\Delta}_{k+1}$ , which satisfies the desired conditions:

$$\bar{\partial}\eta_{k+1} = \omega \text{ on } \Delta_{k+1} \quad \text{and} \quad \left| f_{\alpha}^{(k+1)}(z) - f_{\alpha}^{(k)}(z) \right| < \frac{1}{2^k} \quad \text{for all } \alpha \text{ and } z \in \bar{\Delta}_k$$

Next, fix one  $\alpha$ . Then we note that<sup>21</sup> the sequence  $\{f_{\alpha}^{(j)}\}$  of smooth functions converges uniformly on each  $\Delta_j$  to a function  $f_{\alpha}$  defined on  $\Delta$ . Moreover,  $f_{\alpha}^{(j)} - f_{\alpha}^{(k)}$  is holomorphic on an open neighborhood of  $\bar{\Delta}_k$  for each  $j > k$  since  $\bar{\partial}(\eta_j - \eta_k) = 0$ . Thus for each fixed  $k$ ,  $\{f_{\alpha}^{(j)} - f_{\alpha}^{(k)}\}$  is a sequence of holomorphic functions on an open neighborhood of  $\bar{\Delta}_k$  which is uniformly convergent on  $\bar{\Delta}_k$ . Therefore, by Morera's theorem [[3](#), Exercise IV.5.8], the limit function  $f_{\alpha} - f_{\alpha}^{(k)}$  is holomorphic on  $\Delta_k$ . Hence,  $f_{\alpha}$  is smooth on  $\Delta_k$ . Since this is true for each  $k$  and  $\bigcup_k \Delta_k = \Delta$ , we conclude that  $f_{\alpha}$  is a complex-valued smooth function on the whole of  $\Delta$ .

Finally we define the  $(p, 0)$ -form

$$\eta = \sum_{\alpha} f_{\alpha} dz_{\alpha} = \lim_{j \rightarrow \infty} \eta_j$$

Note that for a fixed  $k$  we have

$$\eta - \eta_k = \lim_{j \rightarrow \infty} (\eta_j - \eta_k)$$

---

<sup>21</sup>Recall the following three facts from real analysis: (1). If a sequence  $(x_n)_{n=1}^{\infty}$  in  $\mathbb{R}^n$  satisfies  $\sum_{n \geq 1} |x_{n+1} - x_n| < \infty$ , then it is Cauchy.; (2). A sequence  $\{f_n\}$  converges uniformly if and only if  $\{f_n\}$  is uniformly Cauchy; (3). A sequence of functions  $\{f_n\}$  from a set  $A$  to a metric space  $X$  is said to be uniformly Cauchy if for all  $\varepsilon > 0$ , there exists  $N > 0$  such that for all  $a \in A$  we have  $|f_n(a) - f_m(a)| < \varepsilon$  whenever  $m, n > N$ .



Since  $\eta_j - \eta_k$  have coefficients holomorphic in  $\Delta_k$ , it follows that in  $\Delta_k$ ,  $\boxed{\eta = \eta_k + \sigma_k}$  for some holomorphic form  $\sigma_k$  given by

$$\sigma_k = \sum_{\alpha} (f_{\alpha} - f_{\alpha}^{(k)}) dz_{\alpha}$$

Hence  $\bar{\partial}\eta = \bar{\partial}\eta_k = \omega$  in each  $\Delta_k$ , which completes the proof. □

**Remark 1.22.** If we consider  $\omega = f d\bar{z} \in \Omega^{0,1}(U)$  for some open set  $U \subset \mathbb{C}$ , then [Theorem 1.2](#) gives us the “ $\bar{\partial}$ -Poincaré lemma in one variable.” However, due to the lack of purely topological or intrinsic analytical description of the domains in  $\mathbb{C}^n$  for  $n \geq 2$  on which approximation theorems (like Runge’s theorem) hold, we confine ourselves to the simple case of polydiscs [9, §I.F].

**Remark 1.23.** Unlike the Poincaré lemma we proved in the previous report [12, Theorem 1.2], we cannot give a simple topological condition on the domain which will ensure that the  $\bar{\partial}$ -closed forms are also  $\bar{\partial}$ -exact. This is because the failure of *Riemann mapping theorem* in  $\mathbb{C}^n$  for  $n \geq 2$  implies that there is no canonical topologically trivial domain in  $\mathbb{C}^n$  for  $n \geq 2$ , as there is in  $\mathbb{C}$  (namely, the disc) [13, §0.3.2].

### 1.3 Differential forms on complex manifolds

In this section some basic definitions and facts from [10, §2.1, 2.2 and 2.6], [24, §I.2, I.3], [21, §2.1, 2.2, 2.3] and [6, §IV.1] will be stated.

**Definition 1.12** (Complex manifold). A *complex manifold*  $M$  of dimension  $n$  is a second countable Hausdorff space together with a holomorphic structure on it. A *holomorphic structure*  $\mathcal{U}$  is the collection of charts  $\{(U_{\alpha}, \phi_{\alpha})\}_{\alpha \in A}$  where  $U_{\alpha}$  is an open set of  $M$  and  $\phi_{\alpha}$  is a homeomorphism of  $U_{\alpha}$  onto an open set of  $\mathbb{C}^n$  such that

1. the open sets  $\{U_{\alpha}\}_{\alpha \in A}$  cover  $M$ .
2. for every pair of indices  $\alpha, \beta \in A$  with  $U_{\alpha} \cap U_{\beta} \neq \emptyset$  the homeomorphisms

$$\begin{aligned} \phi_{\alpha} \circ \phi_{\beta}^{-1} &: \phi_{\beta}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\alpha}(U_{\alpha} \cap U_{\beta}), \\ \phi_{\beta} \circ \phi_{\alpha}^{-1} &: \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\beta}(U_{\alpha} \cap U_{\beta}) \end{aligned}$$

are holomorphic maps<sup>22</sup> between open subsets of  $\mathbb{C}^n$ .

3. the family  $\mathcal{U}$  is maximal in the sense that it contains all possible pairs  $(U_{\alpha}, \phi_{\alpha})$  satisfying the properties 1. and 2.

**Example 1.1.** Following two complex manifolds will be used throughout this report:

1. The complex space  $\mathbb{C}^n$  is a complex manifold with single chart  $(\mathbb{C}^n, \mathbb{1}_{\mathbb{C}^n})$ , where  $\mathbb{1}_{\mathbb{C}^n}$  is the identity map. In other words,  $(\mathbb{C}^n, \mathbb{1}_{\mathbb{C}^n}) = (\mathbb{C}^n, z_1, \dots, z_n)$  where  $z_1, \dots, z_n$  are the standard coordinates on  $\mathbb{C}^n$ .
2. Any open subset  $V$  of a complex manifold  $M$  is also a smooth manifold. If  $\{(U_{\alpha}, \phi_{\alpha})\}$  is an atlas for  $M$ , then  $\{(U_{\alpha} \cap V, \phi_{\alpha}|_{U_{\alpha} \cap V})\}$  is an atlas for  $V$ , where  $\phi_{\alpha}|_{U_{\alpha} \cap V} : U_{\alpha} \cap V \rightarrow \mathbb{C}^n$  denotes the restriction of  $\phi_{\alpha}$  to the subset  $U_{\alpha} \cap V$ .

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<sup>22</sup>For the definition of several complex variables holomorphic mapping, see [Definition B.8](#).

**Remark 1.24.** Every complex manifold  $M$  is paracompact [6, §IV.1].

**Definition 1.13** (Holomorphic function on a manifold). Let  $M$  be a complex manifold of dimension  $n$ . A function  $f : M \rightarrow \mathbb{C}$  is said to be a *holomorphic function at a point  $w$*  in  $M$  if there is a chart  $(U, \phi)$  about  $w$  in  $M$  such that  $f \circ \phi^{-1}$ , a function defined on the open subset  $\phi(U)$  of  $\mathbb{C}^n$ , is holomorphic<sup>23</sup> at  $\phi(w)$ . The function  $f$  is said to be holomorphic in  $M$  if it is holomorphic at every point of  $M$ .

$$\begin{array}{ccc} (U, w) & \xrightarrow{\phi} & (\mathbb{C}^n, \phi(w)) \\ & \searrow f & \downarrow f \circ \phi^{-1} \\ & & (\mathbb{C}, f(w)) \end{array}$$

**Definition 1.14** (Holomorphic map between complex manifolds). Let  $M$  and  $N$  be complex manifolds of dimension  $m$  and  $n$ , respectively. A continuous map  $F : M \rightarrow N$  is said to be *holomorphic at a point  $w$*  of  $M$  if there are charts  $(V, \psi)$  about  $F(w)$  in  $N$  and  $(U, \phi)$  about  $w$  in  $M$  such that the composition  $\psi \circ F \circ \phi^{-1}$ , a map from the open subset  $\phi(F^{-1}(V) \cap U)$  of  $\mathbb{C}^m$  to  $\mathbb{C}^n$ , is holomorphic at  $\phi(w)$ .

$$\begin{array}{ccc} (U, w) & \xrightarrow{F} & (V, F(w)) \\ \downarrow \phi & & \downarrow \psi \\ (\mathbb{C}^m, \phi(w)) & \xrightarrow{\psi \circ F \circ \phi^{-1}} & (\mathbb{C}^n, \psi(F(w))) \end{array}$$

The continuous map  $F : M \rightarrow N$  is said to be *holomorphic* if it is holomorphic at every point in  $M$ .

**Definition 1.15** (Biholomorphic manifolds). Two complex manifolds  $M$  and  $N$  are called *biholomorphic* if there exists a holomorphic homeomorphism<sup>24</sup>  $f : X \rightarrow Y$ .

**Theorem 1.4.** If  $(U, \phi)$  is a chart on a complex manifold  $M$  of dimension  $n$ , then  $U$  is biholomorphic to  $\phi(U) \subset \mathbb{C}^n$ .

**Remark 1.25.** If  $(U, \phi)$  is a chart of a manifold, i.e.  $\phi : U \rightarrow \mathbb{C}^n$ , then let  $r_j = z_j \circ \phi$  be the  $j^{\text{th}}$  component of  $\phi$  and write  $\phi = (r_1, \dots, r_n)$  and  $(U, \phi) = (U, r_1, \dots, r_n)$ . Thus, for  $w \in U$ ,  $(r_1(w), \dots, r_n(w))$  is a point in  $\mathbb{C}^n$ . The functions  $r_1, \dots, r_n$  are called *coordinates* or *local coordinates* on  $U$ .

### 1.3.1 Complex differential forms

**Definition 1.16** (Complex vector bundle). A *complex vector bundle* of rank  $k$  over a smooth manifold  $M$  is a smooth manifold  $E$  equipped with a smooth surjective map  $\pi : E \rightarrow M$  such that for an open cover  $\{U_\alpha\}$  of  $M$ , there is a *local trivialization* diffeomorphism

$$\tau_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^k$$

satisfying the following conditions:

<sup>23</sup>For the definition of complex-valued holomorphic function, see [Definition B.3](#).

<sup>24</sup>Note that the inverse of a holomorphic homeomorphism is holomorphic by [Proposition B.5](#).

1. the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\tau_\alpha} & U_\alpha \times \mathbb{C}^k \\ & \searrow \pi & \swarrow p_1 \\ & U_\alpha & \end{array}$$

where  $p_1$  is the projection onto the first factor,

2. the composite maps

$$\tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta(\pi^{-1}(U_\alpha \cap U_\beta)) \rightarrow \tau_\alpha(\pi^{-1}(U_\alpha \cap U_\beta))$$

are  $\mathbb{C}$ -linear for each  $w \in U_\alpha \cap U_\beta$ .

**Remark 1.26.** For a fixed  $w \in U_\alpha \cap U_\beta$ , the linear transformation

$$\left(\tau_\alpha \circ \tau_\beta^{-1}\right)_w : \{w\} \times \mathbb{C}^k \rightarrow \{w\} \times \mathbb{C}^k$$

must respect the projection onto the first factor, by the first condition above, and is thus described by a complex  $k \times k$ -matrix, whose coefficients are smooth functions of  $w$ . These matrices are called *transition matrices*. In particular, the map  $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$  is given by

$$\tau_{\alpha\beta}(w, v) = (w, \sigma_{\alpha\beta}(w)v) \quad \forall w \in U_\alpha \cap U_\beta, v \in \mathbb{C}^k$$

and is completely determined by the map  $\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{C})$ , called the *transition map*. Since  $\tau_{\alpha\beta}$  is smooth, so is  $\sigma_{\alpha\beta}$ . From now on, we will assume that the transition maps can in fact be used to define a vector bundle. For proof, see [25, §9] and [24, §I.2].

**Definition 1.17** (Fiber of a complex vector bundle). If  $\pi : E \rightarrow M$  is a complex vector bundle and  $w \in M$ , then  $E_w = \pi^{-1}(w)$  is called the *fiber* of  $E$  at the point  $w$ . It is canonically a vector space, with structure given by any of the trivializations of  $E$  in the neighborhood of  $w$ .

**Remark 1.27.** A complex vector bundle is a smooth vector bundle whose fibers are complex vector spaces and the transition maps are complex linear [12, Definition 1.36].

**Definition 1.18** (Almost complex structure). An *almost complex structure* on a smooth manifold  $M$  is a vector bundle endomorphism  $J$  of (real) tangent bundle  $T_{\mathbb{R}}M$ , such that  $J^2 = -\mathbf{1}_{T_{\mathbb{R}}M}$ , i.e. for all  $w \in M$ , the linear map  $J_w : T_{w, \mathbb{R}}M \rightarrow T_{w, \mathbb{R}}M$  is a linear complex structure for  $T_{w, \mathbb{R}}M$ .

**Remark 1.28.** Equivalently, the almost complex structure is the structure of a complex vector bundle on  $T_{\mathbb{R}}M$  [21, Definition 2.11]. Also, if an almost complex structure exists, then the real dimension of  $M$  is even [10, Definition 2.6.1]. However, not every smooth manifold of even dimension admits an almost complex structure [10, Remark 2.6.3].

**Definition 1.19** (Almost complex manifold). An *almost complex manifold* is a smooth manifold together with an almost complex structure.

**Proposition 1.6.** *Let  $M$  be an almost complex manifold. Then there exists a direct sum decomposition of the complexified tangent bundle  $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$  into complex vector bundles*

$$T_{\mathbb{C}}M = (T_{\mathbb{R}}M)^{1,0} \oplus (T_{\mathbb{R}}M)^{0,1}$$

such that the  $\mathbb{C}$ -linear extension  $\tilde{J} = J \otimes \mathbf{1}_{\mathbb{C}}$  satisfies

$$\tilde{J}|_{(T_{\mathbb{R}}M)^{1,0}} = i \cdot \mathbf{1}_{T_{\mathbb{C}}M} \quad \text{and} \quad \tilde{J}|_{(T_{\mathbb{R}}M)^{0,1}} = -i \cdot \mathbf{1}_{T_{\mathbb{C}}M}$$

**Proposition 1.7.** *Let  $M$  be an almost complex manifold. Then the dual of complexified tangent bundle  $T_{\mathbb{C}}^*M = T_{\mathbb{R}}^*M \otimes_{\mathbb{R}} \mathbb{C}$  decomposes as a direct sum of complex vector bundles*

$$T_{\mathbb{C}}^*M = (T_{\mathbb{R}}^*M)^{1,0} \oplus (T_{\mathbb{R}}^*M)^{0,1}$$

such that the  $\mathbb{C}$ -linear extension  $\tilde{\mathcal{J}} = \mathcal{J} \otimes \mathbb{1}_{\mathbb{C}}$  satisfies

$$\tilde{\mathcal{J}}|_{(T_{\mathbb{R}}^*M)^{1,0}} = i \cdot \mathbb{1}_{T_{\mathbb{C}}^*M} \quad \text{and} \quad \tilde{\mathcal{J}}|_{(T_{\mathbb{R}}^*M)^{0,1}} = -i \cdot \mathbb{1}_{T_{\mathbb{C}}^*M}$$

**Remark 1.29.** As in [Proposition A.8](#), we get the almost complex structure  $\mathcal{J}$  on  $T_{w,\mathbb{R}}^*M$  from the almost complex structure  $J$  on  $T_{w,\mathbb{R}}M$ . We will regard this  $\mathcal{J}$  as a vector bundle endomorphism of the smooth vector bundle  $T_{\mathbb{R}}^*M$  over  $M$ .

**Definition 1.20** (Differential  $(p, q)$ -form). Let  $M$  be an almost complex manifold. Over  $M$  we define the complex vector bundle of rank  $\binom{n}{p} \binom{n}{q}$

$$\bigwedge^{p,q} T_{\mathbb{R}}^*M := \bigwedge^p ((T_{\mathbb{R}}^*M)^{1,0}) \otimes_{\mathbb{C}} \bigwedge^q ((T_{\mathbb{R}}^*M)^{0,1})$$

whose fiber is  $\bigwedge^{p,q} T_{w,\mathbb{R}}^*M$ . The smooth sections of this vector bundle are called the *differential forms of type  $(p, q)$*  on  $M$ . The space of all smooth differential forms of type  $(p, q)$  on  $M$  is denoted by  $\Omega^{p,q}(M)$ .

**Remark 1.30.** Let  $\Omega_{\mathbb{C}}^k(M)$  be the space of sections of vector bundle  $\bigwedge^k T_{\mathbb{C}}^*M$ . By [Remark A.12](#) we have

$$\bigwedge^k T_{\mathbb{C}}^*M \cong \bigoplus_{p+q=k} \bigwedge^{p,q} T_{\mathbb{R}}^*M \implies \Omega_{\mathbb{C}}^k(M) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

Thus we have natural projection operators  $\bigwedge^k T_{\mathbb{C}}^*M \rightarrow \bigwedge^{p,q} T_{\mathbb{R}}^*M$  and  $\Omega_{\mathbb{C}}^k(M) \rightarrow \Omega^{p,q}(M)$ , denoted by  $\Pi^{p,q}$  for  $p+q=k$ .

**Definition 1.21** (Differential of a  $(p, q)$ -form). Let  $M$  be an almost complex manifold, and  $d : \Omega_{\mathbb{C}}^k(M) \rightarrow \Omega_{\mathbb{C}}^{k+1}(M)$  be the complex linear extension of the usual exterior differential<sup>25</sup>. Then

$$\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M) \quad \text{and} \quad \bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$$

are defined as  $\partial := \Pi^{p+1,q} \circ d$  and  $\bar{\partial} := \Pi^{p,q+1} \circ d$ .

**Lemma 1.3.** *For an almost complex manifold  $M$ , the differential operators  $\partial$  and  $\bar{\partial}$  satisfy the Leibniz's rule, i.e.*

$$\begin{aligned} \partial(\omega \wedge \eta) &= \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta \\ \bar{\partial}(\omega \wedge \eta) &= \bar{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \bar{\partial}\eta \end{aligned}$$

for  $\omega \in \Omega^{p,q}(M)$  and  $\eta \in \Omega^{r,s}(M)$ .

*Proof.* We will use the properties of  $d$  studied earlier<sup>26</sup>. Recall that for  $\omega \in \Omega_{\mathbb{C}}^{p+q}(M)$  and  $\eta \in \Omega_{\mathbb{C}}^{r+s}(M)$  we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{p+q}\omega \wedge d\eta \in \Omega_{\mathbb{C}}^{p+q+r+s+1}(U)$$

Since  $\partial := \Pi^{p+r+1,q+s} \circ d$ , taking the  $(p+r+1, q+s)$ -parts on both sides one obtains

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$

Similarly, taking the  $(p+r, q+s+1)$ -parts one obtains

$$\bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \bar{\partial}\eta$$

Hence completing the proof. □

<sup>25</sup>For definition, see [12, Definition 1.44].

<sup>26</sup>In the previous report, see [12, Theorem IX, Theorem XXVI].

**Definition 1.22** (Integrable almost complex structure). An almost complex structure  $J$  on  $M$  is called *integrable* if  $d\omega = \partial\omega + \bar{\partial}\omega$  for all  $\omega \in \Omega_{\mathbb{C}}^k(M)$ .

**Remark 1.31.** By [Lemma 1.1](#) we know that the almost complex structures on the open sets in  $\mathbb{C}^n$  are integrable. For more details about this definition, see [\[10, Proposition 2.6.15\]](#), [\[24, p. 34\]](#) and [\[21, Theorem 2.24\]](#).

**Definition 1.23** (Complex manifold). A *complex manifold*  $M$  of dimension  $n$  is a smooth manifold of dimension  $2n$  equipped with a holomorphic structure, i.e. if  $M$  is covered by open sets  $U_{\alpha}$  which are diffeomorphic via maps called  $\phi_{\alpha}$  to open sets in  $\mathbb{C}^n$ , in such a way that the transition diffeomorphisms

$$\phi_{\alpha} \circ \phi_{\beta}^{-1} : \phi_{\beta}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\alpha}(U_{\alpha} \cap U_{\beta})$$

are holomorphic.

**Proposition 1.8.** *A complex manifold  $M$  induces an almost complex structure on its underlying smooth manifold.*

*Proof.* This follows from [Definition A.4](#) and [Remark A.5](#). For details, see [\[24, Proposition I.3.4\]](#).  $\square$

**Theorem 1.5.** *The induced almost complex structure on a complex manifold is integrable.*

*Proof.* This follows by looking at the local coordinates as in [Lemma 1.1](#). For details, see [\[24, Theorem I.3.7\]](#).  $\square$

**Corollary 1.6.** *If  $M$  is a complex manifold, then  $\bar{\partial}^2 = 0$ .*

**Definition 1.24** (Pullback of a  $k$ -form). Let  $F : M \rightarrow N$  be a holomorphic map between complex manifolds. Then the  $\mathbb{C}$ -linear extension of the pullback map defined on the underlying smooth manifolds [\[12, Definition 1.45\]](#)

$$F^* : \Omega_{\mathbb{C}}^k(N) \rightarrow \Omega_{\mathbb{C}}^k(M)$$

is called the pullback of a complex  $k$ -form

**Remark 1.32.** Pullback of the identity map is an identity map, i.e.  $(\mathbb{1}_M)^* = \mathbb{1}_{\Omega_{\mathbb{C}}^k(M)}$ .

**Proposition 1.9.** *If  $F : M \rightarrow N$  and  $G : N \rightarrow N'$  are holomorphic maps between complex manifolds, then  $(G \circ F)^* = F^* \circ G^*$ .*

$$\begin{array}{ccc} \Omega_{\mathbb{C}}^k(N') & \xrightarrow{G^*} & \Omega_{\mathbb{C}}^k(N) \\ & \searrow & \downarrow F^* \\ & & \Omega_{\mathbb{C}}^k(M) \end{array}$$

**Proposition 1.10.** *Let  $F : M \rightarrow N$  be a holomorphic map between complex manifolds. If  $\omega$  is a differential form on  $N$ , then  $F^*(d\omega) = d(F^*\omega)$ , i.e. the following diagram commutes*

$$\begin{array}{ccc} \Omega_{\mathbb{C}}^k(N) & \xrightarrow{d} & \Omega_{\mathbb{C}}^{k+1}(N) \\ \downarrow F^* & & \downarrow F^* \\ \Omega_{\mathbb{C}}^k(M) & \xrightarrow{d} & \Omega_{\mathbb{C}}^{k+1}(M) \end{array}$$

**Theorem 1.6.** Let  $F : M \rightarrow N$  be a holomorphic map between complex manifolds. Then the pullback of differential forms  $F^* : \Omega_{\mathbb{C}}^k(N) \rightarrow \Omega_{\mathbb{C}}^k(M)$  induces natural  $\mathbb{C}$ -linear maps  $F^* : \Omega^{p,q}(N) \rightarrow \Omega^{p,q}(M)$ . These maps are compatible with  $\partial$  and  $\bar{\partial}$ .

*Proof.* If  $F$  is holomorphic then  $F^*$  is compatible with the decomposition [10, Proposition 2.6.10]

$$\Omega_{\mathbb{C}}^k(M) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

In particular,  $F^*(\Omega^{p,q}(N)) \subset \Omega^{p,q}(M)$  and  $\Pi^{p+1,q} \circ F^* = F^* \circ \Pi^{p+1,q}$ . Thus, for  $\omega \in \Omega^{p,q}(M)$  we have

$$\bar{\partial}(F^*(\omega)) = \Pi^{p+1,q}(d(F^*(\omega))) = \Pi^{p+1,q}(F^*(d(\omega))) = F^*(\Pi^{p+1,q}(d(\omega))) = F^*(\bar{\partial}(\omega))$$

where, as usual, we are abusing the notations  $\bar{\partial}$  and  $d$ . Analogously, we can show that  $\partial \circ F^* = F^* \circ \partial$ .  $\square$

### 1.3.2 Holomorphic differential forms

**Definition 1.25** (Holomorphic vector bundle). A *holomorphic vector bundle* of rank  $k$  is a triple  $(E, M, \pi)$  consisting of a pair of complex manifolds  $E$  and  $M$ , and a holomorphic surjective map  $\pi : E \rightarrow M$  satisfying the following conditions

1. for each  $w \in M$ , the inverse image  $E_w = \pi^{-1}(w)$  is an  $k$ -dimensional vector space over  $\mathbb{C}$ ,
2. for each  $w \in M$ , there is an open neighborhood  $U$  of  $w$  and a biholomorphic map  $\tau : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^k$  such that

- (a) the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tau} & U \times \mathbb{C}^k \\ & \searrow \pi & \swarrow p_1 \\ & U & \end{array}$$

where  $p_1$  is the projection onto the first factor,

- (b) for each  $v \in U$ , the induced map  $\tau_v : \pi^{-1}(v) \rightarrow \mathbb{C}^k$ , defined by  $\tau_v(z) = (v, \tau(z))$ , is a  $\mathbb{C}$ -linear isomorphism.

**Remark 1.33.** We can also define it the way we defined the complex vector bundle in [Remark 1.26](#). That is, we have biholomorphic local trivializations

$$\tau_\alpha : \pi_\alpha^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^k$$

such that the transition maps  $\sigma_{\alpha\beta} = U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{C})$  are holomorphic.

**Definition 1.26** (Pullback of holomorphic vector bundle). Let  $f : M \rightarrow N$  be a holomorphic map between complex manifolds and let  $E$  be a holomorphic vector bundle on  $N$  given by transition maps  $\sigma_{\alpha\beta}$  corresponding to an open cover  $\{U_\alpha\}$ . Then the *pullback*  $f^*E$  of  $E$  is the holomorphic vector bundle over  $M$  that is given by the transition maps  $\sigma_{\alpha\beta} \circ f$  corresponding to an open cover  $\{f^{-1}(U_\alpha)\}$ .

**Definition 1.27** (Holomorphic tangent bundle). Let  $M$  be a complex manifold of dimension  $n$  which is covered by open sets  $U_\alpha$  biholomorphic, via maps called  $\phi_\alpha$ , to open sets  $V_\alpha$  of  $\mathbb{C}^n$ .

Then the *holomorphic tangent bundle*  $\mathcal{T}M$  of  $M$  is a holomorphic vector bundle of rank  $n$  with the transition maps  $\sigma_{\alpha\beta}$  given by

$$\sigma_{\alpha\beta}(w) := \text{Jac}(\phi_{\alpha\beta})(w) = \left[ \frac{\partial \phi_{\alpha\beta}^\ell}{\partial z_j} \Big|_w \right]_{\substack{1 \leq \ell \leq n \\ 1 \leq j \leq n}}$$

is the Jacobian matrix at the point  $w$  (see [Definition B.10](#)).

**Remark 1.34.** In this definition, if we replace the complex manifold with the smooth manifold, and the holomorphic Jacobian matrix with the real Jacobian matrix, we will get the definition of smooth tangent bundle [21, §2.1.2]. This definition is equivalent to the one used in the previous report, where it was defined using *derivations* [12, Definition 1.31].

**Theorem 1.7.** *If  $M$  is a complex manifold, then  $(T_{\mathbb{R}}M)^{1,0}$  is naturally isomorphic (as a complex vector bundle) to the holomorphic tangent bundle  $\mathcal{T}M$ .*

*Proof.* Let  $U, V \subset \mathbb{C}^n$  be open subsets and  $f : U \rightarrow V$  be a biholomorphic map. Then by [Proposition 1.2](#) we get the linear isomorphism

$$\tilde{f}_* : (T_{w, \mathbb{R}}U)^{1,0} \oplus (T_{w, \mathbb{R}}U)^{0,1} \rightarrow (T_{f(w), \mathbb{R}}V)^{1,0} \oplus (T_{f(w), \mathbb{R}}V)^{0,1}$$

Also, from [Remark B.10](#) we know that

$$\tilde{f}_*(w) = \begin{bmatrix} \text{Jac}(f)(w) & 0 \\ 0 & \overline{\text{Jac}(f)(w)} \end{bmatrix}$$

Let  $\{(U_\alpha, \phi_\alpha)\}$  be a holomorphic atlas of  $M$ , i.e.  $U_\alpha$  is biholomorphic to  $\phi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}^n$ . Then  $(\phi_\alpha^{-1})^* \left( (T_{\mathbb{R}}U_\alpha)^{1,0} \right) \cong (T_{\mathbb{R}}V_\alpha)^{1,0}$ . With respect to the canonical trivialization the induced isomorphisms  $\left( T_{\phi_\beta(w), \mathbb{R}}V_\beta \right)^{1,0} \cong \left( T_{\phi_\alpha(w), \mathbb{R}}V_\alpha \right)^{1,0}$  are given by the transition maps of  $\mathcal{T}M$  [10, Definition 2.2.14, Proposition 2.6.4(ii)]. Therefore, both  $(T_{\mathbb{R}}M)^{1,0}$  and  $\mathcal{T}M$  are naturally isomorphic.  $\square$

**Remark 1.35.** We call the bundles  $(T_{\mathbb{R}}M)^{1,0}$  and  $(T_{\mathbb{R}}M)^{0,1}$  the *holomorphic* and *antiholomorphic* tangent bundle of the complex manifold  $M$ .

**Definition 1.28** (Holomorphic cotangent bundle). The *holomorphic cotangent bundle*  $\mathcal{T}^*M$  is the dual of  $\mathcal{T}M$ . That is, for all  $w \in M$  we have  $\mathcal{T}_w^*M = \text{Hom}_{\mathbb{C}}(\mathcal{T}_wM, \mathbb{C})$ .

**Definition 1.29** (Holomorphic  $p$ -forms). Over  $M$  we consider the holomorphic vector bundle  $\bigwedge^p \mathcal{T}^*M$  whose fiber is  $\bigwedge^p \mathcal{T}_w^*M$ . The holomorphic sections<sup>27</sup> of this vector bundle are called the *holomorphic  $p$ -forms* on  $M$ . The space of all holomorphic  $p$ -forms on  $M$  is denoted by  $\mathcal{O}^p(M)$ .

**Remark 1.36.** We note that holomorphic 0-forms on  $M$  are the holomorphic complex-valued functions on  $M$ , i.e.  $\mathcal{O}^0(M) = \mathcal{O}(M)$ . As in [Remark 1.25](#), let  $(U, r_1, \dots, r_n)$  be a coordinate chart of  $M$ . Then the differentials  $\{dr_1, \dots, dr_n\}$  are 1-forms on  $U$ . At each point  $w \in U$ , the 1-forms  $\{dr_1|_w, \dots, dr_n|_w\}$  form a basis of  $\bigwedge^1(\mathcal{T}_w^*M) = \mathcal{T}_w^*M$ , dual to the basis  $\{\partial/\partial r_1|_w, \dots, \partial/\partial r_n|_w\}$  for the tangent space  $\mathcal{T}_wM$ . Hence, a 1-form on  $U$  is a linear combination  $\omega = \sum_{\alpha=1}^n f_\alpha dr_\alpha$  where  $f_\alpha$  are complex-valued holomorphic functions on  $U$ . In general<sup>28</sup>, any holomorphic  $p$ -form  $\omega \in \mathcal{O}^p(M)$  can be written uniquely as

$$\omega = \sum_{|\alpha|=p} f_\alpha dr_\alpha$$

where  $\alpha = (\alpha_1, \dots, \alpha_p)$  are multi-indices with  $1 \leq \alpha_j \leq n$ ,  $dr_\alpha = dr_{\alpha_1} \wedge \dots \wedge dr_{\alpha_p}$  and  $f_\alpha$  is a complex-valued holomorphic function on  $U$ , i.e.  $f_\alpha \in \mathcal{O}(U)$ .

<sup>27</sup>Replace “smooth” by “holomorphic” in [12, Definition 1.38].

<sup>28</sup>In the case of  $M = \mathbb{C}^n$  the expression was much more straightforward because  $\mathcal{T}_wM \cong \mathbb{C}^n$  (vector space isomorphism) and we could replace  $r_j$  by  $z_j$ .

## 1.4 $\bar{\partial}$ -closed and exact forms on complex manifolds

In this section some basic definitions and facts from [8, p. 25], [21, §2.3.3], [10, §2.6] and [13, §6.3] will be stated.

**Definition 1.30** ( $\bar{\partial}$ -closed forms). Let  $M$  be a complex manifold. Then a differential form  $\omega \in \Omega^{p,q}(M)$  is called  $\bar{\partial}$ -closed if  $\bar{\partial}\omega = 0$ .

**Remark 1.37.** Given a complex manifold  $M$ , denote the set of all  $\bar{\partial}$ -closed  $(p, q)$ -forms on  $M$  by  $\mathcal{Z}^{p,q}(M)$ . The sum of two such  $(p, q)$ -forms is another  $\bar{\partial}$ -closed  $(p, q)$ -form, and so is the product of a  $\bar{\partial}$ -closed  $(p, q)$ -form by a scalar. Hence  $\mathcal{Z}^{p,q}(M)$  is the vector sub-space of  $\Omega^{p,q}(M)$ . Also, if we write the elements of  $\mathcal{Z}^{p,0}(M)$  in terms of local coordinates, then from [Theorem B.2](#) it follows that it is the space of  $(p, 0)$ -forms whose coefficients are complex-valued holomorphic functions in  $M$ , i.e.  $\mathcal{O}^p(M) = \mathcal{Z}^{p,0}(M)$  by [Remark 1.36](#). In particular, note that  $\mathcal{Z}^{0,0}(M) = \mathcal{O}(M)$ , the space of complex-valued functions holomorphic in  $M$ .

**Definition 1.31** ( $\bar{\partial}$ -exact forms). Let  $M$  be a complex manifold. Then a differential form  $\omega \in \Omega^{p,q}(M)$ , for  $q > 0$ , is called  $\bar{\partial}$ -exact if  $\omega = \bar{\partial}\eta$  for some differential form  $\eta \in \Omega^{p,q-1}(M)$ .

**Remark 1.38.** Given a complex manifold  $M$ , denote the set of all  $\bar{\partial}$ -exact  $(p, q)$ -forms on  $M$  by  $\mathcal{B}^{p,q}(M)$ . The sum of two such  $(p, q)$ -forms is another  $\bar{\partial}$ -exact  $(p, q)$ -form, and so is the product of a  $\bar{\partial}$ -exact  $(p, q)$ -form by a scalar. Hence  $\mathcal{B}^{p,q}(M)$  is the vector sub-space of  $\Omega^{p,q}(M)$ . Moreover, the trivial form  $\omega \equiv 0$  is the only  $(p, 0)$ -form which is  $\bar{\partial}$ -exact for any value of  $p = 0, 1, \dots, n$ . That is,  $\mathcal{B}^{p,0}(M)$  consists only of zero.

**Theorem 1.8.** *On a complex manifold  $M$ , every  $\bar{\partial}$ -exact form is  $\bar{\partial}$ -closed.*

*Proof.* Let  $M$  be a complex manifold and  $\omega \in \mathcal{B}^{p,q}(M)$  such that  $\omega = \bar{\partial}\eta$  for some  $\eta \in \Omega^{p,q-1}(M)$ . From [Corollary 1.6](#) we know that  $\bar{\partial}\omega = \bar{\partial}(\bar{\partial}\eta) = 0$  hence  $\omega \in \mathcal{Z}^{p,q}(M)$  for all  $q \geq 1$ . For  $q = 0$ , the statement is trivially true.  $\square$

**Lemma 1.4.** *Let  $F : M \rightarrow N$  be a holomorphic map of complex manifolds, then the pullback map  $F^*$  sends  $\bar{\partial}$ -closed forms to  $\bar{\partial}$ -closed forms, and  $\bar{\partial}$ -exact forms to  $\bar{\partial}$ -exact forms.*

*Proof.* Suppose  $\omega$  is  $\bar{\partial}$ -closed. From [Theorem 1.6](#) we know that  $F^*$  commutes with  $\bar{\partial}$

$$\bar{\partial}F^*\omega = F^*\bar{\partial}\omega = 0$$

Hence,  $F^*\omega$  is also  $\bar{\partial}$ -closed. Next suppose  $\theta = \bar{\partial}\eta$  is  $\bar{\partial}$ -exact. Then

$$F^*\theta = F^*\bar{\partial}\eta = \bar{\partial}F^*\eta$$

Hence,  $F^*\theta$  is  $\bar{\partial}$ -exact.  $\square$

### 1.4.1 Dolbeault cohomology

**Definition 1.32** (Dolbeault cohomology of a complex manifold). The  $(p, q)^{th}$  Dolbeault cohomology group<sup>29</sup> of a complex manifold  $M$  is the quotient group

$$H_{\bar{\partial}}^{p,q}(M) := \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$

**Remark 1.39.** Hence, the Dolbeault cohomology of a complex manifold measures the extent to which  $\bar{\partial}$ -closed forms are not  $\bar{\partial}$ -exact on that manifold.

<sup>29</sup>It is also a vector space over  $\mathbb{C}$ .



**Proposition 1.11.** *If  $M$  is a complex manifold then its Dolbeault cohomology group in degree  $(p, 0)$  is the group of holomorphic  $p$ -forms on  $M$ .*

*Proof.* Since there are no non-zero  $\bar{\partial}$ -exact  $(0, p)$ -forms

$$H_{\bar{\partial}}^{p,0}(M) = \mathcal{Z}^{p,0}(M) = \mathcal{O}^p(M)$$

□

**Remark 1.40.** Though the definitions of de Rham and Dolbeault cohomology are similar, they measure different things. The de Rham cohomology is a topological invariant, whereas the Dolbeault cohomology measures the holomorphic complexity<sup>30</sup>.

**Proposition 1.12.** *On a complex manifold  $M$  of dimension  $n$ , the Dolbeault cohomology  $H_{\bar{\partial}}^{p,q}(M)$  vanishes for  $q > n$ .*

*Proof.* It follows from the fact that if  $q > n$  then  $\bigwedge^{p,q}(T_{\mathbb{R}}^*M) = 0$ . Hence for  $q > n$ , the only  $(p, q)$ -form on  $M$  is the zero form. □

### 1.4.2 $\bar{\partial}$ -Poincaré lemma for complex manifolds

**Definition 1.33** (Pullback map in cohomology). Let  $F : M \rightarrow N$  be a holomorphic map of complex manifolds, then its pullback  $F^*$  induces<sup>31</sup> a linear map of quotient spaces, denoted by  $F^\#$

$$F^\# : \frac{\mathcal{Z}^{p,q}(N)}{\mathcal{B}^{p,q}(N)} \rightarrow \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$

$$[\omega] \mapsto [F^*(\omega)]$$

This is a map in cohomology  $F^\# : H_{\bar{\partial}}^{p,q}(N) \rightarrow H_{\bar{\partial}}^{p,q}(M)$  called the *pullback map in cohomology*.

**Remark 1.41.** From **Remark 1.32** and **Proposition 1.9** it follows that:

1. If  $\mathbb{1}_M : M \rightarrow M$  is the identity map, then  $\mathbb{1}_M^\# : H_{\bar{\partial}}^{p,q}(M) \rightarrow H_{\bar{\partial}}^{p,q}(M)$  is also the identity map.
2. If  $F : M \rightarrow N$  and  $G : N \rightarrow N'$  are holomorphic maps, then  $(G \circ F)^\# = F^\# \circ G^\#$ .

**Proposition 1.13** (Invariance of Dolbeault cohomology for biholomorphic manifolds). *Let  $F : M \rightarrow N$  be a biholomorphic map of manifolds, then the pullback map in cohomology  $F^\# : H_{\bar{\partial}}^{p,q}(N) \rightarrow H_{\bar{\partial}}^{p,q}(M)$  is an isomorphism.*

*Proof.* Since  $F$  is a biholomorphic map,  $F^{-1} : N \rightarrow M$  is also a holomorphic map of manifolds. Therefore, using **Remark 1.41** we have

$$\mathbb{1}_{H_{\bar{\partial}}^{p,q}(M)} = \mathbb{1}_M^\# = (F^{-1} \circ F)^\# = F^\# \circ (F^{-1})^\#$$

This implies that  $(F^{-1})^\#$  is the inverse of  $F^\#$ , i.e.  $F^\#$  is an isomorphism. □

**Theorem 1.9** ( $\bar{\partial}$ -Poincaré lemma for complex manifolds). *Let  $M$  be a complex manifold, then for all  $w \in M$  there exists an open neighborhood  $U$  such that every  $\bar{\partial}$ -closed  $(p, q)$ -form on  $U$  is  $\bar{\partial}$ -exact for  $q \geq 1$ .*

*Proof.* Let  $(U, \phi)$  be a chart on the complex manifold  $M$  of dimension  $n$  such that  $w \in U$ . By **Theorem 1.4** we know that the coordinate map  $\phi : U \rightarrow \phi(U) \subset \mathbb{C}^n$  is biholomorphic. We choose  $U$  such that  $\phi(U)$  is an open polydisc in  $\mathbb{C}^n$ . Then by **Theorem 1.3** every  $\bar{\partial}$ -closed  $(p, q)$ -form on  $\phi(U)$  is exact for  $q \geq 1$ , i.e.  $H_{\bar{\partial}}^{p,q}(\phi(U)) = 0$  for  $q \geq 1$ . Now we can use **Proposition 1.13** to conclude that  $H_{\bar{\partial}}^{p,q}(U) = 0$  for  $q \geq 1$ , i.e. every  $\bar{\partial}$ -closed  $(p, q)$ -form on  $U$  is  $\bar{\partial}$ -exact for  $q \geq 1$ . □

<sup>30</sup>Donu Arapura, “de Rham vs Dolbeault Cohomology”, <https://mathoverflow.net/q/95432>, 28 April 2012.

<sup>31</sup>Follows from **Lemma 1.4**.

## Chapter 2

# Cousin problems

### 2.1 Cousin problems for $\mathbb{C}$

In this section some basic definitions and facts from [19, §1.6], [14, §13.1], [13, §0.3.4] and [3, §VII.5, VIII.3] will be stated.

#### 2.1.1 Mittag-Leffler theorem

Consider the following problem:

Let  $U$  be an open subset of  $\mathbb{C}$  and  $\{a_k\}$  be a sequence of distinct points in  $U$  such that  $\{a_k\}$  has no limit points in  $U$ . For each integer  $k \geq 1$  consider the rational function

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}$$

where  $m_k$  is some positive integer and  $A_{1k}, \dots, A_{m_k k}$  are arbitrary complex coefficients. Is there a meromorphic function  $f$  on  $U$  whose poles are exactly the points  $\{a_k\}$  and such that the singular part<sup>1</sup> of  $f$  at  $z = a_k$  is  $S_k(z)$ ?

The answer to this problem is yes and was solved by Gösta Mittag-Leffler during 1876-1884, building on the work of his mentor Karl Weierstrass [20]. Here we will discuss a proof which will illustrate the general method for solving the Cousin problems.

**Theorem 2.1** (Single variable Cousin I). *Let  $U \subset \mathbb{C}$  be an open set with an open covering  $\{U_\alpha\}$ . Suppose that for each  $U_\alpha, U_\beta$  with nonempty intersection there is a holomorphic function  $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$  satisfying*

1.  $g_{\alpha\beta} + g_{\beta\alpha} = 0$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

*Then there exist  $f_\alpha \in \mathcal{O}(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = f_\beta - f_\alpha$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty.*

*Proof.* As in [Theorem C.1](#), let  $\{V_k\}$  be a locally finite refinement of  $\{U_\alpha\}$  and  $\{\psi_k\}$  be a smooth partition of unity of  $U$  subordinate to the open cover  $\{V_k\}$ . Then for a fixed  $k$ ,  $\psi_k$  has a compact

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<sup>1</sup>Let  $f$  has a pole of order  $m$  at  $z = a$  such that  $f$  has the Laurent series expansion in an open neighborhood  $V$  of  $a$  give by

$$f(z) = \frac{A_m}{(z-a)^m} + \dots + \frac{A_{m-1}}{(z-a)^{m-1}} + \dots + \frac{A_1}{(z-a)} + g(z)$$

where  $g$  is analytic in  $V$  and  $A_m \neq 0$ . Then  $\sum_{j=1}^m \frac{A_j}{(z-a)^j}$  is called *singular part* or *principal part* of  $f$  at  $z = a$ .

support contained in  $V_k \subset U_{r(k)}$ . We can then define the smooth functions  $\{h_{k\alpha}\}$  in the open sets  $\{U_\alpha\}$  by

$$h_{k\alpha}(z) = \begin{cases} \psi_k(z)g_{r(k)\alpha}(z) & \text{if } z \in V_k \cap U_\alpha \\ 0 & \text{if } z \in U_\alpha \setminus (V_k \cap U_\alpha) \end{cases}$$

Since  $\psi_k$  vanishes in an open neighborhood of  $U \setminus V_k$ ,  $\psi_k$  will also vanish in an open neighborhood of  $U_\alpha \setminus (U_\alpha \cap V_k)$ . Therefore, the function  $\boxed{h_{k\alpha} = \psi_k g_{r(k)\alpha}}$  is a smooth function  $U_\alpha$ , and for each  $\alpha$  we have the smooth function

$$h_\alpha := \sum_k h_{k\alpha} \quad \text{on } U_\alpha$$

Then, on  $U_\alpha \cap U_\beta$ , using the properties of  $\{g_{\alpha\beta}\}$  we get

$$\begin{aligned} h_\beta - h_\alpha &= \sum_k (h_{k\beta} - h_{k\alpha}) = \sum_k \psi_k (g_{r(k)\beta} - g_{r(k)\alpha}) \\ &= \sum_k \psi_k (-g_{\beta r(k)} - g_{r(k)\alpha}) = \sum_k \psi_k (g_{\alpha\beta}) = g_{\alpha\beta} \end{aligned}$$

since  $\sum_k \psi_k = 1$ . This gives us a smooth solution  $\{h_\alpha\}$  to the first Cousin problem.

Next, since  $g_{\alpha\beta}$  is holomorphic, by [Theorem B.2](#) we have

$$\frac{\partial h_\alpha}{\partial \bar{z}} = \frac{\partial h_\beta}{\partial \bar{z}} \quad \text{on } U_\alpha \cap U_\beta$$

Hence there exists a function  $h \in C^\infty(U)$  such that

$$h = \frac{\partial h_\alpha}{\partial \bar{z}} \quad \text{on } U_\alpha \text{ for each } \alpha \quad (2.1)$$

Also, from [Theorem 1.2](#) we get  $f \in C^\infty(U)$  such that

$$\frac{\partial f}{\partial \bar{z}} = h \quad (2.2)$$

Comparing (2.1) and (2.2) we get that

$$f_\alpha = h_\alpha - f \in \mathcal{O}(U_\alpha) \text{ for each } \alpha$$

Since  $f_\beta - f_\alpha = g_{\alpha\beta}$ , the set  $\{f_\alpha\}$  is the required holomorphic solution to the Cousin problem.  $\square$

**Theorem 2.2** (Mittag-Leffler theorem). *Let  $U$  be an open subset of  $\mathbb{C}$  and  $\{a_k\}$  be a sequence of distinct points in  $U$  such that  $\{a_k\}$  has no limit points in  $U$ . For each integer  $k \geq 1$  consider the rational function*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}$$

where  $m_k$  is some positive integer and  $A_{1k}, \dots, A_{m_k k}$  are arbitrary complex coefficients. Then there is a meromorphic function  $f$  on  $U$  whose poles are exactly the points  $\{a_k\}$  and such that the singular part of  $f$  at  $z = a_k$  is  $S_k(z)$ .

*Proof.* Choose an open cover  $\{U_\alpha\}$  of  $U$  with the property that each  $U_\alpha$  contains at most one point of  $\{a_k\}$ . Assign a meromorphic function  $h_\alpha$  on  $U_\alpha$  for each  $\alpha$  such that  $h_\alpha = S_k$  if  $U_\alpha$  contains  $a_k$ , otherwise  $f_\alpha \equiv 0$ . We can then define the Cousin data for the cover  $\{U_\alpha\}$  by setting

$$g_{\alpha\beta} = h_\beta - h_\alpha \quad \text{on } U_\alpha \cap U_\beta \quad (2.3)$$

Note that for each  $U_\alpha, U_\beta$  with nonempty intersection  $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$  since there doesn't exist any pole  $a_k \in U_\alpha \cap U_\beta$ . Moreover,  $\{g_{\alpha\beta}\}$  satisfies the conditions

1.  $g_{\alpha\beta} + g_{\beta\alpha} = 0$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

Therefore, by [Theorem 2.1](#), there exist  $f_\alpha \in \mathcal{O}(U_\alpha)$  for each  $\alpha$  such that

$$g_{\alpha\beta} = f_\beta - f_\alpha \quad \text{on } U_\alpha \cap U_\beta \tag{2.4}$$

Comparing [\(2.3\)](#) and [\(2.4\)](#) we get that

$$h_\beta - h_\alpha = f_\beta - f_\alpha \quad \text{on } U_\alpha \cap U_\beta$$

for each pair  $(\alpha, \beta)$ . Hence, we can define a meromorphic function  $f$  on  $U$  such that

$$f(z) = h_\alpha(z) - f_\alpha(z) \quad \text{for } z \in U_\alpha$$

for each  $\alpha$ . Since subtracting a holomorphic function  $f_\alpha$  from  $h_\alpha$  doesn't affect the poles and singular parts,  $f$  is the desired meromorphic function on  $U$  whose poles are exactly the points  $\{a_k\}$  and the singular part at  $z = a_k$  is  $S_k$ .  $\square$

### 2.1.2 Weierstrass theorem

Consider the following problem:

Let  $U$  be an open subset of  $\mathbb{C}$  and  $\{a_k\}$  be a sequence of distinct points in  $U$  such that  $\{a_k\}$  has no limit points in  $U$ . Given a sequence of integers  $\{m_k\}$ , is there a function  $f$  which is holomorphic on  $U$  such that the only zeros of  $f$  are the points  $a_k$  with multiplicity  $m_k$ ?

The answer to this problem is yes and was solved by Karl Weierstrass in 1876. Though this problem was solved before Mittag-Leffler theorem, we will deduce it from Cousin I following [\[14, Theorem 13.1.6\]](#).

**Lemma 2.1.** *Let  $U \subset \mathbb{C}$  be simply connected open set and  $f : U \rightarrow \mathbb{C}$  be a holomorphic and non-vanishing function. Then there is a holomorphic function  $g$  on  $U$  such that  $\exp(g) = f$ .*

*Proof.* This is a standard result in single variable complex analysis, see [\[3, Theorem VIII.2.2\(g\)\]](#) or [\[14, Lemma 13.1.5\]](#).  $\square$

**Theorem 2.3** (Single variable Cousin II). *Let  $U \subset \mathbb{C}$  be an open set with an open covering  $\{U_\alpha\}$ . Suppose that for each  $U_\alpha, U_\beta$  with nonempty intersection there is a non-vanishing holomorphic function  $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$  satisfying*

1.  $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

*Then there exist  $f_\alpha \in \mathcal{O}^*(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty.*

*Proof.* Let  $\{V_j\}$  be a refinement of  $\{U_\alpha\}$  such that for each  $j$ ,  $V_j$  is an open ball and  $V_j \subset U_{r(j)}$ . Next, we define  $h_{jk} : V_j \cap V_k \rightarrow \mathbb{C}$  by  $h_{jk}(z) = g_{r(j)r(k)}(z)$ . Then  $\{h_{jk}\}$  is a set of holomorphic functions satisfying

1.  $h_{jk} \cdot h_{kj} = 1$  for each pair  $(j, k)$ ;
2.  $h_{jk} \cdot h_{k\ell} \cdot h_{\ell j} = 1$  on  $V_j \cap V_k \cap V_\ell$  for each triple  $(j, k, \ell)$ .

**Step 1:** *There exist  $u_j \in \mathcal{O}^*(V_j)$  for each  $j$  such that  $h_{jk} = u_k/u_j$  on  $V_j \cap V_k$  whenever the intersection is nonempty.*

Since each open ball  $V_j$  is simply connected, by [Lemma 2.1](#), there exists  $\tilde{h}_{jk} \in \mathcal{O}(V_j \cap V_k)$  such that  $h_{jk} = \exp(\tilde{h}_{jk})$ . Then  $\{\tilde{h}_{jk}\}$  satisfies the condition of Cousin I data for the covering  $\{V_j\}$ , and by [Theorem 2.1](#) there exist  $\tilde{u}_j \in \mathcal{O}(V_j)$  for each  $j$  such that  $\tilde{h}_{jk} = \tilde{u}_k - \tilde{u}_j$  on  $V_j \cap V_k$  whenever the intersection is nonempty. Then the set  $\{u_j\}$  for  $u_j = \exp(\tilde{u}_j)$  is the required holomorphic solution to the Cousin problem.

**Step 2:** *There exist  $f_\alpha \in \mathcal{O}^*(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = f_\beta/f_\alpha$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty.*

Note that, for  $z \in U_\alpha \cap V_j \cap V_k$  we have

$$\begin{aligned} \left( \frac{u_k}{u_j} g_{r(k)\alpha} g_{\alpha r(j)} \right) (z) &= \left( \frac{u_k}{u_j} \frac{1}{g_{r(j)r(k)}} \right) (z) \\ &= \left( \frac{u_k}{u_j} g_{r(k)r(j)} \right) (z) \\ &= \left( \frac{u_k}{u_j} h_{kj} \right) (z) \\ &= 1 \end{aligned}$$

Therefore, we have  $u_k g_{r(k)\alpha}(z) = u_j g_{r(j)\alpha}(z)$  on  $U_\alpha \cap V_j \cap V_k$ . Since this is true for any pair  $(j, k)$ , for any  $\alpha$  we define non-vanishing holomorphic function  $f_\alpha \in \mathcal{O}^*(U_\alpha)$  such that

$$f_\alpha(z) = u_j g_{r(j)\alpha}(z) \quad \text{for } z \in U_\alpha \cap V_j$$

Finally,  $\{f_\alpha\}$  is the required holomorphic solution to the Cousin problem since

$$\frac{f_\beta}{f_\alpha}(z) = \frac{u_j g_{r(j)\beta}}{u_j g_{r(j)\alpha}}(z) = \frac{1}{g_{\beta r(j)} g_{r(j)\alpha}}(z) = g_{\alpha\beta}(z) \quad \text{for } z \in U_\alpha \cap U_\beta \cap V_j$$

where  $j$  is arbitrary. □

**Theorem 2.4** (Weierstrass theorem). *Let  $U$  be an open subset of  $\mathbb{C}$  and  $\{a_k\}$  be a sequence of distinct points in  $U$  such that  $\{a_k\}$  has no limit points in  $U$ . Given a sequence of integers  $\{m_k\}$ , there is a function  $f$  which is holomorphic on  $U$  such that the only zeros of  $f$  are the points  $a_k$  with multiplicity  $m_k$ .*

*Proof.* Choose an open cover  $\{U_\alpha\}$  of  $U$  with the property that each  $U_\alpha$  contains at most one point of  $\{a_k\}$ . Assign a holomorphic function  $h_\alpha$  on  $U_\alpha$  for each  $\alpha$  such that  $h_\alpha = (z - a_k)^{m_k}$  if  $U_\alpha$  contains  $a_k$ , otherwise  $h_\alpha \equiv 1$ . We can then define the Cousin data for the cover  $\{U_\alpha\}$  by setting

$$g_{\alpha\beta} = \frac{h_\beta}{h_\alpha} \quad \text{on } U_\alpha \cap U_\beta \tag{2.5}$$

Note that for each  $U_\alpha, U_\beta$  with nonempty intersection  $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$  since there doesn't exist any zero  $a_k \in U_\alpha \cap U_\beta$ , and  $\{g_{\alpha\beta}\}$  satisfies the conditions

1.  $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

Therefore, by [Theorem 2.3](#), there exist  $f_\alpha \in \mathcal{O}^*(U_\alpha)$  for each  $\alpha$  such that

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta \tag{2.6}$$

Comparing (2.5) and (2.6) we get that

$$\frac{h_\beta}{h_\alpha} = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta$$

for each pair  $(\alpha, \beta)$ . Hence, we can define a holomorphic function  $f$  on  $U$  such that

$$f(z) = \frac{h_\alpha(z)}{f_\alpha(z)} \quad \text{for } z \in U_\alpha$$

for each  $\alpha$ . Since dividing  $h_\alpha$  by a non-vanishing holomorphic function  $f_\alpha$  doesn't affect the zeros of  $h_\alpha$  and their multiplicities,  $f$  is the desired holomorphic function on  $U$  whose only zeros are the points  $a_k$  with multiplicity  $m_k$ .  $\square$

**Corollary 2.1.** *Let  $U \subset \mathbb{C}$  be any open set. Let  $Y \subset U$  be a discrete set. Then there is a holomorphic function  $f$  on all of  $U$  such that  $Y = \{z \in U : f(z) = 0\}$ .*

## 2.2 Cousin problems for $\mathbb{C}^n$

In this section some basic definitions and facts from [13, §6.1] and [9, §I.E] will be stated.

### 2.2.1 Cousin I

Consider the following problem:

Let  $U \subset \mathbb{C}^n$  be an open set with an open covering  $\{U_\alpha\}$ . Suppose that for each  $U_\alpha, U_\beta$  with nonempty intersection there is a holomorphic function  $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$  satisfying

1.  $g_{\alpha\beta} + g_{\beta\alpha} = 0$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

Then does there exist  $f_\alpha \in \mathcal{O}(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = f_\beta - f_\alpha$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty?

The answer to this problem is yes when  $U$  is a polydisc. Moreover, in general, this is true when  $U$  is a domain of holomorphy<sup>2</sup>, for details see [13, Proposition 6.1.8]. In fact, the solution to Cousin I is exactly same as the single variable case since in the theory of single variable holomorphic functions, every open set is a domain of holomorphy.

**Theorem 2.5** (Cousin I for a polydisc). *Let  $\Delta \subset \mathbb{C}^n$  be an open polydisc with an open covering  $\{U_\alpha\}$ . Suppose that for each  $U_\alpha, U_\beta$  with nonempty intersection there is a holomorphic function  $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$  satisfying*

1.  $g_{\alpha\beta} + g_{\beta\alpha} = 0$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

*Then there exist  $f_\alpha \in \mathcal{O}(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = f_\beta - f_\alpha$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty.*

---

<sup>2</sup>An open set  $U \subset \mathbb{C}^n$  is called a *domain of holomorphy* if there doesn't exist non-empty open sets  $U_1, U_2$  with  $U_2$  connected,  $U_2 \not\subset U_1$ ,  $U_1 \subset U_2 \cap U$ , such that for every holomorphic function  $f$  on  $U$  there is a holomorphic function  $f_2$  on  $U_2$  such that  $f = f_2$  on  $U_1$ , see [13, §0.3.1] and [19, §2.5].

*Proof.* As in [Theorem C.1](#), let  $\{V_k\}$  be a locally finite refinement of  $\{U_\alpha\}$  and  $\{\psi_k\}$  be a smooth partition of unity of  $\Delta$  subordinate to the open cover  $\{V_k\}$ . Then for a fixed  $k$ ,  $\psi_k$  has a compact support contained in  $V_k \subset U_{r(k)}$ . We can then define the smooth functions  $\{h_{k\alpha}\}$  in the open sets  $\{U_\alpha\}$  by

$$h_{k\alpha}(z) = \begin{cases} \psi_k(z)g_{r(k)\alpha}(z) & \text{if } z \in V_k \cap U_\alpha \\ 0 & \text{if } z \in U_\alpha \setminus (V_k \cap U_\alpha) \end{cases}$$

Since  $\psi_k$  vanishes in an open neighborhood of  $\Delta \setminus V_k$ ,  $\psi_k$  will also vanish in an open neighborhood of  $U_\alpha \setminus (U_\alpha \cap V_k)$ . Therefore, the function  $\boxed{h_{k\alpha} = \psi_k g_{r(k)\alpha}}$  is a smooth function  $U_\alpha$ , and for each  $\alpha$  we have the smooth function

$$h_\alpha = \sum_k h_{k\alpha} \quad \text{on } U_\alpha$$

Then, on  $U_\alpha \cap U_\beta$ , using the properties of  $\{g_{\alpha\beta}\}$  we get

$$\begin{aligned} h_\beta - h_\alpha &= \sum_k (h_{k\beta} - h_{k\alpha}) = \sum_k \psi_k (g_{r(k)\beta} - g_{r(k)\alpha}) \\ &= \sum_k \psi_k (-g_{\beta r(k)} - g_{r(k)\alpha}) = \sum_k \psi_k (g_{\alpha\beta}) = g_{\alpha\beta} \end{aligned}$$

since  $\sum_k \psi_k = 1$ . This gives us a smooth solution  $\{h_\alpha\}$  to the first Cousin problem.

Next, for each set  $U_\alpha$  consider the differential form  $\omega_\alpha \in \bar{\partial}h_\alpha \in \Omega^{0,1}(U_\alpha)$ . In each intersection  $U_\alpha \cap U_\beta$  we note that  $\omega_\alpha = \bar{\partial}(h_\beta + g_{\alpha\beta}) = \omega_\beta$ , since  $g_{\alpha\beta}$  are holomorphic functions. Hence there exists a global differential form  $\omega \in \Omega^{p,q}(\Delta)$  such that

$$\omega = \bar{\partial}h_\alpha \quad \text{on } U_\alpha \text{ for each } \alpha \quad (2.7)$$

Also, since  $\bar{\partial}\omega = 0$ , from [Theorem 1.3](#) we get  $f \in \Omega^{0,0}(\Delta) = C^\infty(\Delta)$  such that

$$\bar{\partial}f = \omega \quad (2.8)$$

Comparing (2.7) and (2.8) we get that

$$f_\alpha = h_\alpha - f \in \mathcal{O}(U_\alpha) \text{ for each } \alpha$$

Since  $f_\beta - f_\alpha = g_{\alpha\beta}$ , the set  $\{f_\alpha\}$  is the required holomorphic solution to the Cousin problem.  $\square$

### 2.2.2 Cousin II

Consider the following problem:

Let  $U \subset \mathbb{C}^n$  be an open set with an open covering  $\{U_\alpha\}$ . Suppose that for each  $U_\alpha, U_\beta$  with nonempty intersection there is a non-vanishing holomorphic function  $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$  satisfying

1.  $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

Then does there exist  $f_\alpha \in \mathcal{O}^*(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty?

The answer to this problem is yes when  $U$  is a polydisc. However, in general, this is not true when  $U$  is any domain of holomorphy. Unlike the single variable case, Cousin I doesn't imply Cousin II for  $n \geq 2$ . For the counterexample given by Kiyoshi Oka, see [13, pp. 250-253].

**Lemma 2.2.** *Let  $U \subset \mathbb{C}^n$  be simply connected open set and  $f : U \rightarrow \mathbb{C}$  be a holomorphic and non-vanishing function. Then there is a holomorphic function  $g$  on  $U$  such that  $\exp(g) = f$ .*

*Proof.* Since this is a topological fact, we are able to generalize the proof of [Lemma 2.1](#). For details, see [[14](#), Lemma 13.1.5] and [[13](#), Lemma 6.1.10].  $\square$

**Theorem 2.6** (Cousin II for a polydisc). *Let  $\Delta \subset \mathbb{C}^n$  be an open polydisc with an open covering  $\{U_\alpha\}$ . Suppose that for each  $U_\alpha, U_\beta$  with nonempty intersection there is a non-vanishing holomorphic function  $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$  satisfying*

1.  $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

*Then there exist  $f_\alpha \in \mathcal{O}^*(U_\alpha)$  for each  $\alpha$  such that  $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$  on  $U_\alpha \cap U_\beta$  whenever the intersection is nonempty.*

Instead of proving this theorem<sup>3</sup>, we will directly prove the generalization of [Corollary 2.1](#) in the next section.

## 2.3 Cousin problem for analytic hypersurface in $\mathbb{C}^n$

Consider the following problem:

Is any analytic subvariety  $Y$  of a complex manifold  $M$  the zero-locus of some global holomorphic functions defined on  $M$ ?

The answer to this problem is yes when  $Y$  is a hypersurface and  $M$  is  $\mathbb{C}^n$ .

### 2.3.1 Analytic subvariety of a complex manifold

In this subsection some definitions and properties from [[10](#), §2.1, 2.3] and [[6](#), §I.8, IV.1] will be discussed.

**Definition 2.1** (Analytic subvariety). Let  $M$  be a  $n$ -dimensional complex manifold. An *analytic subvariety* of  $M$  is a closed subset  $Y \subset M$  such that for every point  $w \in Y$  there exists an open neighborhood  $w \in U \subset M$  and  $f_1, \dots, f_m \in \mathcal{O}(U)$  with

$$U \cap Y = \{z \in U : f_j(z) = 0 \text{ for } j = 1, \dots, m\}$$

**Remark 2.1.** A more natural definition of an analytic subvariety of  $M$  is that it is a subset  $Y \subset M$  such that for every point  $w \in M$  there exists an open neighborhood  $w \in U \subset M$  and  $f_1, \dots, f_m \in \mathcal{O}(U)$  with

$$U \cap Y = \{z \in U : f_j(z) = 0 \text{ for } j = 1, \dots, m\}$$

This definition is equivalent to the earlier one because we can prove that  $w \in M \setminus Y$  if and only if  $Y$  is a closed subset of  $M$  [[6](#), p. 36].

**Definition 2.2** (Analytic hypersurface). An analytic subvariety  $Y$  of  $M$  is called *analytic hypersurface* if we can always take  $m = 1$ , i.e. for every point  $w \in Y$  there exists an open neighborhood  $w \in U \subset M$  and  $f \in \mathcal{O}(U)$  with

$$U \cap Y = \{z \in U : f(z) = 0\}$$

---

<sup>3</sup>For an outline of the proof, see [[9](#), pp. 33-36]. Here, unlike the single variable case, we also need to show the existence of non-vanishing continuous solution before proving the existence of non-vanishing smooth solution. For details, see [[13](#), Proposition 6.1.11(Part I)].



**Remark 2.2.** In general, analytic subvariety cannot be given by global equations. For example, if  $M$  is compact and connected, there are no non-constant holomorphic functions on  $M$ . For an example in which the ambient manifold  $M$  is not compact, consider the complex manifold  $M := U_1 \cup U_2$  with

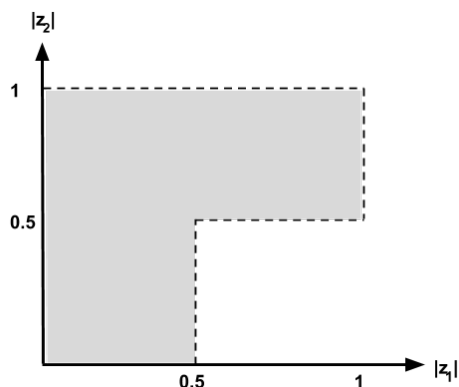
$$U_1 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < \frac{1}{2} \text{ and } |z_2| < 1 \right\}$$

$$U_2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \text{ and } \frac{1}{2} < |z_2| < 1 \right\}$$

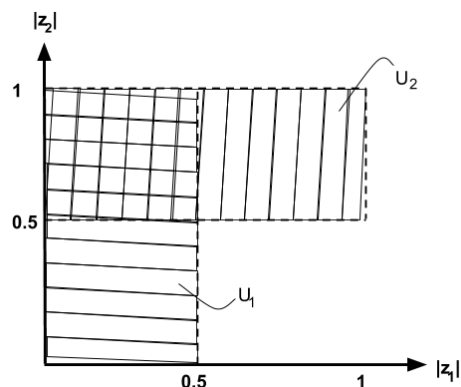
Next, consider the closed subset<sup>4</sup>  $Y = \{(z_1, z_2) \in U_2 : z_1 = z_2\} \subset M$ . Note that  $U_1$  and  $U_2$  give an open covering of  $M$  with  $Y \subset U_2$ , i.e. for all  $p \in Y$  we can use  $U_2$  since  $Y \cap U_2 = \{(z_1, z_2) \in U_2 : f(z_1, z_2) = z_1 - z_2 = 0\}$  where  $f \in \mathcal{O}(U_2)$ . Therefore,  $Y$  is an analytic hypersurface of  $M$ .

**Claim:** There does not exist  $g \in \mathcal{O}(M)$  such that  $Y = \{(z_1, z_2) \in M : g(z_1, z_2) = 0\}$ .

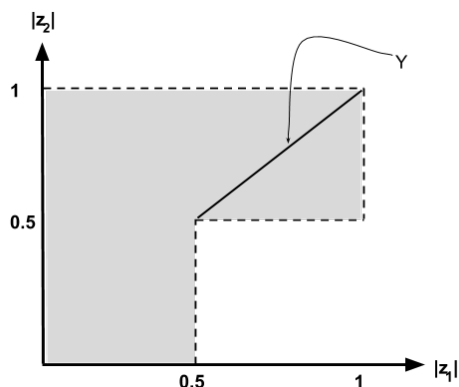
On the contrary, let there exist  $g \in \mathcal{O}(M)$  such that  $g$  vanishes exactly on  $Y$ . Note that  $M \subset \Delta(0;1)$ . Hence, by **Theorem B.4**, there exists  $G \in \mathcal{O}(\Delta(0;1))$  such that  $G|_M = g$ . In particular, for  $z_1 = z_2 = z$ ,  $G(z, z) = h(z)$  is a single variable holomorphic functions which vanishes for  $\frac{1}{2} < |z| < 1$  in  $M$ . Since zero function is the only single variable holomorphic function with uncountably many zeros,  $h(z)$  vanishes for  $0 \leq |z| < 1$  in  $\Delta(0;1)$ , i.e.  $G|_M = g$  vanishes on  $Z = \{(z_1, z_1) \in M : z_1 = z_2\} \supset Y$ . Contradiction.



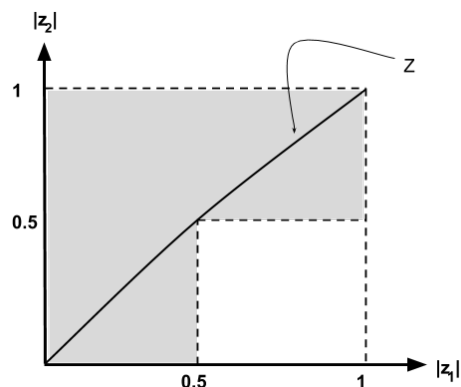
The region corresponding to  $M$



The regions corresponding to  $U_1$  and  $U_2$ .



The analytic subvariety  $Y$  of  $X$



The vanishing set  $Z$  of  $G$  in  $M$

<sup>4</sup>Recall that for a continuous function the inverse image of a closed set is closed. In particular, the set of zeros of a continuous function is closed.

### 2.3.2 Sheaf theory and Čech cohomology

In this subsection we will revisit the results from sheaf theory and Čech cohomology that were discussed in the previous report [12, Chapter 2].

**Example 2.1** (Sheaves on complex manifold). Recall that if one has a presheaf of functions (or forms) on a topological space  $M$  which is defined by some local property, then the presheaf is also a sheaf [12, Example 2.4]. In particular, if  $M$  is a complex manifold then:

- $\mathcal{O}$  is the sheaf of holomorphic functions on  $M$  such that for every open subset  $U$  of  $M$  we have the additive abelian group  $\mathcal{O}(U)$  of holomorphic functions on  $U$  along with the natural restriction maps as the group homomorphisms for the nested open subsets.
- $\mathcal{O}^*$  is the sheaf of non-vanishing holomorphic functions on  $M$  such that for every open subset  $U$  of  $M$  we have the multiplicative abelian group  $\mathcal{O}^*(U)$  of non-vanishing holomorphic functions on  $U$  along with the natural restriction maps as the group homomorphisms for the nested open subsets.
- $\Omega^{p,q}$  is the sheaf of complex  $(p, q)$ -forms on  $M$  such that for every open subset  $U$  of  $M$  we have the additive abelian group  $\Omega^{p,q}(U)$  of smooth  $(p, q)$ -forms on  $U$  (smooth sections of an exterior power of a vector bundle, i.e. smooth maps of manifolds) along with the natural restriction maps as the group homomorphisms for the nested open subsets.
- $\mathcal{O}^p$  is the sheaf of holomorphic  $p$ -forms on  $M$  such that for every open subset  $U$  of  $M$  we have the additive abelian group  $\mathcal{O}^p(U)$  of holomorphic  $p$ -forms on  $U$  (holomorphic sections of an exterior power of holomorphic cotangent bundle, i.e. holomorphic maps of manifolds) along with the natural restriction maps as the group homomorphisms for the nested open subsets.

**Example 2.2** (Sheaf maps). Recall that a sheaf map is collection of group homomorphisms which commute with the restriction maps. Then for a complex manifold  $M$  we have:

- The exponential map  $\exp : \mathcal{O} \rightarrow \mathcal{O}^*$  defined by the collection of group homomorphisms  $\{\exp_U : \mathcal{O}(U) \rightarrow \mathcal{O}^*(U)\}_{U \subset M}$  where  $\exp_U(f) = \exp(f)$  is defined via charts. This is a sheaf map since for  $U \subset V \subset M$ ,  $\exp_U$  and  $\exp_V$  commute with the restriction maps.
- Since the exterior derivative is a local operator, it commutes with restriction [12, Remark 1.29]. Therefore,  $d : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1}$  is a map of sheaves [12, Example 2.6]. In particular,  $\partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q}$  is a sheaf map between the sheaf of complex differential forms on a complex manifold  $M$ .

**Example 2.3** (Kernel sheaf). For a complex manifold  $M$  we have the sheaf of closed  $(p, q)$ -forms on  $M$  given by  $\ker(\partial) = \mathcal{Z}^{p,q}$  corresponding to the sheaf map  $\bar{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}$ . In particular,  $\mathcal{Z}^{p,0} = \mathcal{O}^p$  is the sheaf of holomorphic  $p$ -forms on  $M$ .

**Example 2.4** (Exact sequence of sheaves). For a complex manifold  $M$  we have:

- The short exact sequence, called *exponential sheaf sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

Note that the sheaf map  $\exp$  is surjective by [Lemma 2.2](#), since locally  $M$  is biholomorphic to an open set in  $\mathbb{C}^n$  and for every point  $w \in \mathbb{C}^n$  we can find a simply connected open neighborhood  $U$  such that every  $f \in \mathcal{O}^*(U)$  can be written as  $\exp(g) = f$  for some  $g \in \mathcal{O}(U)$ .

- The exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \Omega^{p,2} \xrightarrow{\bar{\partial}} \dots$$

where the exactness follows from [Theorem 1.8](#), [Theorem 1.9](#), and [Remark 1.37](#).

**Remark 2.3** (Long exact sequence of Čech cohomology). By [Remark 1.24](#) we know that complex manifolds are paracompact. Hence we can use Serre's theorem [[12](#), Theorem 2.1] to get the long exact sequence of Čech cohomology corresponding to a short exact sequence of sheaves of a complex manifold.

- The exponential sheaf sequence on  $M$  will induce the following long exact sequence of cohomology

$$\dots \longrightarrow \check{H}^q(M, \mathcal{O}) \longrightarrow \check{H}^q(M, \mathcal{O}^*) \xrightarrow{\Delta} \check{H}^{q+1}(M, \underline{\mathbb{Z}}) \longrightarrow \check{H}^{q+1}(M, \mathcal{O}) \longrightarrow \dots$$

- Using [Example 2.3](#) and [Example 2.4](#) we get the following short exact sequence of sheaves on a complex manifold  $M$

$$0 \longrightarrow \mathcal{Z}^{p,\ell} \hookrightarrow \Omega^{p,\ell} \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,\ell+1} \longrightarrow 0$$

for every  $\ell \geq 0$ . This induces the following long exact sequence of cohomology

$$\dots \longrightarrow \check{H}^q(M, \Omega^{p,\ell}) \longrightarrow \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow \check{H}^{q+1}(M, \Omega^{p,\ell}) \longrightarrow \dots$$

for each  $\ell$ .

**Remark 2.4** (Fine sheaves). Note that, for  $p, \ell \geq 0$ ,  $\Omega^{p,\ell}$  are smooth sections of vector bundles and hence are fine sheaves. Therefore we can use [[12](#), Theorem 2.2] to get  $\check{H}^q(M, \Omega^{p,\ell}) = 0$  for all  $\ell \geq 1$ .

**Theorem 2.7** (Homotopy invariance of Čech cohomology). *Let  $M$  and  $N$  be two smooth manifolds, and assume that  $f : M \rightarrow N$  is a homotopy equivalence. If  $\underline{G}$  is a constant sheaf on  $N$ , then  $\check{H}^q(M, f^{-1}\underline{G}) \cong \check{H}^q(N, \underline{G})$  for all  $q \geq 0$ . In other words, the Čech cohomology of locally constant sheaves on smooth manifolds is a homotopy invariant.*

*Proof.* In the previous report [[12](#), Theorem 2.1, 2.2], we proved that Čech cohomology of a manifold is isomorphic to its sheaf cohomology [[22](#), §5.18, 5.33]. Moreover, it is a well known fact that sheaf cohomology of locally constant sheaves is a homotopy invariant [[23](#), §10.2, 11.3]. Therefore, sheaf cohomology of locally constant sheaves is a homotopy invariant [[18](#), §6.3].  $\square$

**Corollary 2.2.** *If a smooth manifold  $M$  is contractible and  $\underline{G}$  is a constant sheaf on  $M$ , then  $\check{H}^0(M, \underline{G}) \cong \underline{G}(M)$  and  $\check{H}^q(M, \underline{G}) \cong 0$  for  $q > 0$ .*

*Proof.* Since  $M \simeq \{*\}$ , form some point  $* \in M$ , we know that  $\check{H}^k(M, \underline{G}) \cong \check{H}^k(\{*\}, f^{-1}\underline{G})$  for all  $k \geq 0$ . We also know that  $\check{H}^k(M, \underline{G}) \cong \underline{G}(M)$  [[12](#), Proposition 2.6]. Therefore, we just need to show that  $\check{H}^k(\{*\}, \underline{G}) = 0$  for any constant sheaf  $\underline{G}$ . Fortunately, when we calculate Čech cohomology of a point we don't need to take direct limit because the system is trivial, i.e. there is only one covering with only one open subset:

$$\check{H}^q(\{*\}, \underline{G}) = \frac{\check{Z}^q(\{\{*\}\}, \underline{G})}{\check{B}^q(\{\{*\}\}, \underline{G})} = \frac{\ker\{\delta : \check{C}^q(\{\{*\}\}, \underline{G}) \rightarrow \check{C}^{q+1}(\{\{*\}\}, \underline{G})\}}{\operatorname{im}\{\delta : \check{C}^{q-1}(\{\{*\}\}, \underline{G}) \rightarrow \check{C}^q(\{\{*\}\}, \underline{G})\}}$$

Note that  $\check{C}^q(\{\{*\}\}, \underline{G}) = \{f|f : \{*\} \rightarrow \underline{G} \text{ is a constant map}\}$ . Hence for  $q > 0$  we have

$$\check{Z}^q(\{\{*\}\}, \underline{G}) = \begin{cases} \check{C}^q(\{\{*\}\}, \underline{G}) & \text{if } q \text{ is odd} \\ \{f|f \equiv 0 \text{ where } 0 \text{ is the identity element of } G\} & \text{if } q \text{ is even} \end{cases}$$

and

$$\check{B}^q(\{\{*\}\}, \underline{G}) = \begin{cases} \{f|f \equiv 0 \text{ where } 0 \text{ is the identity element of } G\} & \text{if } q-1 \text{ is odd} \\ \check{C}^q(\{\{*\}\}, \underline{G}) & \text{if } q-1 \text{ is even} \end{cases}$$

Therefore,  $\check{Z}^q(\{\{*\}\}, \underline{G}) = \check{B}^q(\{\{*\}\}, \underline{G})$  for all  $q > 0$ . Hence completing the proof.  $\square$

**Remark 2.5.** In the previous report we proved de Rham-Čech isomorphism, which says that if  $M$  is a smooth manifold then for each  $k \geq 0$  there exists a group isomorphism  $H_{dR}^k(M) \cong \check{H}^k(M, \mathbb{R})$  [12, Theorem 3.1]. By the above theorem we can conclude that de Rham cohomology is in fact a homotopy invariant.

### 2.3.3 Dolbeault isomorphism

In this subsection we will prove Dolbeault's theorem, following [13, §6.3] and [8, p. 45]. This is a complex analogue of de Rham's theorem [12, Theorem 3.1], and asserts that the Dolbeault cohomology is isomorphic to the Čech cohomology of the sheaf of holomorphic differential forms.

**Theorem 2.8.** *Let  $M$  be a complex manifold. Then for each  $p, q \geq 0$  there exists a group isomorphism*

$$H_{\bar{\partial}}^{p,q}(M) \cong \check{H}^q(M, \mathcal{O}^p)$$

*Proof.* For  $q = 0$ , from [Proposition 1.11](#) and [12, Lemma 2.4], we know that both  $H_{\bar{\partial}}^{p,0}(M)$  and  $\check{H}^0(M, \mathcal{O}^p)$  are isomorphic to the group of holomorphic  $p$ -forms on  $M$ . That is

$$\boxed{H_{\bar{\partial}}^{p,0}(M) \cong \check{H}^0(M, \mathcal{O}^p)}$$

Now let's restrict our attention to  $q \geq 1$ . From [Example 2.4](#) we know that the  $\bar{\partial}$ -Poincaré lemma implies the existence of the following long exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \Omega^{p,2} \xrightarrow{\bar{\partial}} \dots$$

Then, as noted in [Remark 2.3](#), we have a family of short exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^p & \hookrightarrow & \Omega^{p,0} & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,1} & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathcal{Z}^{p,1} & \hookrightarrow & \Omega^{p,1} & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,2} & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & \mathcal{Z}^{p,\ell} & \hookrightarrow & \Omega^{p,\ell} & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,\ell+1} & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

which will induce the respective long exact sequences of Čech cohomology

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \check{H}^q(M, \Omega^{p,0}) & \longrightarrow & \check{H}^q(M, \mathcal{Z}^{p,1}) & \xrightarrow{\Delta} & \check{H}^{q+1}(M, \mathcal{O}^p) & \longrightarrow & \check{H}^{q+1}(M, \Omega^{p,0}) & \longrightarrow & \dots \\ \dots & \longrightarrow & \check{H}^q(M, \Omega^{p,1}) & \longrightarrow & \check{H}^q(M, \mathcal{Z}^{p,2}) & \xrightarrow{\Delta} & \check{H}^{q+1}(M, \mathcal{Z}^{p,1}) & \longrightarrow & \check{H}^{q+1}(M, \Omega^{p,1}) & \longrightarrow & \dots \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \\ \dots & \longrightarrow & \check{H}^q(M, \Omega^{p,\ell}) & \longrightarrow & \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) & \xrightarrow{\Delta} & \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) & \longrightarrow & \check{H}^{q+1}(M, \Omega^{p,\ell}) & \longrightarrow & \dots \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

Now let's study one of these long exact sequence of Čech cohomology. By [12, Lemma 2.4] we have  $\check{H}^0(M, \Omega^{p,\ell}) \cong \Omega^{p,\ell}(M)$  and  $\check{H}^0(M, \mathcal{Z}^{p,\ell}) \cong \mathcal{Z}^{p,\ell}(M)$ . Also by Remark 2.4 we have  $\check{H}^q(M, \Omega^{p,\ell}) = 0$  for all  $q \geq 1$  and  $\ell \geq 0$ . Hence for any  $\ell \geq 0$  we get the exact sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{Z}^{p,\ell}(M) & \hookrightarrow & \Omega^{p,\ell}(M) & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,\ell+1}(M) & \xrightarrow{\Delta} & \check{H}^1(M, \mathcal{Z}^{p,\ell}) & \longrightarrow & 0 & \longrightarrow & \check{H}^1(M, \mathcal{Z}^{p,\ell+1}) \\ & & & & & & & & & & & & & \downarrow \Delta \\ & & & & & & \dots & \longleftarrow & 0 & \longleftarrow & \check{H}^3(M, \mathcal{Z}^{p,\ell}) & \xleftarrow{\Delta} & \check{H}^2(M, \mathcal{Z}^{p,\ell+1}) & \longleftarrow & 0 & \longleftarrow & \check{H}^2(M, \mathcal{Z}^{p,\ell}) \end{array}$$

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{H}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

Since this sequence is exact, the map  $\Delta : \mathcal{Z}^{p,\ell+1}(M) \rightarrow \check{H}^1(M, \mathcal{Z}^{p,\ell})$  is a surjective group homomorphism and  $\text{im}\{\bar{\partial} : \Omega^{p,\ell}(M) \rightarrow \mathcal{Z}^{p,\ell+1}(M)\} = \ker(\Delta)$ . Hence by the *first isomorphism theorem* we get

$$\check{H}^1(M, \mathcal{Z}^{p,\ell}) \cong \frac{\mathcal{Z}^{p,\ell+1}(M)}{\ker(\Delta)} \quad \text{for all } \ell \geq 0$$

Since  $\text{im}\{\bar{\partial} : \Omega^{p,\ell}(M) \rightarrow \mathcal{Z}^{p,\ell+1}(M)\} = \text{im}\{\bar{\partial} : \Omega^{p,\ell}(M) \rightarrow \Omega^{p,\ell+1}(M)\} = \mathcal{B}^{p,\ell+1}(M)$ , we get

$$\check{H}^1(M, \mathcal{Z}^{p,\ell}) \cong H_{\bar{\partial}}^{p,\ell+1}(M) \quad \text{for all } \ell \geq 0 \quad (2.9)$$

Note that  $\mathcal{Z}^{p,0} = \mathcal{O}^p$ , hence from (2.9) we get

$$\boxed{\check{H}^1(M, \mathcal{O}^p) \cong H_{\bar{\partial}}^{p,1}(M)}$$

Next we consider the remaining parts of the long exact sequence, i.e. for  $q \geq 1$  and  $\ell \geq 0$  we have

$$0 \longrightarrow \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

The group homomorphism  $\Delta$  is an isomorphism since this is an exact sequence of abelian groups

$$\check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \cong \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \quad \text{for all } q \geq 1, \ell \geq 0 \quad (2.10)$$

Again substituting  $\mathcal{Z}^{p,0} = \mathcal{O}^p$  and restricting our attention to  $q \geq 2$ , we apply (2.10) recursively to get

$$\begin{aligned} \check{H}^q(M, \mathcal{O}^p) &\cong \check{H}^{q-1}(M, \mathcal{Z}^{p,1}) \\ &\cong \check{H}^{q-2}(M, \mathcal{Z}^{p,2}) \\ &\vdots \\ &\cong \check{H}^1(M, \mathcal{Z}^{p,q-1}) \end{aligned}$$

Then using (2.9) we get

$$\boxed{\check{H}^q(M, \mathcal{O}^p) \cong H_{\bar{\partial}}^{p,q}(M)} \quad \text{for all } q \geq 2$$

Hence completing the proof.  $\square$

### 2.3.4 Solution of the problem

We can now solve the Cousin problem, following the solution outlined in [8, p. 47].

**Lemma 2.3.**  $\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$  for  $q > 0$ .

*Proof.* Consider the long exact sequence associated to the exponential sheaf sequence on  $\mathbb{C}^n$

$$\dots \longrightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{H}^{q+1}(\mathbb{C}^n, \mathbb{Z}) \longrightarrow \check{H}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \dots$$

By the  $\bar{\partial}$ -Poincaré lemma (Theorem 1.3), we get  $H_{\bar{\partial}}^{p,q}(\mathbb{C}^n) = 0$  for all  $p \geq 0$  and  $q > 0$ . Then using Dolbeault isomorphism (Theorem 2.8) for  $p = 0$ , we get  $\check{H}^q(\mathbb{C}^n, \mathcal{O}) = 0$  for  $q > 0$ . Moreover, since  $\mathbb{C}^n$  is contractible, we can use Corollary 2.2 to get  $\check{H}^q(\mathbb{C}^n, \mathbb{Z}) = 0$  for  $q > 0$ . Substituting these in the sequence and using exactness, we conclude that  $\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$  for  $q > 0$ .  $\square$

**Theorem 2.9.** Any analytic hypersurface in  $\mathbb{C}^n$  is the zero locus of an entire function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ .

*Proof.* Let  $H$  be the analytic hypersurface in  $\mathbb{C}^n$ , then  $H \subset \mathbb{C}^n$  such that for every point  $w \in \mathbb{C}^n$  there exists an open neighborhood  $w \in U \subset \mathbb{C}^n$  and  $f \in \mathcal{O}(U)$  with

$$U \cap H = \{z \in U : h(z) = 0\}$$

By Theorem B.7 we know that  $\mathcal{O}_w$  is a unique factorization domain. Therefore, if  $h$  is a representative element of the equivalence classes in  $\mathcal{O}_w$ , then  $h = h_1 \cdots h_k$  for some irreducible representative functions in  $\mathcal{O}_w$ . Hence we can choose  $h$  such that it is not divisible by the square of any non-unit<sup>5</sup> in  $\mathcal{O}_w$ .

Next, choose an open cover  $\mathcal{U} = \{U_\alpha\}$  of  $\mathbb{C}^n$  and functions  $h_\alpha \in \mathcal{O}(U_\alpha)$  such that

$$U_\alpha \cap H = \{z \in U : h_\alpha(z) = 0\}$$

where  $h_\alpha$  is not divisible by the square of any non-unit. We can then define the Cousin data for the cover  $\mathcal{U} = \{U_\alpha\}$  by setting

$$g_{\alpha\beta} = \frac{h_\beta}{h_\alpha} \quad \text{on } U_\alpha \cap U_\beta \quad (2.11)$$

Note that for each  $U_\alpha, U_\beta$  with nonempty intersection  $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$  since<sup>6</sup>  $h_\alpha$  and  $h_\beta$  vanish at the same points in  $U_\alpha \cap U_\beta$ , and  $\{g_{\alpha\beta}\}$  satisfies the conditions

1.  $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$  for each pair  $(\alpha, \beta)$ ;
2.  $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  for each triple  $(\alpha, \beta, \gamma)$ .

Therefore,  $(g_{\alpha\beta}) \in \check{Z}^1(\mathcal{U}, \mathcal{O}^*)$ . But since  $\check{H}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$  by Lemma 2.3, after some refinement of  $\mathcal{U}$  if necessary<sup>7</sup>, there exists a cochain  $(f_\alpha) \in \check{C}^0(\mathcal{U}, \mathcal{O}^*)$  such that

$$g_{\alpha\beta} = \delta(f_\alpha) = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta \quad (2.12)$$

<sup>5</sup>Recall that the non-vanishing functions at  $w \in \mathbb{C}^n$  are the unit elements in  $\mathcal{O}_w$ .

<sup>6</sup>We can prove this by contradiction. On the contrary assume that there exists  $z \in U_\alpha \cap U_\beta$  such that  $h_\alpha(z) = 0$  but  $h_\beta(z) \neq 0$ . Then  $z \in U_\alpha \cap H$  but  $z \notin U_\beta \cap H$ . Which contradicts our assumption that  $z \in U_\alpha \cap U_\beta$ .

<sup>7</sup>If we need a refinement  $\mathcal{V}$  of  $\mathcal{U}$ , then just start whole argument with the open cover  $\mathcal{V}$  instead of  $\mathcal{U}$ .

Comparing (2.11) and (2.12) we get that

$$\frac{h_\beta}{h_\alpha} = \frac{f_\beta}{f_\alpha} \quad \text{on } U_\alpha \cap U_\beta$$

for each pair  $(\alpha, \beta)$ . Hence, we can define a global holomorphic function  $f$  on  $\mathbb{C}^n$  such that

$$f(z) = \frac{h_\alpha(z)}{f_\alpha(z)} \quad \text{for } z \in U_\alpha$$

for each  $\alpha$ . Since dividing  $h_\alpha$  by a non-vanishing holomorphic function  $f_\alpha$  doesn't affect the vanishing set of  $h_\alpha$ ,  $f$  is the desired holomorphic function on  $\mathbb{C}^n$  whose vanishing set is  $H$ .  $\square$

# Future work

In the [Remark 1.26](#) and [Remark 1.33](#) we noted that transition maps can be used to define vector bundles. Following is the more precise statement:

**Theorem 1.** *If  $M$  be a smooth manifold and  $\pi : E \rightarrow M$  is a complex<sup>8</sup> vector bundle of rank  $k$ . Then there exists an open cover  $\{U_\alpha\}$  of  $M$  and a collection of smooth transition maps  $\{\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{C})\}$  such that*

1.  $\sigma_{\alpha\alpha} = I_k$

2.  $\sigma_{\alpha\beta} \cdot \sigma_{\beta\gamma} \cdot \sigma_{\gamma\alpha} = I_k$

where  $I_k$  is a  $k \times k$  identity matrix. This collection  $\{\sigma_{\alpha\beta}\}$  is called transition data. Conversely, given an open cover  $\{U_\alpha\}$  of  $M$  and a collection of smooth maps  $\{\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{C})\}$  satisfying the above two conditions, there exists a complex rank  $k$  vector bundle  $\pi' : E' \rightarrow M$  whose transition data is given by  $\{\sigma_{\alpha\beta}\}$ . Moreover, these two processes are well-defined and are inverses of each other when applied to the set of equivalence classes of vector bundles<sup>9</sup> and the set of equivalence classes of transition data<sup>10</sup>.

Now, if we use this result to define vector bundles using transition data, then we get the following [\[24, Lemma III.4.4\]](#):

**Theorem 2.** *There is one-to-one correspondence between the equivalence classes of holomorphic line bundles on a complex manifold  $M$  and the elements of the cohomology group  $\check{H}^1(M, \mathcal{O}^*)$  where  $\mathcal{O}^*$  is the sheaf of non-vanishing holomorphic functions.*

Also, by considering the underlying complex vector bundle of rank 1, we get:

**Theorem 3.** *There is one-to-one correspondence between the equivalence classes of complex line bundles on a smooth manifold  $M$  and the elements of the cohomology group  $\check{H}^1(M, \mathcal{E}^*)$  where  $\mathcal{E}^*$  is the sheaf of non-vanishing smooth functions.*

We can generalize this result by generalizing the definition of Čech cohomology. In the previous report we defined Čech cohomology for a sheaf of abelian groups [\[12, Definition 2.15\]](#). Note that we can't define Čech cohomology in a similar way if  $\mathcal{F}$  is a sheaf of non-abelian groups, since  $\delta \circ \delta \neq 0$  if the sheaf is not abelian. However, we have the following general definition of the zeroth and first Čech cohomology [\[16, Remark 5.5\(2\)\]](#):

- (a).  $\check{H}^0(X, \mathcal{F}) := \mathcal{F}(X)$

---

<sup>8</sup>Same argument is valid for smooth and holomorphic vector bundles. For the case of smooth vector bundles, replace  $\mathbb{C}$  by  $\mathbb{R}$ , and for the case of holomorphic vector bundles consider holomorphic transition maps and holomorphic isomorphism of vector bundles.

<sup>9</sup>Two vector bundles over  $M$  are said to be equivalent if they are isomorphic as vector bundles over  $M$ .

<sup>10</sup>Two sets of transition data  $\{\sigma_{\alpha\beta}\}$  and  $\{\sigma'_{\alpha\beta}\}$  are said to be equivalent if there exists a collection of smooth functions  $\{\rho_\alpha : U_\alpha \rightarrow GL(k, \mathbb{C})\}$  such that  $\sigma'_{\alpha\beta} = \rho_\alpha \cdot \sigma_{\alpha\beta} \cdot \rho_\beta^{-1}$  for all  $\alpha, \beta$ .



- (b).  $\check{H}^1(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F})$  where the direct limit is indexed over all the open covers of  $X$  with order relation induced by refinement, i.e.  $\mathcal{U} < \mathcal{V}$  if  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ , and  $\check{H}^1(\mathcal{U}, \mathcal{F})$  is a pointed set<sup>11</sup> defined as

$$\check{H}^1(\mathcal{U}, \mathcal{F}) := \ker\{\delta : \check{C}^1(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^2(\mathcal{U}, \mathcal{F})\} / \sim$$

$$(g_{\alpha\beta}) \sim (h_{\alpha\beta}) \Leftrightarrow \exists (f_\alpha) \in \check{C}^0(\mathcal{U}, \mathcal{F}) \text{ such that } f_\alpha * g_{\alpha\beta} = h_{\alpha\beta} * f_\beta \text{ on } U_\alpha \cap U_\beta$$

with  $*$  being the group operation. Therefore,  $\check{H}^1(X, \mathcal{F})$  is a group if and only if  $\mathcal{F}$  is an abelian sheaf.

Using this new definition we get the following more general correspondence between vector bundles and Čech cohomology [25, §24]:

**Theorem 4.** *Let  $M$  be a smooth manifold, then*

- (a). *there is one-to-one correspondence between the equivalence classes of rank  $k$  smooth vector bundles over  $M$  and the elements of the first cohomology set  $\check{H}^1(M, O(k))$  where  $O(k)$  is the sheaf of smooth functions to the Lie group  $O(k)$  of orthogonal matrices.*
- (b). *there is one-to-one correspondence between the equivalence classes of rank  $k$  complex vector bundles over  $M$  and the elements of the first cohomology set  $\check{H}^1(M, U(k))$  where  $U(k)$  is the sheaf of smooth functions to the Lie group  $U(k)$  of unitary matrices.*

Clearly this is a generalization of the previous result, since for  $k = 1$  we get  $\check{H}^1(M, U(1)) = \check{H}^1(M, S^1) = \check{H}^1(M, \mathcal{E}^*)$ .

**Definition** (Picard group). The set of isomorphic classes of line bundles on a manifold  $M$  form a group under the tensor product<sup>12</sup> operation, where the inverse of a line bundle is its dual bundle<sup>13</sup>. This group of isomorphism classes of holomorphic line bundles on  $M$  is called the *Picard group* of  $M$ , denoted by  $\text{Pic}(M)$ .

In fact, the one-to-one correspondence that we get in **Theorem 2** is a group isomorphism, i.e.  $\text{Pic}(M) \cong \check{H}^1(M, \mathcal{O}^*)$  [8, p. 133]. This enables us to define the first Chern class of holomorphic line bundles as follows [10, Definition 2.2.13]:

**Definition** (First Chern class of holomorphic line bundle). The exponential sheaf sequence on a complex manifold  $M$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

gives a long exact sequence in cohomology

$$\dots \longrightarrow \check{H}^q(M, \mathcal{O}) \longrightarrow \check{H}^q(M, \mathcal{O}^*) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathbb{Z}) \longrightarrow \check{H}^{q+1}(M, \mathcal{O}) \longrightarrow \dots$$

Therefore, we have the connecting homomorphism

$$\begin{aligned} \Delta : \check{H}^1(M, \mathcal{O}^*) &\rightarrow \check{H}^2(M, \mathbb{Z}) \\ [L] &\mapsto c_1(L) \end{aligned}$$

where  $c_1(L)$  is called the *first Chern class* of the holomorphic line bundle  $L$  on  $M$ .

<sup>11</sup>For details regarding its construction, refer to the lecture notes by Zinger [25, §24].

<sup>12</sup>If  $\pi : L \rightarrow M$  and  $\pi' : L' \rightarrow M$  are smooth line bundles, then their tensor product,  $L \otimes L'$  is defined such that  $(L \otimes L')_w = L_w \otimes L'_w$  for all  $w \in M$  [25, §13].

<sup>13</sup>If  $\pi : L \rightarrow M$  is a smooth line bundles of rank  $k$ , the dual bundle of  $L^*$  is a line bundle  $L^* \rightarrow M$  such that  $(L^*)_w = L_w^* = \text{Hom}_{\mathbb{R}}(L_w, \mathbb{R})$  for all  $w \in M$ . For complex and holomorphic line bundles, replace  $\mathbb{R}$  by  $\mathbb{C}$  [25, §12].

The immediate consequences of this definition are [8, p. 139]:

$$c_1(L \otimes L') = c_1(L) + c_1(L') \quad \text{and} \quad c_1(L^*) = -c_1(L)$$

Note that, in the previous report we only proved the existence of connecting homomorphism  $\Delta$  [12, Theorem 2.1]. However, to be able to calculate the first Chern class of a holomorphic line bundle we must know the exact definition of  $\Delta$ , which turns out to be a challenging task [24, p. 104].

Similarly, the one-to-one correspondence that we get in [Theorem 3](#) is a group isomorphism. This enables us to define the first Chern class of complex line bundles as follows [24, p. 105]:

**Definition** (First Chern class of complex line bundle). The exponential sheaf sequence on a smooth manifold  $M$

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{E} \xrightarrow{\exp} \mathcal{E}^* \longrightarrow 0$$

gives a long exact sequence in cohomology

$$\dots \longrightarrow \check{H}^q(M, \mathcal{E}) \longrightarrow \check{H}^q(M, \mathcal{E}^*) \xrightarrow{\Delta} \check{H}^{q+1}(M, \underline{\mathbb{Z}}) \longrightarrow \check{H}^{q+1}(M, \mathcal{E}) \longrightarrow \dots$$

Therefore, we have the connecting homomorphism

$$\begin{aligned} \Delta : \check{H}^1(M, \mathcal{E}^*) &\rightarrow \check{H}^2(M, \underline{\mathbb{Z}}) \\ [L] &\mapsto c_1(L) \end{aligned}$$

where  $c_1(L)$  is called the *first Chern class* of the complex line bundle  $L$  on  $M$ .

Since  $\mathcal{E}$  is a fine sheaf, by [12, Theorem 2.2],  $\check{H}^k(M, \mathcal{E}) = 0$  for  $k > 0$ . Therefore, the connecting homomorphism  $\Delta : \check{H}^1(M, \mathcal{E}^*) \rightarrow \check{H}^2(M, \underline{\mathbb{Z}})$  is a group isomorphism, and the equivalence classes of complex line bundles are determined by their first Chern class in  $\check{H}^2(M, \underline{\mathbb{Z}})$  [8, p. 140].

**Theorem 5.** *There is a natural group isomorphism between the equivalence classes of complex line bundles on a smooth manifold  $M$  and the elements of the cohomology group  $\check{H}^2(M, \underline{\mathbb{Z}})$ . That is, a complex line bundle is determined upto smooth vector bundle isomorphism by its first Chern class.*

In the previous report we proved that  $H_{dR}^k(M) \cong \check{H}^k(M, \underline{\mathbb{R}})$  for  $k \geq 0$  [12, Theorem 3.1]. Also note that there is a natural homomorphism  $j : \check{H}^2(M, \underline{\mathbb{Z}}) \rightarrow \check{H}^2(M, \underline{\mathbb{R}})$  induced by the inclusion of constant sheaves  $\underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{R}}$ . Combining these with the fact that  $\check{H}^1(M, \mathcal{E}^*) \cong \check{H}^2(M, \underline{\mathbb{Z}})$ , we can compute the Chern classes of complex line bundles using differential forms [24, Theorem III.4.5].

$$\begin{aligned} c_1 : \{\text{isomorphism classes of complex line bundles over } M\} &\rightarrow H_{dR}^2(M) \\ [L] &\mapsto c_1(L) \end{aligned}$$

Since a complex vector bundle  $L$  of rank 1 over a smooth manifold  $M$  can be thought of as a smooth vector bundle  $L$  of rank 2 over  $M$ , we can use the following result for computing the first Chern class of a complex line bundle [1, pp. 71-73]:

**Theorem 6.** *Let  $\pi : L \rightarrow M$  be an oriented smooth oriented rank 2 vector bundle over  $M$ , and  $\{U_\alpha\}$  be a coordinate open cover of  $M$  that trivializes  $E$ . If  $\{\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow SO(2)\}$  are the transition functions<sup>14</sup> of  $L$  and  $\{\eta_\gamma\}$  is a partition of unity of  $M$  subordinate to  $\{U_\gamma\}$ , then*

$$c_1(L) = -\frac{1}{2\pi i} \sum_{\gamma} d(\eta_\gamma d \log(\sigma_{\gamma\alpha})) \quad \text{on } U_\alpha \text{ for each } \alpha$$

where  $\sigma_{\alpha\beta}$  are thought of as complex valued functions by identifying  $SO(2)$  with  $S^1$  via  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = e^{i\theta}$  and  $c_1(L)$  is a closed form representing a cohomology class in  $H_{dR}^2(M)$ .

<sup>14</sup>The structure group of every smooth rank  $k$  vector bundle  $\pi : E \rightarrow M$  can be reduced to the orthogonal group  $O(k)$  using Gram-Schmidt process. This is also a key step of the proof of [Theorem 4\(a\)](#). Moreover, if the vector bundle is orientable then the structure group can be further reduced to  $SO(k)$  [1, Proposition 6.4].

# Appendix A

## Algebra

### A.1 Complexification of vector space

In this section some definitions and facts from [17, Chapter 14] will be stated.

**Definition A.1** (Tensor product of vector spaces). Let  $U$  and  $V$  be vector spaces over a field  $F$ . The tensor product  $U \otimes_F V$  is a vector space over  $F$  equipped with a bilinear map  $f : U \times V \rightarrow U \otimes_F V$  such that for each bilinear map from  $U \times V$  to any vector space  $W$  over  $F$  there is a unique linear map  $h : U \otimes_F V \rightarrow W$  making the following diagram commute.

$$\begin{array}{ccc} U \times V & \xrightarrow{f} & U \otimes_F V \\ & \searrow g & \downarrow h \\ & & W \end{array}$$

**Remark A.1.** We use the symbol  $\otimes$  to denote the image of any ordered pair  $(u, v)$  under the tensor map, i.e.  $u \otimes v = f(u, v)$  for any  $u \in U$  and  $v \in V$ . Not all members of  $U \otimes_F V$  are of this form. In general, if  $\{u_i : i \in I\}$  is a basis for  $U$  and  $\{v_j : j \in J\}$  is a basis for  $V$ , then any vector  $w \in U \otimes_F V$  has a unique expression as a sum

$$w = \sum_{i \in I} \sum_{j \in J} r_{i,j} (u_i \otimes v_j)$$

where only a finite number of the coefficients  $r_{i,j}$  are non-zero.

**Proposition A.1.** For finite dimensional vector spaces  $U$  and  $V$  over a field  $F$

$$\dim_F(U \otimes_F V) = \dim_F(U) \dim_F(V)$$

**Proposition A.2** (Bilinearity on  $U \times V$  equals linearity on  $U \otimes_F V$ ). Let  $U, V$  and  $W$  be vector spaces over a field  $F$ . Let  $\text{Hom}_F(U, V; W)$  be the set of all bilinear maps from  $U \times V$  to  $W$ , and  $\text{Hom}_F(U \otimes_F V; W)$  be the set of all linear maps from  $U \otimes_F V$  to  $W$ . Then the mediating map

$$\begin{aligned} \phi : \text{Hom}_F(U, V; W) &\rightarrow \text{Hom}_F(U \otimes_F V; W) \\ g &\mapsto h \end{aligned}$$

where  $h$  is the unique linear map satisfying  $g = h \circ f$  for the tensor map  $f : U \times V \rightarrow U \otimes_F V$ , is an isomorphism.

**Proposition A.3** (Linear functionals on tensor product). Let  $U$  and  $V$  be finite dimensional vector spaces over a field  $F$ . Then the linear transformation

$$\psi : U^* \otimes_F V^* \rightarrow (U \otimes_F V)^*$$

defined by  $\psi(f \otimes g)(u \otimes v) = f(u)g(v)$ , is an isomorphism. Thus, the tensor product of linear functionals is a linear functional on tensor products.

**Corollary A.1.** For a finite dimensional vector spaces  $U$  and  $V$  over a field  $F$ , we have

$$U^* \otimes_F V^* \cong \text{Hom}_F(U, V; F)$$

*Proof.* From [Proposition A.3](#) we know that  $U^* \otimes_F V^* \cong (U \otimes_F V)^*$ . Note that  $(U \otimes_F V)^* = \text{Hom}_F(U \otimes_F V; F)$ , hence we can use [Proposition A.2](#) to conclude that  $U^* \otimes_F V^* \cong (U \otimes_F V)^* \cong \text{Hom}_F(U, V; F)$   $\square$

**Theorem A.1** (Extending the base field). Let  $V$  be vector space over a field  $F$  and  $K$  be a finite extension of  $F$ . Then  $W = V \otimes_F K$  is a vector space over  $K$  such  $\dim_K(W) = \dim_F(V)$ . Moreover, if  $W_F$  is the vector space obtained by restricting the the scalar multiplication for  $W$  to scalars from  $F$ , then  $W_F$  contains an isomorphic copy of  $V$ .

*Proof.* Since  $K$  is a vector space over  $F$ , we can form the tensor product

$$W_F = V \otimes_F K$$

where all relevant maps are  $F$ -bilinear and  $F$ -linear. By definition of tensor product  $W_F$  is a vector space over  $F$ . However, since  $V$  is not a  $K$ -space, we can't have a  $K$ -tensor product. We just need to show that  $W_F$  can be made into a vector space over  $K$ .

**Claim:** For  $\alpha \in K$ , the scalar multiplication operation  $\alpha(v \otimes \beta) = v \otimes (\alpha\beta)$  is well defined.

To prove the claim, we need to check that

$$v \otimes \beta = w \otimes \gamma \quad \Rightarrow \quad v \otimes (\alpha\beta) = w \otimes (\alpha\gamma)$$

Note that for a fixed  $\alpha$ , the map

$$\begin{aligned} g : V \times K &\rightarrow V \otimes_F K \\ (v, \beta) &\mapsto v \otimes (\alpha\beta) \end{aligned}$$

is  $F$ -bilinear. Now the definition of tensor product implies that there exists a unique  $F$ -linear map

$$\begin{aligned} h : V \otimes_F K &\rightarrow V \otimes_F K \\ v \otimes \beta &\mapsto v \otimes (\alpha\beta) \end{aligned}$$

since the following diagram commutes

$$\begin{array}{ccc} V \times K & \xrightarrow{f} & V \otimes_F K \\ & \searrow g & \downarrow h \\ & & V \otimes_F K \end{array}$$

We define this map  $h$  to be scalar multiplication by  $\alpha$ , under which  $W = V \otimes_F K$  is a vector space over the field  $K$ . Note that  $W_F$  and  $W$  are identical as sets and as abelian groups, only the scalar multiplication operation is different. Moreover, we recover  $W_F$  from  $W$  simply by restricting scalar multiplication to scalars from  $F$ .

If  $K$  is a degree  $d$  field extension of  $F$ , then using [Proposition A.1](#) we get

$$\dim_F(W_F) = \dim_F(V \otimes_F K) = \dim_F(V) \cdot d$$

Hence, if  $\{v_i : i \in I\}$  is a basis for  $V$ , then  $\{v_i \otimes 1\}$  is a basis for  $W$ , that is,

$$\dim_K(W) = \dim_F(V)$$

The map  $\mu : V \rightarrow W_F$  defined by  $\mu(v) = v \otimes 1$  is an injective  $F$ -linear map, so  $W_F$  contains an isomorphic copy of  $V$ .  $\square$

**Remark A.2.** We can also think of  $\mu$  as mapping of  $V$  into  $W$ , in which case  $\mu$  is called the  $K$ -extension map of  $V$ .

**Theorem A.2** (Extending the linear map). *Let  $U$  and  $V$  be two vector spaces over the field  $F$ , with  $K$ -extension maps  $\mu_U$  and  $\mu_V$ , respectively. Then for any  $F$ -linear map  $\tau : U \rightarrow V$ , the map  $\tau \otimes \mathbf{1}_K : U \otimes_F K \rightarrow V \otimes_F K$  is the unique  $K$ -linear map that makes the following diagram commute*

$$\begin{array}{ccc} U & \xrightarrow{\tau} & V \\ \downarrow \mu_U & & \downarrow \mu_V \\ U \otimes_F K & \xrightarrow{\tau \otimes \mathbf{1}_K} & V \otimes_F K \end{array}$$

Thus,  $\tau \otimes \mathbf{1}_K$  is the extension of the  $F$ -linear map  $\tau$  to a  $K$ -linear map.

**Definition A.2** (Complexification of a real vector space). To each real vector space  $V$ , we can associate a complex vector space  $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$  called the *complexification of  $V$* .

**Proposition A.4.** *Let  $V$  be a real vector space, and  $\tilde{V} = V \oplus V$  be a complex vector space with multiplication law  $(a + ib)(v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$ . Then there is a unique isomorphism  $\phi : \tilde{V} \rightarrow V_{\mathbb{C}}$  of  $\mathbb{C}$ -vector spaces which makes the diagram*

$$\begin{array}{ccc} & V & \\ \swarrow & & \searrow \mu \\ \tilde{V} & \xrightarrow{\phi} & V_{\mathbb{C}} \end{array}$$

commute. Explicitly,

$$\phi(v_1, v_2) = v_1 \otimes 1 + v_2 \otimes i$$

*Proof.* Firstly we will verify that  $\phi$  is  $\mathbb{C}$ -linear

$$\begin{aligned} \phi((a + ib)(v_1, v_2)) &= \phi(av_1 - bv_2, bv_1 + av_2) \\ &= (av_1 - bv_2) \otimes 1 + (bv_1 + av_2) \otimes i \\ &= a(v_1 \otimes 1) - b(v_2 \otimes 1) + b(v_1 \otimes i) + a(v_2 \otimes i) \\ &= a(v_1 \otimes 1) + ib(v_2 \otimes i) + ib(v_1 \otimes 1) + a(v_2 \otimes i) \\ &= a(v_1 \otimes 1 + v_2 \otimes i) + ib(v_2 \otimes i + v_1 \otimes 1) \\ &= (a + ib) \phi(v_1, v_2) \end{aligned}$$

To show that  $\phi$  is an isomorphism, we will write down the inverse map:

$$\begin{aligned} \psi : V_{\mathbb{C}} &\rightarrow \tilde{V} \\ v \otimes \alpha &\mapsto \alpha(v, 0) \end{aligned}$$

which is extended by linearity. Using the definition of scalar multiplication for  $V_{\mathbb{C}}$  we verify that  $\psi$  is  $\mathbb{C}$ -linear. Let  $\beta \in \mathbb{C}$  then

$$\begin{aligned} \psi(\beta(v \otimes \alpha)) &= \psi(v \otimes \beta\alpha) \\ &= \beta\alpha(v, 0) \\ &= \beta \psi(v \otimes \alpha) \end{aligned}$$

Finally, we show that  $\phi$  and  $\psi$  are inverse of each other:

$$\begin{aligned} \psi(\phi(v_1, v_2)) &= \psi(v_1 \otimes 1 + v_2 \otimes i) = (v_1, 0) + i(v_2, 0) = (v_1, 0) + (0, v_2) = (v_1, v_2) \\ \phi(\psi(v \otimes \alpha)) &= \phi(\alpha(v, 0)) = \alpha\phi(v, 0) = \alpha(v \otimes 1) = v \otimes \alpha \end{aligned}$$

Note that it suffices to verify  $\phi \circ \psi = \mathbf{1}_{V_{\mathbb{C}}}$  for elementary tensors. □

**Proposition A.5.** *The complexification of the dual space  $V^*$  of a real vector space  $V$  is naturally isomorphic to the space of all  $\mathbb{R}$ -linear maps from  $V$  to  $\mathbb{C}$ . That is,  $(V^*)_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ .*

*Proof.* The isomorphism is given by

$$\begin{aligned} \Phi : (V^*)_{\mathbb{C}} &\rightarrow \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \\ \varphi_1 \otimes 1 + \varphi_2 \otimes i &\mapsto \varphi_1 + i\varphi_2 \end{aligned}$$

where  $\varphi_1$  and  $\varphi_2$  are elements of  $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ .  $\square$

**Corollary A.2.** *The complexification of the dual space  $V^*$  of a real vector space  $V$  is naturally isomorphic to the dual of the dual space of  $V_{\mathbb{C}}$ . That is,  $(V^*)_{\mathbb{C}} = (V_{\mathbb{C}})^*$ .*

*Proof.* Given a  $\mathbb{R}$ -linear map  $\varphi : V \rightarrow \mathbb{C}$ , we can extend by linearity to obtain a  $\mathbb{C}$ -linear map

$$\begin{aligned} \tilde{\varphi} : V_{\mathbb{C}} &\rightarrow \mathbb{C} \\ v \otimes \alpha &\mapsto \alpha\varphi(v) \end{aligned}$$

This extension gives an isomorphism from  $\text{Hom}_{\mathbb{R}}(V, \mathbb{C})$  to  $\text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ . The latter is just the complex dual space to  $V_{\mathbb{C}}$ , hence giving the isomorphism  $(V^*)_{\mathbb{C}} \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$ .  $\square$

**Remark A.3.** More generally, given real vector spaces  $V$  and  $W$  there is a natural isomorphism

$$\text{Hom}_{\mathbb{R}}(V, W)_{\mathbb{C}} \cong \text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$$

**Proposition A.6.** *Complexification commutes with the operations of taking tensor products. That is, if  $V$  and  $W$  are real vector spaces then there is a natural isomorphism  $(V \otimes_{\mathbb{R}} W)_{\mathbb{C}} \cong V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$ , where the left-hand tensor product is taken over  $\mathbb{R}$  while the right-hand one is taken over  $\mathbb{C}$ .*

## A.2 Linear complex structure

In this section some definitions and facts from [24, §I.3] and [10, §1.2] will be stated.

**Definition A.3** (Complex structure). Let  $V$  be a real vector space and suppose that  $J$  is an  $\mathbb{R}$ -linear endomorphism  $J : V \rightarrow V$  such that  $J^2 = -\mathbf{1}_V$ . Then  $J$  is called a *complex structure* on  $V$ .

**Lemma A.1.** *If  $J$  is a complex structure on a real vector space  $V$ , then  $V$  admits in a natural way the structure of a complex vector space.*

*Proof.* We can equip  $V$  with the structure of a complex vector space in the following manner:

$$(\alpha + i\beta)v := \alpha v + \beta J(v), \quad \alpha, \beta \in \mathbb{R}, i = \sqrt{-1}$$

Thus scalar multiplication on  $V$  by complex numbers is well defined, and  $V$  becomes a complex vector space.  $\square$

**Lemma A.2.** *If  $V$  is a complex vector space, then we can define a complex structure  $J$  on  $V$  when it is considered as a real vector space.*

*Proof.* Since  $V$  is a complex vector space and  $\mathbb{R} \subset \mathbb{C}$ , it can also be considered as a vector space over  $\mathbb{R}$ , and the operation of multiplication by  $i = \sqrt{-1}$  is an  $\mathbb{R}$ -linear endomorphism of  $V$  onto itself, which we can call  $J$ ,

$$\begin{aligned} J : V &\rightarrow V \\ v &\mapsto iv \end{aligned}$$

This is a complex structure.  $\square$



*Proof.* Note that  $\tilde{J} = J \otimes \mathbf{1}_{\mathbb{C}}$  is the  $\mathbb{C}$ -linear extension of the  $\mathbb{R}$ -linear map  $J$ , which still has the property that  $\tilde{J}^2 = -\mathbf{1}_{V_{\mathbb{C}}}$ . It follows that  $\tilde{J}$  has two eigenvalues  $\{i, -i\}$ . Also,  $V^{1,0}$  is the eigenspace corresponding to the eigenvalue  $i$  and  $V^{0,1}$  is the eigenspace corresponding to  $-i$ . Since the minimal polynomial  $p(t) = t^2 + 1$  of  $\tilde{J}$  is product of distinct linear factors,  $\tilde{J}$  is diagonalizable [17, Theorem 8.11]. Hence  $V_{\mathbb{C}}$  is the direct sum of eigenspaces corresponding to the distinct eigenvalues [17, Theorem 8.10].

In particular, every vector  $w$  of  $V_{\mathbb{C}}$  can be written as :

$$w = \frac{w - i\tilde{J}(w)}{2} + \frac{w + i\tilde{J}(w)}{2}$$

where  $(w - i\tilde{J}(w))/2$  is an eigenvector with eigenvalue  $i$  while  $(w + i\tilde{J}(w))/2$  is an eigenvector with eigenvalue  $-i$ . Note that

$$\overline{\left(\frac{w - i\tilde{J}(w)}{2}\right)} = \frac{\bar{w} + i\tilde{J}(\bar{w})}{2}$$

Hence, complex conjugation interchanges the two factors, and induces  $\mathbb{R}$ -linear isomorphism  $V^{1,0} \cong V^{0,1}$ .  $\square$

**Remark A.7.** Note that the complex vector space obtained from  $V$  by means of the complex structure  $J$ , denoted by  $V_J$ , is  $\mathbb{C}$ -linear isomorphic to  $V^{1,0}$ . Hence we can identify  $V_J$  with  $V^{1,0}$ .

**Proposition A.8.** *Let  $V$  be a real vector space endowed with a complex structure  $J$ . Then the dual space  $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$  has a natural complex structure given by  $\mathcal{J}(f)(v) = f(J(v))$  for all  $f \in V^*$  and  $v \in V$ . The induced decomposition on  $(V^*)_{\mathbb{C}} \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$  is given by*

$$(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

where

$$\begin{aligned} (V^*)^{1,0} &\cong \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(J(v)) = if(v)\} \cong (V^{1,0})^* \\ (V^*)^{0,1} &\cong \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(J(v)) = -if(v)\} \cong (V^{0,1})^* \end{aligned}$$

### A.3 Multilinear algebra

By replacing bilinearity with multilinearity in **Definition A.1**, we can extend the definition of tensor product to more than two vector spaces. In this section some facts about tensor spaces will be stated from [17, Chapter 14] and [10, §1.2]. Unlike the rest of the report, here the letter  $T$  denote ‘‘tensor’’ space instead of ‘‘tangent’’ space.

**Definition A.5** ( $(p, q)$ -tensor). Let  $V$  be a finite dimensional vector space over a field  $F$ . For non-negative integers  $p$  and  $q$ , the tensor product

$$T_q^p(V) = \underbrace{V \otimes_F \cdots \otimes_F V}_{p \text{ factors}} \otimes_F \underbrace{V^* \otimes_F \cdots \otimes_F V^*}_{q \text{ factors}} = V^{\otimes p} \otimes (V^*)^{\otimes q}$$

where  $V^{\otimes k}$  is  $k$ -fold tensor product of  $V$  with itself, is called the space of tensors of type  $(p, q)$ , where  $p$  is the *contravariant type* and  $q$  is the *covariant type*. If  $p = q = 0$ , then  $T_q^p(V) = F$ .

**Remark A.8.** For a finite dimensional vector space  $V$  over a field  $F$ , we have  $V \cong V^{**}$ , hence we can generalize **Corollary A.1** to get:

$$T_q^p(V) = V^{\otimes p} \otimes_F (V^*)^{\otimes q} \cong ((V^*)^{\otimes p} \otimes_F V^{\otimes q})^* \cong \text{Hom}_F((V^*)^{\times p} \times V^{\times q}, F)$$

where  $V^{\times k}$  is  $k$ -fold cartesian product of  $V$  with itself. Therefore, the  $k$ -tensor defined in the previous report [12, Definition 1.6, 1.7] is in fact a  $(0, k)$ -tensor, i.e. a vector belonging to  $(V^*)^{\otimes k}$ . In other words,  $T_k^0(V) = T_0^k(V^*) = \mathcal{L}^k(V)$  [12, Remark 1.4].



**Proposition A.9.** Let  $V$  be a finite dimensional vector space over a field  $F$ . Then

1.  $\dim_F(T_q^p(V)) = (\dim_F(V))^{p+q}$
2.  $T_q^p(V) \otimes T_s^r(V) \cong T_{q+s}^{p+r}(V)$

**Definition A.6** (Tensor algebra). The external direct sum

$$T(V) = \bigoplus_{p=0}^{\infty} T_0^p(V)$$

is a graded algebra, where  $T_0^p(V)$  are the elements of grade  $p$ . This graded algebra  $T(V)$  is called the *tensor algebra* over  $V$ .

**Remark A.9.** Since

$$T_q^0(V) = (V^*)^{\otimes q} = T_0^q(V^*)$$

there is no need to look separately at  $T_q^0(V)$ .

**Definition A.7** (Antisymmetric tensor). Let  $V$  be a finite dimensional vector space and  $\tau \in T_0^p(V)$ . For each  $\sigma \in S_p$ , we have the isomorphism on  $T_0^p(V)$  defined as

$$\begin{aligned} \lambda_p : T_0^p(V) &\rightarrow T_0^p(V) \\ x_1 \otimes \cdots \otimes x_p &\mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} \end{aligned}$$

which we extend by linearity. A tensor  $\tau \in T_0^p(V)$  is said to be *antisymmetric*  $(p, 0)$ -tensor if  $\lambda_\sigma(\tau) = (\text{sgn } \sigma)\tau$  for all permutations  $\sigma \in S_p$ .

**Remark A.10.** The set of all antisymmetric  $(p, 0)$ -tensors

$$\bigwedge^p(V) := \{\tau \in T_0^p(V) \mid \lambda_\sigma(\tau) = (\text{sgn } \sigma)\tau \text{ for all } \sigma \in S_p\}$$

is a subspace of  $T_0^p(V)$ , called the *antisymmetric tensor space* or *exterior product space* of degree  $(p, 0)$  over  $V$ .

**Remark A.11.** Note that if  $\text{char}(F) \neq 2$  then alternating and skew symmetric tensors are the same [17, pp. 391, 398]. Since we have  $F = \mathbb{R}$  or  $\mathbb{C}$ , the alternating  $k$ -tensor defined in the previous report [12, Definition 1.9] is in fact an antisymmetric  $(0, k)$ -tensor, i.e. a vector belonging to  $\bigwedge^k(V^*)$ . In other words,  $\bigwedge^k(V^*) = \mathcal{A}^k(V)$  [12, Remark 1.5, Definition 1.37]. Hence the definition and properties of wedge product (or exterior product) stated in the previous report [12, p. 6], like  $\dim_F(\bigwedge^p(V)) = \binom{n}{p}$  and  $\bigwedge^p(V) = 0$  for  $p > n$  where  $n = \dim_F(V)$ , hold here also.

**Definition A.8** (Antisymmetric tensor algebra). The graded algebra

$$\bigwedge(V) = \bigoplus_{p=0}^n \bigwedge^p(V)$$

where  $\dim_F(V) = n$ , is called *antisymmetric tensor algebra* or *exterior algebra* of  $V$ .

**Proposition A.10.** The exterior algebra of a direct sum is isomorphic to the tensor product of the exterior algebras. That is, if  $V$  and  $W$  are vector spaces over a field  $F$ , then

$$\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes_F \bigwedge(W)$$

This is a graded isomorphism; i.e.,

$$\bigwedge^k(V \oplus W) \cong \bigoplus_{p+q=k} \bigwedge^p(V) \otimes_F \bigwedge^q(W)$$

**Proposition A.11.** *Complexification commutes with the operations of taking exterior powers. That is, if  $V$  is a real vector space there is a natural isomorphism  $(\bigwedge_{\mathbb{R}}^p V)_{\mathbb{C}} \cong \bigwedge_{\mathbb{C}}^p (V_{\mathbb{C}})$ , where the left-hand exterior power is taken over  $\mathbb{R}$  while the right-hand one is taken over  $\mathbb{C}$ .*

**Remark A.12.** If  $V$  is endowed with a complex structure  $J$ , then we introduce the notation

$$\bigwedge^{p,q} V := \bigwedge^p (V^{1,0}) \otimes_{\mathbb{C}} \bigwedge^q (V^{0,1})$$

where  $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$  as shown in [Proposition A.7](#). Hence we have

$$\bigwedge^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \bigwedge^{p,q} V$$

**Definition A.9** (Natural projection). With respect to the direct sum decomposition of  $\bigwedge V_{\mathbb{C}} = \bigoplus_{k=0}^n \bigwedge^k V_{\mathbb{C}}$  one defines the *natural projections*

$$\Pi^k : \bigwedge V_{\mathbb{C}} \rightarrow \bigwedge^k V_{\mathbb{C}} \quad \text{and} \quad \Pi^{p,q} : \bigwedge V_{\mathbb{C}} \rightarrow \bigwedge^{p,q} V$$

**Remark A.13.** The operator  $\Pi^k$  does not depend on the complex structure  $J$ , but the operator  $\Pi^{p,q}$  certainly do.

# Appendix B

## Analysis

### B.1 Several variable holomorphic functions

In this section some definitions and facts from [9, §I.A], [10, §1.1] and [13, §1.2] will be stated.

**Definition B.1** (Open polydisc). An *open polydisc* or *open polycylinder* in  $\mathbb{C}^n$  is a subset  $\Delta(z; r) \subset \mathbb{C}^n$  of the form

$$\Delta(z; r) = \Delta(z_1, \dots, z_n; r_1, \dots, r_n) = \{w \in \mathbb{C}^n : |w_j - z_j| < r_j, 1 \leq j \leq n\}$$

**Definition B.2** (Closed polydisc). The closure of  $\Delta(z; r)$  is called the *closed polydisc* with center  $z$  and polyradius  $r$ , and is denoted by  $\bar{\Delta}(z; r)$ .

**Remark B.1.** The open polydiscs form a basis for the product topology on  $\mathbb{C}^n$ . Considered only as a topological space (or as a real vector space),  $\mathbb{C}^n$  is the same as  $\mathbb{R}^{2n}$ , the ordinary Euclidean space of  $2n$  dimensions.

**Definition B.3** (Several variable holomorphic function). A complex-valued function  $f$  defined on an open subset  $U \subset \mathbb{C}^n$  is called *holomorphic* in  $U$  if each point  $w = (w_1, \dots, w_n) \in U$  has an open neighborhood  $W, w \in W \subset U$ , such that the function  $f$  has a power series expansion

$$f(z) = f(z_1, \dots, z_n) = \sum_{j_1, \dots, j_n=0}^{\infty} a_{j_1 \dots j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

which converges for all  $z \in W$ .

**Remark B.2.** The set of all complex-valued functions holomorphic in  $U$  is denoted by  $\mathcal{O}(U)$ . Clearly, if  $f$  is holomorphic in  $U \subset \mathbb{C}^n$ , then  $f$  is smooth in  $U$ , i.e.  $f \in \mathcal{O}(U)$  implies that  $f \in C^\infty(U)$ .

**Proposition B.1.** *If a complex-valued function  $f$  is holomorphic in an open subset  $U \subset \mathbb{C}^n$ , then it is continuous in  $U$  and is holomorphic in each variable separately.*

*Proof.* The function  $f$  has a power series expansion of the form

$$f(z) = f(z_1, \dots, z_n) = \sum_{j_1, \dots, j_n=0}^{\infty} a_{j_1 \dots j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

which is absolutely uniformly convergent in all suitably small open polydiscs  $\Delta(w; r)$  [3, Theorem III.1.3]. Therefore, the function  $f$  is continuous in such polydiscs  $\Delta(w; r)$ , and hence any function holomorphic in  $U$  is also continuous in  $U$ . Moreover, the power series can be rearranged arbitrarily and will still represent the function  $f$ . In particular, if the coordinates

$z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$  are given any fixed values  $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ , then this power series can be rearranged as a convergent power series in the variable  $z_j$  alone, for  $z_j$  sufficiently close to  $w_j$  and each  $a_k$  sufficiently close to  $z_k$  for  $k = 1, \dots, j-1, j+1, \dots, n$ . Therefore, the function  $f$  is holomorphic in each variable separately throughout the domain in which it is analytic.  $\square$

**Definition B.4** (Complex partial differential operators). As in [Definition 1.4](#), we define the following two first-order linear partial differential operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

for  $z_j = x_j + iy_j$  and  $j = 1, \dots, n$ .

**Remark B.3.** The previous result implies that the operation  $\partial/\partial z_j$  is well-defined for each complex-valued holomorphic function. Therefore, when applied to holomorphic functions, the operator  $\partial/\partial z_j$  coincides with the ordinary complex derivative with respect to one of the variables  $z_j$ . For example,

$$\begin{aligned} \frac{\partial}{\partial z_j} z_j^n &= \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) (x_j + iy_j)^n \\ &= \frac{1}{2} \left( \frac{\partial}{\partial x_j} (x_j + iy_j)^n - i \frac{\partial}{\partial y_j} (x_j + iy_j)^n \right) \\ &= \frac{1}{2} (n(x_j + iy_j)^{n-1} - i \cdot n(x_j + iy_j)^{n-1}i) \\ &= n(x_j + iy_j)^{n-1} \\ &= nz_j^{n-1} \end{aligned}$$

**Proposition B.2** (Cauchy formula for polydisc). *Let  $w \in \mathbb{C}^n$  and  $f$  be a complex-valued holomorphic function in an open neighborhood of a closed polydisc  $\bar{\Delta}(w; r)$ . Then, for any  $z \in \Delta(w; r)$ , it holds that*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r_n} \cdots \int_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

*Proof.* From the previous result we know that  $f$  is holomorphic in each variable in an open neighborhood of  $\bar{\Delta}(w; r)$ . By repeated application of Cauchy integral formula for functions of one variable leads to the formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta_n - w_n| = r_n} \frac{f(z_1, \dots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} d\zeta_n \\ &= \frac{1}{(2\pi i)^2} \int_{|\zeta_n - w_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} \int_{|\zeta_{n-1} - w_{n-1}| = r_{n-1}} \frac{f(z_1, \dots, \zeta_{n-1}, \zeta_n)}{\zeta_{n-1} - z_{n-1}} d\zeta_{n-1} \\ &\quad \vdots \\ &\quad \vdots \\ &= \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} \cdots \int_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_{n-1}, \zeta_n)}{\zeta_1 - z_1} d\zeta_1 \end{aligned}$$

for all  $z \in \Delta(w; r)$ . For any fixed point  $z = (z_1, \dots, z_n)$ , from the the previous result, it follows that this integrand is continuous on the compact domain of integration. Hence the iterated integral can be replaced by a single multiple integral

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r_n} \cdots \int_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

completing the proof.  $\square$

**Theorem B.1** (Osgood's lemma). *If a complex-valued function  $f$  is continuous in an open set  $U \subset \mathbb{C}^n$  and is holomorphic in each variable separately, then it is holomorphic in  $U$ .*

*Proof.* Select any point  $w \in U$  and any closed polydisc  $\bar{\Delta}(w; r) \subset U$ . Since  $f$  is holomorphic in each variable separately in an open neighborhood of  $\bar{\Delta}(w; r)$ , a repeated application of Cauchy integral formula leads to the formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = r_1} \frac{d\zeta_1}{\zeta_1 - z_1} \cdots \int_{|\zeta_n - w_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} f(\zeta)$$

for all  $z \in \Delta(w; r)$ . For any fixed point  $z = (z_1, \dots, z_n)$ , this integrand is continuous on the compact domain of integration. Hence the iterated integral can be replaced by a single multiple integral

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = r_1} \cdots \int_{|\zeta_n - w_n| = r_n} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \quad (\text{B.1})$$

Note that  $|z_j - w_j| < |\zeta_j - w_j|$  for all  $j = 1, \dots, n$ . Therefore, we have

$$\sum_{k=0}^{\infty} \left( \frac{z_j - w_j}{\zeta_j - w_j} \right)^k = \frac{1}{1 - \frac{z_j - w_j}{\zeta_j - w_j}} = \frac{\zeta_j - w_j}{\zeta_j - z_j} \quad \forall j = 1, \dots, n$$

Hence for a fixed  $z \in \Delta(w; r)$ , we have the following absolutely uniformly convergent series expansion for all points  $\zeta$  on the domain of integration

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}}{(\zeta_1 - w_1)^{k_1+1} \cdots (\zeta_n - w_n)^{k_n+1}} \quad (\text{B.2})$$

Using (B.2) in (B.1), and interchanging the orders of summation and integration, we get the power series expansion of  $f$

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = r_1} \cdots \int_{|\zeta_n - w_n| = r_n} f(\zeta) d\zeta_1 \cdots d\zeta_n \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}}{(\zeta_1 - w_1)^{k_1+1} \cdots (\zeta_n - w_n)^{k_n+1}} \\ &= \sum_{k_1, \dots, k_n=0}^{\infty} a_{k_1 \dots k_n} (z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n} \\ &\quad \text{where } a_{k_1 \dots k_n} = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = r_1} \cdots \int_{|\zeta_n - w_n| = r_n} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - w_1)^{k_1+1} \cdots (\zeta_n - w_n)^{k_n+1}} \end{aligned}$$

Therefore,  $f$  is a holomorphic function in  $U$ .  $\square$

**Remark B.4.** The hypothesis that the function  $f$  be continuous in  $U$  is not required, i.e. Goursat's theorem [3, §IV.8] can be generalized to several variables. However, this stronger result, called Hartogs's theorem, is much more difficult to prove [13, Theorem 1.2.5].

**Corollary B.1.** *The power series expansion of a holomorphic function  $f : U \rightarrow \mathbb{C}$  at  $w \in U \subset \mathbb{C}^n$  is uniquely determined by that function and it converges within the polydisc  $\Delta(w; r)$  contained in  $U$ .*

*Proof.* By differentiating (B.1) it follows that

$$\frac{\partial^{k_1+\dots+k_n} f(z)}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} = \frac{k_1! \dots k_n!}{(2\pi i)^n} \int_{|\zeta_1 - w_1|=r_1} \dots \int_{|\zeta_n - w_n|=r_n} \frac{f(\zeta) d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1)^{k_1+1} \dots (\zeta_n - z_n)^{k_n+1}}$$

Comparing this with the final statement of the above theorem, we get

$$a_{k_1 \dots k_n} = \frac{1}{k_1! \dots k_n!} \frac{\partial^{k_1+\dots+k_n} f(w)}{\partial w_1^{k_1} \dots \partial w_n^{k_n}}$$

Therefore, all the power series expansion convergent within any fixed compact subset of  $\Delta(w; r)$  must coincide.  $\square$

**Theorem B.2** (Cauchy-Riemann criterion). *A complex-valued smooth<sup>1</sup> function  $f$  defined in an open subset  $U \subset \mathbb{C}^n$  is holomorphic in  $U$  if and only if it satisfies the system of partial differential equations*

$$\frac{\partial}{\partial \bar{z}_j} f(z) = 0, \quad \forall j = 1, \dots, n$$

*Proof.* At any point in  $U$ , we consider  $f(z)$  to be a function of the single variable  $z_j$ , holding the other variables constant. Next, we decompose  $f$  into its real and imaginary parts by writing  $f(z) = u(z) + iv(z)$ , and observe that

$$\frac{\partial}{\partial \bar{z}_j} f(z) = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) (u(z) + iv(z)) = \frac{1}{2} \left( \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} \right) + \frac{i}{2} \left( \frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j} \right)$$

Therefore,  $\partial f(z)/\partial \bar{z}_j = 0$ , for all  $j = 1, \dots, n$  is equivalent to the classical Cauchy-Riemann equations for each variable separately. This is equivalent to the function  $f$  being holomorphic in each variable separately. The desired result follows from [Proposition B.1](#) and [Theorem B.1](#).  $\square$

**Remark B.5.** The transition from the real partial differentials to the complex partial differentials can be illustrated for the simplest case. For some open set  $U \subset \mathbb{C} = \mathbb{R}^2$ , consider the differentiable map  $f : U \rightarrow \mathbb{R}^2$  such that  $f(x, y) = (u(x, y), v(x, y))$ . Then the total derivative<sup>2</sup>  $Df(w)$  at point  $w = (r, s) \in U$  is a  $\mathbb{R}$ -linear map between tangent spaces  $Df(w) : T_w \mathbb{R}^2 \rightarrow T_{f(w)} \mathbb{R}^2$ . With respect to the standard basis we get the real Jacobian matrix

$$Df(w) = \begin{bmatrix} \left. \frac{\partial u}{\partial x} \right|_w & \left. \frac{\partial u}{\partial y} \right|_w \\ \left. \frac{\partial v}{\partial x} \right|_w & \left. \frac{\partial v}{\partial y} \right|_w \end{bmatrix}$$

Next, we extend  $Df(w)$  to a  $\mathbb{C}$ -linear map  $\widetilde{Df}(w) : T_w \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T_{f(w)} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$ . If we consider  $f = u + iv$  and  $z = x + iy$ , then with respect to the new basis we get the complexified Jacobian matrix

$$\widetilde{Df}(w) = \begin{bmatrix} \left. \frac{\partial f}{\partial z} \right|_w & \left. \frac{\partial f}{\partial \bar{z}} \right|_w \\ \left. \frac{\partial f}{\partial z} \right|_w & \left. \frac{\partial f}{\partial \bar{z}} \right|_w \end{bmatrix} = \begin{bmatrix} \left. \frac{\partial f}{\partial z} \right|_w & \left. \frac{\partial f}{\partial \bar{z}} \right|_w \\ \left( \left. \frac{\partial f}{\partial \bar{z}} \right|_w \right) & \left( \left. \frac{\partial f}{\partial z} \right|_w \right) \end{bmatrix}$$

<sup>1</sup>That is, continuously differentiable in the underlying real coordinates of  $\mathbb{C}^n$ . In other words,  $f \in C^\infty(U)$ .

<sup>2</sup>This was called *pushforward of a vector* in the previous report [[12](#), Definition 1.5].

Therefore, if  $f$  is holomorphic, then the differential in the new base system is given by the diagonal matrix

$$\begin{bmatrix} \frac{\partial f}{\partial z} \Big|_w & 0 \\ 0 & \frac{\partial \bar{f}}{\partial \bar{z}} \Big|_w \end{bmatrix}$$

**Proposition B.3.** *Let  $U$  be an open set in  $\mathbb{C}^n$ . Then:*

1.  $\mathcal{O}(U)$  is a ring under the operations  $(f + g)(z) = f(z) + g(z)$  and  $(fg)(z) = f(z)g(z)$ .
2. If  $f \in \mathcal{O}(U)$  and is nowhere vanishing, then  $1/f \in \mathcal{O}(U)$
3. If  $f \in \mathcal{O}(U)$  and is real-valued or has constant modulus, then  $f$  is constant.

**Theorem B.3** (Identity theorem). *Let  $U$  be a connected open set in  $\mathbb{C}^n$  and  $f, g \in \mathcal{O}(U)$ . If  $f(z) = g(z)$  for all points  $z$  in an open subset  $V \subset U$ , then  $f(z) = g(z)$  for all points  $z \in U$ .*

*Proof.* This is a straight-forward generalization of the single-variable identity theorem, see [9, Theorem I.A.6] for the proof.  $\square$

**Theorem B.4** (Hartogs's extension theorem). *Let  $U \subset \mathbb{C}^n$  for  $n > 1$  be a bounded open set and  $K$  be a compact subset  $U$  with the property that  $U \setminus K$  is connected. If  $f$  is a complex-valued holomorphic function on  $U \setminus K$ , then there is a unique complex-valued holomorphic function  $F$  on  $U$  such that  $F|_{U \setminus K} = f$ .*

*Proof.* The proof involves a typical  $\bar{\partial}$ -argument as seen in the proof of  $\bar{\partial}$ -Poincaré lemma, see [13, Theorem 1.2.6] and [19, §2.2].  $\square$

**Remark B.6.** This extension does not hold when  $n = 1$ . For example, consider the function  $f(z) = 1/z$ , which is clearly holomorphic in  $\mathbb{C} \setminus \{0\}$ , but cannot be continued as a holomorphic function on the whole  $\mathbb{C}$ .

This extension also does not hold when  $U \setminus K$  is not connected. For example, consider the open ball  $U = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$  and the compact set  $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1/2\}$ . Then  $U \setminus K = U_1 \cup U_2$  where

$$\begin{aligned} U_1 &:= \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1/2\} \\ U_2 &:= \{(z_1, z_2) \in \mathbb{C}^2 : 1/2 < |z_1|^2 + |z_2|^2 < 1\} \end{aligned}$$

such that  $U_1 \cap U_2 = \emptyset$ . Now consider the holomorphic function  $f$  defined on  $U \setminus K$  as

$$f(z) = \begin{cases} 0 & \text{if } z \in U_1 \\ 1 & \text{if } z \in U_2 \end{cases}$$

But this clearly can't be extended to a holomorphic function on  $U$ .

## B.2 Algebraic properties of $\mathcal{O}_w$

In this section some definitions and facts from [13, §6.4], [9, §II.A, II.B] and [10, §1.1] will be stated.

**Definition B.5** (Ring of germs of holomorphic functions). For  $w \in \mathbb{C}^n$ , consider the set

$$\mathcal{O}_w := \{(U, f) | w \in U \subset \mathbb{C}^n \text{ open}, f \in \mathcal{O}(U)\} / \sim$$

where  $(U, f) \sim (V, g)$  if  $\exists W$  open,  $w \in W$  such that  $W \subset V \cap U$  and  $f|_W = g|_W$ . The representative function of an equivalence class is called a *germ of holomorphic functions at  $w$*  and  $\mathcal{O}_w$  is called the *ring of germs of holomorphic functions at  $w \in \mathbb{C}^n$*  with the following operations:  $[(U, f)] + [(V, g)] := [(U \cap V, f + g)]$  and  $[(U, f)] \cdot [(V, g)] := [(U \cap V, fg)]$ .

**Remark B.7.** The ring  $\mathcal{O}_w$  is a commutative ring with an identity element. The zero of this ring is the germ of the function which vanishes identically, and the identity of the ring is the germ of the function which is identically one.

**Lemma B.1.**  $\mathcal{O}_w$  is an integral domain.

*Proof.* Consider two arbitrary germs  $[(U, f)]$  and  $[(V, g)]$  such that

$$[(U, f)] \cdot [(V, g)] = [(U \cap V, fg)] = [(W, 0)]$$

for some open neighborhood  $W$  of  $w$ . Hence  $f(z)g(z) = 0$  in some connected open neighborhood  $W' \subset W \cap V \cap U$  of  $w$ . If  $f(z_0) \neq 0$  for a single point  $z_0 \in W'$ , then by continuity  $f(z) \neq 0$  in an open neighborhood of  $z_0$  and therefore  $g(z) = 0$  in that open neighborhood. By [Theorem B.3](#), therefore, it follows that  $g(z) = 0$  in  $W'$ , hence that  $(V, g) \sim (W', 0)$ .  $\square$

**Lemma B.2.** A germ  $[(U, f)] \in \mathcal{O}_w$  is a unit if and only if  $f(w) \neq 0$ .

*Proof.* We need to show that the multiplicative inverse of  $[(U, f)]$  exists if and only if  $f$  does not vanish at  $w$ . Suppose that  $[(U, f)] \in \mathcal{O}_w$  such  $f(w) \neq 0$ . By continuity,  $f(z) \neq 0$  in an open neighborhood  $V \subset U$  of  $w$ ; and hence  $1/f(z)$  is continuous in  $V$  and is holomorphic in each variable separately in  $V$ . An application of [Proposition B.3\(2\)](#) shows that  $1/f(z)$  is holomorphic in  $V$ , hence  $[(V, 1/f)] \in \mathcal{O}_w$ .  $\square$

**Lemma B.3.**  $\mathcal{O}_w$  is a local ring.

*Proof.* Since a germ  $[(U, f)]$  is a unit if and only if  $f(w) \neq 0$ , any proper ideal  $\mathfrak{a}$  of  $\mathcal{O}_w$  consists only of germs which vanish at  $w$ . So the unique maximal ideal in  $\mathcal{O}_w$  is

$$\mathfrak{m} := \{[(U, f)] \in \mathcal{O}_w \mid f(w) = 0\}$$

Therefore,  $\mathcal{O}_w$  is a local ring.  $\square$

**Definition B.6** (Order of a holomorphic function). Let  $f$  be a holomorphic function in a neighborhood of  $w$  in  $\mathbb{C}^n$  such that

$$f(z) = f(z_1, \dots, z_n) = \sum_{j_1, \dots, j_n=0}^{\infty} a_{j_1 \dots j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

Then the order of  $f$  is defined to be the least value of  $j_1 + \dots + j_n$  for which  $a_{j_1 \dots j_n} \neq 0$ , i.e.

$$\text{ord}(f) := \min\{j_1 + \dots + j_n \mid a_{j_1 \dots j_n} \neq 0\}$$

**Remark B.8.** If  $\text{ord}(f) = k$ , then there exists a non-singular linear change of coordinates so that in the new coordinates, the coefficient of  $z_n^k$  is 1. When  $f$  is of this form it is said to be *normalized* (with respect to the variable  $z_n$ ) of *order*  $k$ .

**Definition B.7** (Weierstrass polynomial). A function  $W$ , holomorphic in a neighborhood of  $w \in \mathbb{C}^n$  is called a *Weierstrass polynomial* of degree  $m$ , if we have

$$W(z_1, \dots, z_n) = W(z', z_n) = z_n^m + a_{m-1}(z')z_n^{m-1} + \dots + a_1(z')z_n + a_0(z')$$

where  $z' = (z_1, \dots, z_{n-1})$  and  $a_j$  are holomorphic functions in a neighborhood of  $w' = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$  and  $a_j(0) = 0$  for  $j = 0, \dots, m-1$ .

**Remark B.9.** If we denote the ring of germs of holomorphic functions in the variables  $z_1, \dots, z_{n-1}$  by  $\mathcal{O}_{w'}$ , then<sup>3</sup> the Weierstrass polynomial  $W \in \mathcal{O}_{w'}[z_n]$  such that the coefficients are non-unit elements of  $\mathcal{O}_{w'}$ . Note that  $\mathcal{O}_{w'} \subset \mathcal{O}_{w'}[z_n] \subset \mathcal{O}_w$ .

<sup>3</sup>From now onwards we will abuse the notation for germs, i.e. instead of writing  $[(U, f)] \in \mathcal{O}_w$  we will simply write  $f \in \mathcal{O}_w$  such that  $f$  is an holomorphic function in an open neighborhood of  $w$ .



**Theorem B.5** (Weierstrass preparation theorem). *Let  $f$  be a normalized holomorphic function of order  $k$  in a neighborhood of  $w \in \mathbb{C}^n$ . Then in a small neighborhood of  $w$ ,  $f$  can be written uniquely as*

$$f(z) = u(z) \cdot W(z)$$

where  $u \in \mathcal{O}_w$  is a unit and  $W \in \mathcal{O}_{w'}[z_n]$  is a Weierstrass polynomial of degree  $k$ .

*Proof.* To prove this we will need Hartogs's extension theorem [13, Theorem 6.4.5] or Riemann extension theorem [9, Theorem II.B.2].  $\square$

**Theorem B.6** (Weierstrass division theorem). *Let  $W \in \mathcal{O}_{w'}[z_n]$  be a Weierstrass polynomial in  $z_n$  of degree  $k$ . Then any  $f \in \mathcal{O}_w$  can be written in a unique manner in the form  $f = g \cdot W + r$ , for some  $g \in \mathcal{O}_w$  and  $r \in \mathcal{O}_{w'}[z_n]$  a polynomial of degree less than  $k$ . Moreover, if  $f \in \mathcal{O}_{w'}[z_n]$  then necessarily  $g \in \mathcal{O}_{w'}[z_n]$ .*

*Proof.* For a proof, see [9, Theorem II.B.3].  $\square$

**Lemma B.4.** *A Weierstrass polynomial  $W \in \mathcal{O}_{w'}[z_n]$  is reducible over  $\mathcal{O}_w$  if and only if it is reducible over  $\mathcal{O}_{w'}[z_n]$ . Moreover, if  $W$  is reducible, then all of its non-unit factors are Weierstrass polynomials of  $\mathcal{O}_{w'}[z_n]$ .*

*Proof.* ( $\Rightarrow$ ) Suppose that  $W$  is reducible over  $\mathcal{O}_w$ , and write  $W = f_1 f_2$  for some non-units  $f_1, f_2 \in \mathcal{O}_w$ . Since  $W$  is a Weierstrass polynomial, it is normalized and hence both  $f_1$  and  $f_2$  are also normalized. Applying **Theorem B.5**, we get  $f_1 = u_1 W_1$  and  $f_2 = u_2 W_2$  for some units  $u_1, u_2 \in \mathcal{O}_w$  and Weierstrass polynomials  $W_1, W_2 \in \mathcal{O}_{w'}[z_n]$ . Thus  $W = (u_1 u_2)(W_1 W_2)$ . But since  $W_1 W_2$  is also a Weierstrass polynomial, the uniqueness part of the **Theorem B.5** implies that<sup>4</sup>  $u_1 u_2 = 1$  and  $W_1 W_2 = W$ . Therefore  $W$  is reducible in the ring of polynomials  $\mathcal{O}_{w'}[z_n]$  as well, and its factors are Weierstrass polynomials.

( $\Leftarrow$ ) Suppose that  $W$  is reducible over  $\mathcal{O}_{w'}[z_n]$ , and write  $W = g_1 g_2$  for some non-units  $g_1, g_2 \in \mathcal{O}_{w'}[z_n]$ . If  $g_1$  was a unit in  $\mathcal{O}_w$ , then  $W/g_1 = g_2$  and by the application of **Theorem B.6** it would follow that  $1/g_1 \in \mathcal{O}_{w'}[z_n]$ . This is impossible, since  $g_1$  is a non-unit element of  $\mathcal{O}_{w'}[z_n]$ . Therefore  $g_1$  is a non-unit element of  $\mathcal{O}_w$ . Similarly,  $g_2$  is non-unit element of  $\mathcal{O}_w$ . Therefore,  $W$  is reducible in  $\mathcal{O}_w$  as well.  $\square$

**Theorem B.7.** *The local ring  $\mathcal{O}_w$  is a unique factorization domain.*

*Proof.* Note that for any fixed point  $w \in \mathbb{C}^n$  the linear change of variable  $\zeta_j = z_j - w_j$  induces a canonical isomorphism between the rings  $\mathcal{O}_w$  and  $\mathcal{O}_0$ . Hence, for the local theory, it is sufficient to consider only the ring  $\mathcal{O}_0$  for  $0 \in \mathbb{C}^n$ . We will proceed by induction on  $n$ .

For  $n = 1$ , the theorem is trivial: if  $f \in \mathcal{O}_0$  has order  $k$  then  $f(z) = z^k g(z)$  where  $g(0) \neq 0$ , so that  $g$  is a unit in  $\mathcal{O}_0$ .

Let  $\mathcal{O}_0^{n-1}$  denote the ring of germs of holomorphic functions at  $0 \in \mathbb{C}^{n-1}$ . We will continue the abuse of notations by writing  $g \in \mathcal{O}_0^{n-1}$  instead of  $[(U, g)] \in \mathcal{O}_0^{n-1}$ . Now assume that the result is true for  $n - 1$ , i.e.  $\mathcal{O}_0^{n-1}$  is a unique factorization domain. Let  $f \in \mathcal{O}_0^n$ . Without loss of generality, we can assume that  $f$  is normalized of order  $k$ . Then by **Theorem B.5** we have  $f = u \cdot W$ , where  $W \in \mathcal{O}_0^{n-1}[z_n]$ . From Gauss Lemma<sup>5</sup> it follows that  $\mathcal{O}_0^{n-1}[z_n]$  is a unique factorization domain, and  $W = W_1 \cdots W_m$  where  $W_j \in \mathcal{O}_0^{n-1}[z_n]$  are irreducible elements. By **Lemma B.4**, it follows that the  $W_j$ 's are Weierstrass polynomials. Therefore,  $f = u \cdot W_1 \cdots W_m$ . If  $f$  could also be written as  $f = V_1 \cdots V_\ell$ , then we apply **Theorem B.5** to each  $V_j \in \mathcal{O}_0^n$  to obtain  $V_j = u'_j \cdot W'_j$ , that is,  $f = u' \cdot W'_1 \cdots W'_\ell$ , where  $u'$  is a unit and  $W'_j \in \mathcal{O}_0^{n-1}[z_n]$  are Weierstrass

<sup>4</sup>Here again we are abusing notations. Actually, the constant function 1 and  $u_1 u_2$  will represent the same equivalence class in  $\mathcal{O}_w$ , and  $W_1 W_2$  and  $W$  will represent same equivalence class in  $\mathcal{O}_{w'}[z_n]$ . That is, in some small enough neighborhood of  $w$ , all these equalities, like  $W = f_1 f_2$ , will hold.

<sup>5</sup>It implies that  $R$  is a unique factorization domain if and only if  $R[x]$  is a unique factorization domain. For proof, see Theorem 9.3.7 on p. 304 of Dummit and Foote's book "Abstract Algebra".

polynomials. Since there is only one way to write  $f$  as a unit times a Weierstrass polynomial, we conclude that

$$W_1 \cdots W_m = W'_1 \cdots W'_\ell$$

By induction hypothesis  $\mathcal{O}_0^{n-1}[z_n]$  is a unique factorization domain, and hence  $\{W_1, \dots, W_m\} = \{W'_1, \dots, W'_\ell\}$ .  $\square$

### B.3 Several variable holomorphic mappings

In this section some definitions and facts from [9, §I.A, I.B], [10, §1.1] and [6, §I.7] will be stated.

**Definition B.8** (Several variable holomorphic mapping). Let  $U \subset \mathbb{C}^n$  be an open set, and  $g : U \rightarrow \mathbb{C}^m$  be any mapping such that

$$g(z) = g(z_1, \dots, z_n) = (g_1(z), \dots, g_m(z))$$

where  $g_j : U \rightarrow \mathbb{C}$  for all  $j = 1, \dots, m$ . The mapping  $g$  is called a *holomorphic mapping* if the  $m$  complex-valued functions  $g_1, \dots, g_m$  are holomorphic in  $U$ .

**Proposition B.4** (Chain rule). Let  $U \subset \mathbb{C}^n$  and  $V \subset \mathbb{C}^m$  be open subsets. If  $g : U \rightarrow V$  is a holomorphic mapping and  $f : V \rightarrow \mathbb{C}$  is a holomorphic function, then

$$\frac{\partial(f \circ g)}{\partial z_j} = \sum_{k=1}^m \left( \frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial \bar{w}_k} \frac{\partial \bar{g}_k}{\partial z_j} \right) \quad \text{and} \quad \frac{\partial(f \circ g)}{\partial \bar{z}_j} = \sum_{k=1}^m \left( \frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial \bar{z}_j} + \frac{\partial f}{\partial \bar{w}_k} \frac{\partial \bar{g}_k}{\partial \bar{z}_j} \right)$$

where  $w_k = g_k(z_1, \dots, z_n)$  for  $k = 1, \dots, m$ .

*Proof.* We have the following composite maps

$$\begin{array}{ccccc} U & \xrightarrow{g} & V & \xrightarrow{f} & \mathbb{C} \\ (z_1, \dots, z_n) & \longmapsto & (w_1, \dots, w_m) & \longmapsto & f(w) \end{array}$$

where  $w_k = g_k(z_1, \dots, z_n)$  for  $k = 1, \dots, m$ . We can separate each  $g_k$  into real and imaginary parts by writing  $g_k(z) = u_k(z) + iv_k(z)$ . Since all the functions involved are differentiable in the underlying real coordinates, the usual chain rule for differentiation can be applied as follows:

$$\begin{aligned} \frac{\partial(f \circ g)}{\partial z_j} &= \sum_{k=1}^m \left( \frac{\partial f}{\partial u_k} \frac{\partial u_k}{\partial z_j} + \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial z_j} \right) \\ &= \sum_{k=1}^m \frac{1}{2} \left( \frac{\partial f}{\partial u_k} - i \frac{\partial f}{\partial v_k} \right) \frac{\partial g_k}{\partial z_j} + \sum_{k=1}^m \frac{1}{2} \left( \frac{\partial f}{\partial u_k} + i \frac{\partial f}{\partial v_k} \right) \frac{\partial \bar{g}_k}{\partial z_j} \\ &= \sum_{k=1}^m \left( \frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial \bar{w}_k} \frac{\partial \bar{g}_k}{\partial z_j} \right) \end{aligned}$$

Similarly we can prove for  $\partial/\partial \bar{z}$ .  $\square$

**Corollary B.2.** Let  $U \subset \mathbb{C}^n$  and  $V \subset \mathbb{C}^m$  be open subsets. If  $g : U \rightarrow V$  is a holomorphic mapping and  $f : V \rightarrow \mathbb{C}$  is a holomorphic function, then the composition  $f \circ g \in \mathcal{O}(U)$ .

**Definition B.9** (Several complex variables biholomorphic mapping). Let  $U, V \subset \mathbb{C}^n$  be two open sets. A holomorphic mapping  $f : U \rightarrow V$  is called *biholomorphic* if it is bijective and its inverse  $f^{-1} : V \rightarrow U$  is also holomorphic.

**Definition B.10** (Jacobian matrix of a holomorphic mapping). Let  $g : U \rightarrow \mathbb{C}^m$  be a holomorphic mapping, where  $U$  is an open subset of  $\mathbb{C}^n$ . The Jacobian matrix of the mapping  $g$  at a point  $w \in U$  is defined to be the matrix

$$\text{Jac}(g)(w) := \left[ \frac{\partial g_j}{\partial z_k} \Big|_w \right]_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}}$$

**Remark B.10.** This Jacobian matrix is also related to the complexified Jacobian matrix for total derivative discussed in [Remark B.5](#). For some open set  $U \subset \mathbb{C}^n = \mathbb{R}^{2n}$ , consider the differentiable map  $f : U \rightarrow \mathbb{C}^m = \mathbb{R}^{2m}$  such that

$$g(z) = g(z_1, \dots, z_n) = g(\underline{x}, \underline{y}) = (u_1(\underline{x}, \underline{y}), \dots, u_m(\underline{x}, \underline{y}), v_1(\underline{x}, \underline{y}), v_m(\underline{x}, \underline{y}))$$

where  $\underline{x} = (x_1, \dots, x_n)$ ,  $\underline{y} = (y_1, \dots, y_n)$ . Then the total derivative  $Dg(z)$  at point  $w = (\underline{x}, \underline{y}) \in U$  is a  $\mathbb{R}$ -linear map between tangent spaces  $Dg(w) : T_w \mathbb{R}^{2n} \rightarrow T_{f(w)} \mathbb{R}^{2m}$ . With respect to the standard basis we get the real Jacobian matrix

$$Dg(w) = \begin{bmatrix} \left[ \frac{\partial u_j}{\partial x_k} \Big|_w \right]_{j,k} & \left[ \frac{\partial u_j}{\partial y_k} \Big|_w \right]_{j,k} \\ \left[ \frac{\partial v_j}{\partial x_k} \Big|_w \right]_{j,k} & \left[ \frac{\partial v_j}{\partial y_k} \Big|_w \right]_{j,k} \end{bmatrix}$$

Next, we extend  $Dg(w)$  to a  $\mathbb{C}$ -linear map  $\widetilde{Dg}(w) : T_w \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T_{f(w)} \mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$ . If we consider  $g_j = u_j + iv_j$  for all  $j = 1, \dots, m$  and  $z_k = x_k + iy_k$  for  $k = 1, \dots, n$ , then with respect to the new basis we get the complexified Jacobian matrix

$$\widetilde{Dg}(w) = \begin{bmatrix} \left[ \frac{\partial g_j}{\partial z_k} \Big|_w \right]_{j,k} & \left[ \frac{\partial g_j}{\partial \bar{z}_k} \Big|_w \right]_{j,k} \\ \left[ \frac{\partial \bar{g}_j}{\partial z_k} \Big|_w \right]_{j,k} & \left[ \frac{\partial \bar{g}_j}{\partial \bar{z}_k} \Big|_w \right]_{j,k} \end{bmatrix}$$

Therefore, if  $g$  is holomorphic, then the differential in the new base system is given by the diagonal matrix

$$\begin{bmatrix} \text{Jac}(g)(w) & 0 \\ 0 & \overline{\text{Jac}(g)(w)} \end{bmatrix}$$

In particular, for a holomorphic function  $g$  we have

$$\det(Dg(w)) = \det(\text{Jac}(g)(w)) \det(\overline{\text{Jac}(g)(w)}) = |\det(\text{Jac}(g)(w))|^2 \geq 0$$

**Proposition B.5.** Let  $g : U \rightarrow V$  be a bijective holomorphic map between two open subsets  $U$  and  $V$  of  $\mathbb{C}^n$ . Then  $\text{Jac}(g)(w) \neq 0$  for all  $w \in U$ . In particular,  $g$  is biholomorphic.

*Proof.* The proof involves the use of Implicit Function Theorem<sup>6</sup>. For complete proof, see [10, Proposition 1.1.13].  $\square$

**Remark B.11.** Recall that the product topology on  $\mathbb{C}^n = \mathbb{R}^{2n}$  is equivalent to the metric topology, i.e. topology generated by open polydiscs is same as the one generated by open balls. Next, observe that the unit open ball  $B(0, 1)$  and unit open polydisc  $\Delta(0; 1)$  are diffeomorphic:

<sup>6</sup>For the exact statement and proof, see [9, Theorem I.B.5], [10, Proposition 1.1.11] and [6, Theorem I.7.6]. The proof of the implicit function theorem is a special case of the Weierstrass preparation theorem, discussed in the previous section.

1.  $B(0, 1)$  is diffeomorphic to  $\mathbb{R}^{2n}$  and the diffeomorphism is given by the map

$$\begin{aligned}\phi : B(0, 1) &\rightarrow \mathbb{R}^{2n} \\ x &\mapsto \frac{x}{\sqrt{1 - \|x\|^2}}\end{aligned}$$

2. If  $g : (-1, 1) \rightarrow \mathbb{R}$  is any diffeomorphism, then

$$\begin{aligned}\psi : \Delta(0; 1) &\rightarrow \mathbb{R}^{2n} \\ (x_1, \dots, x_{2n}) &\mapsto (g(x_1), \dots, g(x_{2n}))\end{aligned}$$

is a smooth map with smooth inverse. Hence  $\Delta(0; 1)$  is also diffeomorphic to  $\mathbb{R}^{2n}$ .

However, they are not biholomorphic for  $n > 1$  [13, §0.3.2].

# Appendix C

## Topology

In this appendix, for the sake of completeness, the proofs of a few simple standard results<sup>1</sup> for  $\mathbb{C}^n$  have been discussed.

**Lemma C.1.** *If  $U$  is an open set in  $\mathbb{C}$ , then there exists a sequence  $\{K_n\}$  of compact subsets of  $U$  such that*

1.  $K_n \subset \text{int}(K_{n+1})$  for each  $n$ ;
2.  $\bigcup_{n \in \mathbb{N}} \text{int}(K_n) = U$ ; and
3. each bounded component of the complement of  $K_n$  contains a point of the complement of  $U$ .

*Proof.* For each  $n \in \mathbb{N}$ , define the open set,

$$V_n := \Delta(\infty; n) \cup \bigcup_{z \in \mathbb{C} \setminus U} \Delta\left(z; \frac{1}{n}\right)$$

where  $\Delta(z; \frac{1}{n}) = \{w \in \mathbb{C} : |z - w| < \frac{1}{n}\}$ , and  $\Delta(\infty, n) = \{w \in \mathbb{C} : |w| > n\}$  is the “disk at  $\infty$ ”. Then we define

$$K_n := \mathbb{C} \setminus V_n$$

which is a closed and bounded (hence compact<sup>2</sup>) subset of  $U$  for all  $n$ . Now we will verify the three desired properties:

1. If  $z \in K_n$  and  $r = \frac{1}{n} - \frac{1}{n+1}$ , then  $\Delta(z; r) \subset K_{n+1}$ . The interior of  $K_{n+1}$  is, by definition, the largest open subset of  $K_{n+1}$ . Therefore,  $K_n \subset \bigcup_{z \in K_n} \Delta(z; r) \subset \text{int}(K_{n+1})$ .
2. As  $n \rightarrow \infty$  we get  $V_n \rightarrow \mathbb{C} \setminus U$ . Therefore,  $\bigcup_{n \in \mathbb{N}} K_n = U$ . Now since  $K_n \subset \text{int}(K_{n+1})$ , we have  $\bigcup_{n \in \mathbb{N}} \text{int}(K_n) = U$ .
3. We need to show that every bounded connected component  $\mathcal{C}$  of  $V_n$  meets  $\mathbb{C} \setminus U$ . To prove this, pick a  $w \in \mathcal{C}$ . Note that  $w$ , being an element of  $V_n$ , must be contained in  $\Delta(z; \frac{1}{n})$  for some  $z \in \mathbb{C} \setminus U$  or in  $\Delta(\infty; n)$ . Since  $\mathcal{C}$  is bounded, we have<sup>3</sup>  $w \in \Delta(z; \frac{1}{n})$  for some  $z \in \mathbb{C} \setminus U$ . Observe that  $\mathcal{C} \cup \Delta(z; \frac{1}{n})$  is a connected subset of  $V_n$ , since it is the union of two connected open subsets of  $V_n$  with non-empty intersection. Since  $\mathcal{C}$  is a connected component of  $V_n$ , we know that  $\mathcal{C}$  is a maximal connected set of  $V_n$ . Therefore,  $\Delta(z, \frac{1}{n})$  must be contained in  $\mathcal{C}$ . Hence  $\mathcal{C}$  contains  $z$ , which is in  $\mathbb{C} \setminus U$ .

---

<sup>1</sup>These results are also true for  $\mathbb{R}^n$ .

<sup>2</sup>Note that  $K_n$  can be empty also.

<sup>3</sup>If  $\mathbb{C} \setminus U \neq \emptyset$  then there is no bounded component of  $V_n$  to begin with.

□

**Remark C.1.** We can't guarantee that the third property will hold for an unbounded component, unless we replace  $\mathbb{C}$  by the Riemann sphere<sup>4</sup>  $\mathbb{C} \cup \{\infty\}$ . For example, if  $U = \{z \in \mathbb{C} : |z| > 1/2\}$  then for  $n = 1$  the unbounded connected component  $\mathcal{C} = \Delta(\infty; 1)$  doesn't intersect with  $\mathbb{C} \setminus U$ .

**Lemma C.2.** *Let  $K$  be a compact subset of an open set  $U \subset \mathbb{C}^n$ . Then there exists a real-valued smooth function  $F(z)$  in  $\mathbb{C}^n$  such that*

1.  $0 \leq F(z) \leq 1$  for all  $z \in \mathbb{C}^n$ ;
2.  $F(z) = 1$  for  $z \in K$ ; and
3.  $F(z) = 0$  for  $z \in \mathbb{C}^n \setminus U$ .

*Proof.* Consider the following smooth<sup>5</sup> function defined on  $\mathbb{R}$ :

$$h(x) = \begin{cases} e^{\frac{-1}{(x-r)}} e^{\frac{-1}{(x-R)}} & \text{for } r < x < R \\ 0 & \text{otherwise} \end{cases}$$

Consequently the function defined as<sup>6</sup>

$$g(x) = \frac{\int_x^R h(t) dt}{\int_r^R h(t) dt}$$

is a smooth function such that

1.  $0 \leq g(x) \leq 1$  for all  $x \in \mathbb{R}$ ;
2.  $g(x) = 1$  for  $x \leq r$ ; and
3.  $g(x) = 0$  for  $x \geq R$ .

Next, consider the special case in which  $K$  is a closed ball of radius  $r$  centered at origin, and  $U$  is an open ball of radius  $R > r$ , i.e.

$$K = \{z \in \mathbb{C}^n : |z| \leq r\} \quad \text{and} \quad U = \{z \in \mathbb{C}^n : |z| < R\}$$

Then for  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , the function

$$f(z) = g(\|z\|) = g\left(\sqrt{|z_1|^2 + \dots + |z_n|^2}\right)$$

satisfies the required conditions

1.  $0 \leq f(z) \leq 1$  for all  $z \in \mathbb{C}^n$ ;
2.  $f(z) = 1$  for  $\|z\| \leq r$ ; and
3.  $f(z) = 0$  for  $\|z\| \geq R$ .

---

<sup>4</sup>The same construction works for the case of Riemann sphere. In fact we can prove a stronger statement: for each  $n \in \mathbb{N}$ , every connected component of  $\mathbb{C} \cup \{\infty\} \setminus K_n$  contains a connected component of  $\mathbb{C} \cup \{\infty\} \setminus U$ . For details, see [3, Proposition VII.1.2], there this theorem is used to prove Runge's theorem.

<sup>5</sup>This is a standard exercise in real analysis, for example, see Problem 1.2 of Loring Tu's book "An Introduction to Manifolds."

<sup>6</sup>The same construction can be used for bump functions on smooth manifolds, see Lemma 2.1.8 of Amiya Mukherjee's book "Differential Topology."

Now for the general case, select a finite number of pairs of concentric balls  $K_j \subset U_j$  such that  $K \subset \bigcup K_j$  and  $U_j \subset U$ . Let  $f_j(z)$  be the functions satisfying the desired conditions on these pairs of balls, as constructed for the special case. Then the function

$$F(z) = 1 - \prod_j (1 - f_j(z))$$

is the desired function, hence completing the proof.  $\square$

**Theorem C.1.** *Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of an open subset  $U \subset \mathbb{C}^n$ , then there is a smooth partition of unity  $\{\psi_k\}_{k=1}^\infty$  with every  $\psi_k$  having compact support such that for each  $k$ ,  $\text{supp}(\psi_k) \subset U_\alpha$  for some  $\alpha \in A$ .*

*Proof.* Since any open subset  $U \subset \mathbb{C}^n$  is paracompact, every open covering  $\{U_\alpha\}$  has a locally finite refinement  $\{V_k\}$ . Then the smooth partition of unity  $\{\psi_k\}$  of  $U$  subordinate to  $\{V_k\}$  will have compact support. For details, see [9, Appendix A].  $\square$

**Remark C.2.** This result was also used in the previous report [12, Theorem XIII]. However, there we didn't require the support to be compact.

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