Sheaf Theory and Complex Geometry

Gaurish Korpal

24 April 2019

Outline

Complex manifolds

Complex manifold

Complex manifold

A *complex manifold* M of dimension n is a smooth manifold of dimension $2n$ equipped with a holomorphic structure, i.e. if M is covered by open sets U_α which are diffeomorphic via maps called ϕ_α to open sets in $\mathbb{C}^n$, then the transition diffeomorphisms

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic for all α, β.

Complex local coordinates

Let $w \in M$ be a point. If (U, ϕ) is a chart of M with $w \in U$, then

$$\begin{aligned} \phi : U &\to \mathbb{C}^n \\ w &\mapsto (z_1(w), \dots, z_n(w)) \end{aligned}$$

where $z_j : U \to \mathbb{C}$ for $j = 1, \dots, n$ are called *local coordinates* at w.

Complex manifold

Complex manifold

A *complex manifold* M of dimension n is a smooth manifold of dimension $2n$ equipped with a holomorphic structure, i.e. if M is covered by open sets U_α which are diffeomorphic via maps called ϕ_α to open sets in $\mathbb{C}^n$, then the transition diffeomorphisms

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic for all α, β.

Complex local coordinates

Let $w \in M$ be a point. If (U, ϕ) is a chart of M with $w \in U$, then

$$\begin{aligned} \phi : U &\to \mathbb{C}^n \\ w &\mapsto (z_1(w), \ldots, z_n(w)) \end{aligned}$$

where $z_j : U \to \mathbb{C}$ for $j = 1, \ldots, n$ are called *local coordinates* at w.

Real tangent space

Real tangent space

Consider the complex manifold M of dimension n as a smooth manifold of dimension $2n$. Then for $w \in M$ we define the *real tangent space* of M at the point w as the real vector space of $\mathbb{R}$-linear derivations on the ring of real-valued smooth functions in a neighborhood of w, i.e.

$$T_{w,\mathbb{R}}M = \{X_w : C_w^\infty(M) \to \mathbb{R} \mid X_w(fg) = X_w(f)g(w) + f(w)X_w(g)\}$$

If we write the local coordinates around $w \in M$ as $z_j = x_j + iy_j$, then a canonical basis of $T_{w,\mathbb{R}}M$ is given by the tangent vectors

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_w, \ldots, \left.\frac{\partial}{\partial x_n}\right|_w, \left.\frac{\partial}{\partial y_1}\right|_w, \ldots, \left.\frac{\partial}{\partial y_n}\right|_w \right\}$$

Clearly, $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}M) = 2n$ as seen in the case of smooth manifolds.

Real tangent space

Real tangent space

Consider the complex manifold M of dimension n as a smooth manifold of dimension $2n$. Then for $w \in M$ we define the *real tangent space* of M at the point w as the real vector space of $\mathbb{R}$-linear derivations on the ring of real-valued smooth functions in a neighborhood of w, i.e.

$$T_{w,\mathbb{R}}M = \{X_w : C_w^\infty(M) \to \mathbb{R} \mid X_w(fg) = X_w(f)g(w) + f(w)X_w(g)\}$$

If we write the local coordinates around $w \in M$ as $z_j = x_j + iy_j$, then a canonical basis of $T_{w,\mathbb{R}}M$ is given by the tangent vectors

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_w, \ldots, \left.\frac{\partial}{\partial x_n}\right|_w, \left.\frac{\partial}{\partial y_1}\right|_w, \ldots, \left.\frac{\partial}{\partial y_n}\right|_w \right\}$$

Clearly, $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}M) = 2n$ as seen in the case of smooth manifolds.

Complexified tangent space

Complexified tangent space

If M is a complex manifold, then we define the *complexified tangent space* of M at the point w to be the complexification of the real tangent space of M at w

$$T_{w,\mathbb{C}}M = T_{w,\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$$

We can view $T_{w,\mathbb{C}}M$ as the complex vector space of $\mathbb{C}$-linear derivations in the ring of complex-valued smooth functions in a neighborhood of w. Using the canonical basis of real tangent space we can define its complexification as the complex vector space with the basis

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_w, \ldots, \frac{\partial}{\partial x_n}\bigg|_w, \frac{\partial}{\partial y_1}\bigg|_w, \ldots, \frac{\partial}{\partial y_n}\bigg|_w \right\}$$

Hence, as expected, we have $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}M) = \dim_{\mathbb{C}}(T_{w,\mathbb{C}}M)$.

Complexified tangent space

Complexified tangent space

If M is a complex manifold, then we define the *complexified tangent space* of M at the point w to be the complexification of the real tangent space of M at w

$$T_{w,\mathbb{C}}M = T_{w,\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$$

We can view $T_{w,\mathbb{C}}M$ as the complex vector space of $\mathbb{C}$-linear derivations in the ring of complex-valued smooth functions in a neighborhood of w. Using the canonical basis of real tangent space we can define its complexification as the complex vector space with the basis

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_w, \ldots, \frac{\partial}{\partial x_n}\bigg|_w, \frac{\partial}{\partial y_1}\bigg|_w, \ldots, \frac{\partial}{\partial y_n}\bigg|_w \right\}$$

Hence, as expected, we have $\dim_{\mathbb{R}}(T_{w,\mathbb{R}}M) = \dim_{\mathbb{C}}(T_{w,\mathbb{C}}M)$.

Almost complex structure

Almost complex structure

An *almost complex structure* on a smooth manifold M is a vector bundle endomorphism J of (real) tangent bundle TM, such that $J^2 = -\mathbb{1}_{TM}$, i.e. for all $w \in M$, the linear map $J_w : T_wM \to T_wM$ is a linear complex structure for T_wM.

A complex manifold M induces an almost complex structure on its underlying smooth manifold, defined on the basis as

$$\begin{aligned} J_w : T_{w,\mathbb{R}}M &\to T_{w,\mathbb{R}}M \\ \left.\frac{\partial}{\partial x_j}\right|_w &\mapsto \left.\frac{\partial}{\partial y_j}\right|_w \\ \left.\frac{\partial}{\partial y_j}\right|_w &\mapsto -\left.\frac{\partial}{\partial x_j}\right|_w \end{aligned}$$

We will regard this J as a vector bundle endomorphism of the real vector bundle $T_{\mathbb{R}}M$ over M.

Almost complex structure

Almost complex structure

An *almost complex structure* on a smooth manifold M is a vector bundle endomorphism J of (real) tangent bundle TM, such that $J^2 = -\mathbb{1}_{TM}$, i.e. for all $w \in M$, the linear map $J_w : T_wM \to T_wM$ is a linear complex structure for T_wM.

A complex manifold M induces an almost complex structure on its underlying smooth manifold, defined on the basis as

$$\begin{aligned} J_w : T_{w,\mathbb{R}}M &\to T_{w,\mathbb{R}}M \\ \left.\frac{\partial}{\partial x_j}\right|_w &\mapsto \left.\frac{\partial}{\partial y_j}\right|_w \\ \left.\frac{\partial}{\partial y_j}\right|_w &\mapsto -\left.\frac{\partial}{\partial x_j}\right|_w \end{aligned}$$

We will regard this J as a vector bundle endomorphism of the real vector bundle $T_{\mathbb{R}}M$ over M.

Almost complex structure

Proposition

The complexified tangent bundle $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles

$$T_{\mathbb{C}}M = (T_{\mathbb{R}}M)^{1,0} \oplus (T_{\mathbb{R}}M)^{0,1}$$

where

$$(T_{\mathbb{R}}M)^{1,0} = \{X \in T_{\mathbb{C}}M : (J \otimes \mathbb{1}_{\mathbb{C}})(X) = i \cdot X\} \quad \text{and}$$

$$(T_{\mathbb{R}}M)^{0,1} = \{X \in T_{\mathbb{C}}M : (J \otimes \mathbb{1}_{\mathbb{C}})(X) = -i \cdot X\}$$

Note that, we have

$$\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \in (T_{\mathbb{R}}M)^{1,0} \quad \text{and} \quad \left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \in (T_{\mathbb{R}}M)^{0,1}$$

Almost complex structure

Proposition

The complexified tangent bundle $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles

$$T_{\mathbb{C}}M = (T_{\mathbb{R}}M)^{1,0} \oplus (T_{\mathbb{R}}M)^{0,1}$$

where

$$(T_{\mathbb{R}}M)^{1,0} = \{X \in T_{\mathbb{C}}M : (J \otimes \mathbb{1}_{\mathbb{C}})(X) = i \cdot X\} \quad \text{and}$$

$$(T_{\mathbb{R}}M)^{0,1} = \{X \in T_{\mathbb{C}}M : (J \otimes \mathbb{1}_{\mathbb{C}})(X) = -i \cdot X\}$$

Note that, we have

$$\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \in (T_{\mathbb{R}}M)^{1,0} \quad \text{and} \quad \left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \in (T_{\mathbb{R}}M)^{0,1}$$

Almost complex structure

Next, observe that

$$\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial y_j} = \frac{i}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) - \frac{i}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

We define the following operators:

Complex partial derivative

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for $j = 1, \ldots, n$.

Almost complex structure

Next, observe that

$$\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial y_j} = \frac{i}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) - \frac{i}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

We define the following operators:

Complex partial derivative

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for $j = 1, \ldots, n$.

Holomorphic and antiholomorphic tangent bundle

Hence we can say that $\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{1,0}$ and $\left\{ \frac{\partial}{\partial \overline{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \overline{z}_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{0,1}$.

Holomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{1,0}$ is called *holomorphic tangent bundle* of M.

Antiholomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{0,1}$ is called *antiholomorphic tangent bundle* of M.

Therefore, the following also forms a basis of $T_{w,\mathbb{C}}M$

$$\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w, \frac{\partial}{\partial \overline{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \overline{z}_n}\Big|_w \right\}$$

Holomorphic and antiholomorphic tangent bundle

Hence we can say that $\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{1,0}$ and $\left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \bar{z}_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{0,1}$.

Holomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{1,0}$ is called *holomorphic tangent bundle* of M.

Antiholomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{0,1}$ is called *antiholomorphic tangent bundle* of M.

Therefore, the following also forms a basis of $T_{w,\mathbb{C}}M$

$$\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w, \frac{\partial}{\partial \bar{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \bar{z}_n}\Big|_w \right\}$$

Holomorphic and antiholomorphic tangent bundle

Hence we can say that $\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{1,0}$ and $\left\{ \frac{\partial}{\partial \overline{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \overline{z}_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{0,1}$.

Holomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{1,0}$ is called *holomorphic tangent bundle* of M.

Antiholomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{0,1}$ is called *antiholomorphic tangent bundle* of M.

Therefore, the following also forms a basis of $T_{w,\mathbb{C}}M$

$$\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w, \frac{\partial}{\partial \overline{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \overline{z}_n}\Big|_w \right\}$$

Holomorphic and antiholomorphic tangent bundle

Hence we can say that $\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{1,0}$ and $\left\{ \frac{\partial}{\partial \overline{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \overline{z}_n}\Big|_w \right\}$ is a basis for the complex vector space $(T_{w,\mathbb{R}}M)^{0,1}$.

Holomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{1,0}$ is called *holomorphic tangent bundle* of M.

Antiholomorphic tangent bundle

The complex vector bundle $(T_{\mathbb{R}}M)^{0,1}$ is called *antiholomorphic tangent bundle* of M.

Therefore, the following also forms a basis of $T_{w,\mathbb{C}}M$

$$\left\{ \frac{\partial}{\partial z_1}\Big|_w, \ldots, \frac{\partial}{\partial z_n}\Big|_w, \frac{\partial}{\partial \overline{z}_1}\Big|_w, \ldots, \frac{\partial}{\partial \overline{z}_n}\Big|_w \right\}$$

Real cotangent space

Real cotangent space

Consider the complex manifold M of dimension n as a smooth manifold of dimension $2n$. Then for $w \in M$ we define the *real cotangent space* of M at the point w as dual space of the real vector space $T_{w,\mathbb{R}}M$, i.e.

$$T^*_{w,\mathbb{R}}M = \mathrm{Hom}_{\mathbb{R}}(T_{w,\mathbb{R}}M, \mathbb{R})$$

If we write the local coordinates around $w \in M$ as $z_j = x_j + iy_j$, then a canonical basis of $T^*_{w,\mathbb{R}}M$ is given by the cotangent vectors

$$\left\{ \mathrm{d}x_1\big|_w, \cdots, \mathrm{d}x_n\big|_w, \mathrm{d}y_1\big|_w, \cdots, \mathrm{d}y_n\big|_w \right\}$$

Clearly, $\dim_{\mathbb{R}}(T^*_{w,\mathbb{R}}M) = 2n$ as seen in the case of smooth manifolds.

Real cotangent space

Real cotangent space

Consider the complex manifold M of dimension n as a smooth manifold of dimension $2n$. Then for $w \in M$ we define the *real cotangent space* of M at the point w as dual space of the real vector space $T_{w,\mathbb{R}}M$, i.e.

$$T^*_{w,\mathbb{R}}M = \mathrm{Hom}_{\mathbb{R}}(T_{w,\mathbb{R}}M, \mathbb{R})$$

If we write the local coordinates around $w \in M$ as $z_j = x_j + iy_j$, then a canonical basis of $T^*_{w,\mathbb{R}}M$ is given by the cotangent vectors

$$\left\{ \mathrm{d}x_1\big|_w, \cdots, \mathrm{d}x_n\big|_w, \mathrm{d}y_1\big|_w, \cdots, \mathrm{d}y_n\big|_w \right\}$$

Clearly, $\dim_{\mathbb{R}}(T^*_{w,\mathbb{R}}M) = 2n$ as seen in the case of smooth manifolds.

Complexified cotangent space

Complexified cotangent space

If M is a complex manifold, then we defined the *complexified cotangent space* of M at the point w to be the complexification of real cotangent space

$$T^*_{w,\mathbb{C}}M = T^*_{w,\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$$

We can also use the canonical basis of real cotangent space to define its complexification. Therefore, $T^*_{w,\mathbb{C}}M$ is the complex vector space with the basis

$$\left\{ \mathrm{d}x_1\Big|_w, \cdots, \mathrm{d}x_n\Big|_w, \mathrm{d}y_1\Big|_w, \cdots, \mathrm{d}y_n\Big|_w \right\}$$

Hence, as expected, we have $\dim_{\mathbb{R}}(T^*_{w,\mathbb{R}}M) = \dim_{\mathbb{C}}(T^*_{w,\mathbb{C}}M)$.

Complexified cotangent space

Complexified cotangent space

If M is a complex manifold, then we defined the *complexified cotangent space* of M at the point w to be the complexification of real cotangent space

$$T^*_{w,\mathbb{C}}M = T^*_{w,\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$$

We can also use the canonical basis of real cotangent space to define its complexification. Therefore, $T^*_{w,\mathbb{C}}M$ is the complex vector space with the basis

$$\left\{ \mathrm{d}x_1\big|_w, \cdots, \mathrm{d}x_n\big|_w, \mathrm{d}y_1\big|_w, \cdots, \mathrm{d}y_n\big|_w \right\}$$

Hence, as expected, we have $\dim_{\mathbb{R}}(T^*_{w,\mathbb{R}}M) = \dim_{\mathbb{C}}(T^*_{w,\mathbb{C}}M)$.

Almost complex structure

We get the linear complex structure $\mathcal{J}_w$ on $T^*_{w,\mathbb{R}}M$ from the linear complex structure J_w on $T_{w,\mathbb{R}}M$ as:

$$\mathcal{J}_w(\tau_w)(X_w) = \tau_w(J_w(X_w)) \ \forall \tau_w \in T^*_{w,\mathbb{R}}M, X_w \in T_{w,\mathbb{R}}M$$

We will regard this $\mathcal{J}$ as a vector bundle endomorphism of the smooth vector bundle $T^*_{\mathbb{R}}U$ over U.

Proposition

The complexified cotangent bundle $T^*_{\mathbb{C}}M = T^*_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles

$$T^*_{\mathbb{C}}M = (T^*_{\mathbb{R}}M)^{1,0} \oplus (T^*_{\mathbb{R}}M)^{0,1}$$

where

$$(T^*_{\mathbb{R}}M)^{1,0} = \{\tau \in T^*_{\mathbb{C}}M \mid (\mathcal{J} \otimes \mathbb{1}_{\mathbb{C}})(\tau) = i \cdot \tau\} \quad \text{and}$$
$$(T^*_{\mathbb{R}}M)^{0,1} = \{\tau \in T^*_{\mathbb{C}}M \mid (\mathcal{J} \otimes \mathbb{1}_{\mathbb{C}})(\tau) = -i \cdot \tau\}$$

Almost complex structure

We get the linear complex structure $\mathcal{J}_w$ on $T^*_{w,\mathbb{R}}M$ from the linear complex structure J_w on $T_{w,\mathbb{R}}M$ as:

$$\mathcal{J}_w(\tau_w)(X_w) = \tau_w(J_w(X_w)) \ \forall \tau_w \in T^*_{w,\mathbb{R}}M, X_w \in T_{w,\mathbb{R}}M$$

We will regard this $\mathcal{J}$ as a vector bundle endomorphism of the smooth vector bundle $T^*_{\mathbb{R}}U$ over U.

Proposition

The complexified cotangent bundle $T^*_{\mathbb{C}}M = T^*_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles

$$T^*_{\mathbb{C}}M = (T^*_{\mathbb{R}}M)^{1,0} \oplus (T^*_{\mathbb{R}}M)^{0,1}$$

where

$$(T^*_{\mathbb{R}}M)^{1,0} = \{\tau \in T^*_{\mathbb{C}}M \mid (\mathcal{J} \otimes \mathbb{1}_{\mathbb{C}})(\tau) = i \cdot \tau\} \quad \text{and}$$
$$(T^*_{\mathbb{R}}M)^{0,1} = \{\tau \in T^*_{\mathbb{C}}M \mid (\mathcal{J} \otimes \mathbb{1}_{\mathbb{C}})(\tau) = -i \cdot \tau\}$$

Almost complex structure

Recall that if V is a real vector space, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. Then we have $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$.

Using this, we can prove that $(T^*_{\mathbb{R}}M)^{1,0} \cong \left((T_{\mathbb{R}}M)^{1,0}\right)^*$ and $(T^*_{\mathbb{R}}M)^{0,1} \cong \left((T_{\mathbb{R}}M)^{0,1}\right)^*$.

Hence we can obtain basis for $T^*_{w,\mathbb{C}}M$ by defining the dual basis of $(T_{w,\mathbb{R}}M)^{1,0}$ and $(T_{w,\mathbb{R}}M)^{0,1}$.

Complex differential

$$dz_j := dx_j + i\, dy_j \quad \text{and} \quad d\bar{z}_j := dx_j - i\, dy_j$$

for $j = 1, \ldots, n$.

We can say that $\left\{ dz_j\big|_w \right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{1,0}$ and $\left\{ d\bar{z}_j\big|_w \right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{0,1}$. The following forms a basis of $T^*_{w,\mathbb{C}}M$

$$\left\{ dz_1\big|_w, \ldots, dz_n\big|_w, d\bar{z}_1\big|_w, \ldots, d\bar{z}_n\big|_w \right\}$$

Almost complex structure

Recall that if V is a real vector space, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. Then we have $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$. Using this, we can prove that $(T^*_{\mathbb{R}}M)^{1,0} \cong \left((T_{\mathbb{R}}M)^{1,0}\right)^*$ and $(T^*_{\mathbb{R}}M)^{0,1} \cong \left((T_{\mathbb{R}}M)^{0,1}\right)^*$.

Hence we can obtain basis for $T^*_{w,\mathbb{C}}M$ by defining the dual basis of $(T_{w,\mathbb{R}}M)^{1,0}$ and $(T_{w,\mathbb{R}}M)^{0,1}$.

Complex differential

$$dz_j := dx_j + i\,dy_j \quad \text{and} \quad d\bar{z}_j := dx_j - i\,dy_j$$

for $j = 1, \ldots, n$.

We can say that $\left\{dz_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{1,0}$ and $\left\{d\bar{z}_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{0,1}$. The following forms a basis of $T^*_{w,\mathbb{C}}M$

$$\left\{dz_1\big|_w, \ldots, dz_n\big|_w, d\bar{z}_1\big|_w, \ldots, d\bar{z}_n\big|_w\right\}$$

Almost complex structure

Recall that if V is a real vector space, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. Then we have $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$. Using this, we can prove that $(T^*_{\mathbb{R}}M)^{1,0} \cong \left((T_{\mathbb{R}}M)^{1,0}\right)^*$ and $(T^*_{\mathbb{R}}M)^{0,1} \cong \left((T_{\mathbb{R}}M)^{0,1}\right)^*$.
Hence we can obtain basis for $T^*_{w,\mathbb{C}}M$ by defining the dual basis of $(T_{w,\mathbb{R}}M)^{1,0}$ and $(T_{w,\mathbb{R}}M)^{0,1}$.

Complex differential

$$\mathrm{d}z_j := \mathrm{d}x_j + i\,\mathrm{d}y_j \quad \text{and} \quad \mathrm{d}\overline{z}_j := \mathrm{d}x_j - i\,\mathrm{d}y_j$$

for $j = 1, \ldots, n$.

We can say that $\left\{\mathrm{d}z_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{1,0}$ and $\left\{\mathrm{d}\overline{z}_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{0,1}$. The following forms a basis of $T^*_{w,\mathbb{C}}M$

$$\left\{\mathrm{d}z_1\big|_w, \ldots, \mathrm{d}z_n\big|_w, \mathrm{d}\overline{z}_1\big|_w, \ldots, \mathrm{d}\overline{z}_n\big|_w\right\}$$

Almost complex structure

Recall that if V is a real vector space, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. Then we have $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$. Using this, we can prove that $(T^*_{\mathbb{R}}M)^{1,0} \cong \left((T_{\mathbb{R}}M)^{1,0}\right)^*$ and $(T^*_{\mathbb{R}}M)^{0,1} \cong \left((T_{\mathbb{R}}M)^{0,1}\right)^*$.
Hence we can obtain basis for $T^*_{w,\mathbb{C}}M$ by defining the dual basis of $(T_{w,\mathbb{R}}M)^{1,0}$ and $(T_{w,\mathbb{R}}M)^{0,1}$.

Complex differential

$$\mathrm{d}z_j := \mathrm{d}x_j + i\,\mathrm{d}y_j \quad \text{and} \quad \mathrm{d}\bar{z}_j := \mathrm{d}x_j - i\,\mathrm{d}y_j$$

for $j = 1, \ldots, n$.

We can say that $\left\{\mathrm{d}z_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{1,0}$ and $\left\{\mathrm{d}\bar{z}_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{0,1}$. The following forms a basis of $T^*_{w,\mathbb{C}}M$

$$\left\{\mathrm{d}z_1\big|_w, \ldots, \mathrm{d}z_n\big|_w, \mathrm{d}\bar{z}_1\big|_w, \ldots, \mathrm{d}\bar{z}_n\big|_w\right\}$$

Almost complex structure

Recall that if V is a real vector space, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. Then we have $(V^*)_{\mathbb{C}} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong (V_{\mathbb{C}})^*$. Using this, we can prove that $(T^*_{\mathbb{R}}M)^{1,0} \cong \left((T_{\mathbb{R}}M)^{1,0}\right)^*$ and $(T^*_{\mathbb{R}}M)^{0,1} \cong \left((T_{\mathbb{R}}M)^{0,1}\right)^*$.
Hence we can obtain basis for $T^*_{w,\mathbb{C}}M$ by defining the dual basis of $(T_{w,\mathbb{R}}M)^{1,0}$ and $(T_{w,\mathbb{R}}M)^{0,1}$.

Complex differential

$$\mathrm{d}z_j := \mathrm{d}x_j + i\,\mathrm{d}y_j \quad \text{and} \quad \mathrm{d}\bar{z}_j := \mathrm{d}x_j - i\,\mathrm{d}y_j$$

for $j = 1, \ldots, n$.

We can say that $\left\{\mathrm{d}z_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{1,0}$ and $\left\{\mathrm{d}\bar{z}_j\big|_w\right\}_{j=1}^n$ is a basis for the complex vector space $(T^*_{w,\mathbb{R}}M)^{0,1}$. The following forms a basis of $T^*_{w,\mathbb{C}}M$

$$\left\{\mathrm{d}z_1\big|_w, \ldots, \mathrm{d}z_n\big|_w, \mathrm{d}\bar{z}_1\big|_w, \ldots, \mathrm{d}\bar{z}_n\big|_w\right\}$$

Differential forms

Differential k-form

Differential k-form

Let M be a complex manifold of dimension n. The smooth sections of rank $\binom{n}{k}$ complex vector bundle $\bigwedge^k T^*_{\mathbb{C}}M$ are called *differential k-forms* on M. The space of all k-forms on M is denoted by $\Omega^k_{\mathbb{C}}(M)$.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then any element $\omega \in \Omega^1_{\mathbb{C}}(U)$ can be written uniquely as

$$\omega = \sum_{j=1}^{n} f_j \, \mathrm{d}z_j + \sum_{k=1}^{n} g_k \, \mathrm{d}\bar{z}_k$$

where f_j, g_k are complex valued smooth functions.
Also, if $\omega \in \Omega^r_{\mathbb{C}}(U)$ and $\eta \in \Omega^s_{\mathbb{C}}(U)$ then

$$\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega \in \Omega^{r+s}_{\mathbb{C}}(U)$$

Differential k-form

Differential k-form

Let M be a complex manifold of dimension n. The smooth sections of rank $\binom{n}{k}$ complex vector bundle $\bigwedge^k T^*_{\mathbb{C}}M$ are called *differential k-forms* on M. The space of all k-forms on M is denoted by $\Omega^k_{\mathbb{C}}(M)$.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then any element $\omega \in \Omega^1_{\mathbb{C}}(U)$ can be written uniquely as

$$\omega = \sum_{j=1}^{n} f_j \, \mathrm{d}z_j + \sum_{k=1}^{n} g_k \, \mathrm{d}\overline{z}_k$$

where f_j, g_k are complex valued smooth functions.

Also, if $\omega \in \Omega^r_{\mathbb{C}}(U)$ and $\eta \in \Omega^s_{\mathbb{C}}(U)$ then

$$\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega \in \Omega^{r+s}_{\mathbb{C}}(U)$$

Differential k-form

Differential k-form

Let M be a complex manifold of dimension n. The smooth sections of rank $\binom{n}{k}$ complex vector bundle $\bigwedge^k T^*_{\mathbb{C}}M$ are called *differential k-forms* on M. The space of all k-forms on M is denoted by $\Omega^k_{\mathbb{C}}(M)$.

Let $(U,\phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then any element $\omega \in \Omega^1_{\mathbb{C}}(U)$ can be written uniquely as

$$\omega = \sum_{j=1}^{n} f_j \, \mathrm{d}z_j + \sum_{k=1}^{n} g_k \, \mathrm{d}\overline{z}_k$$

where f_j, g_k are complex valued smooth functions.
Also, if $\omega \in \Omega^r_{\mathbb{C}}(U)$ and $\eta \in \Omega^s_{\mathbb{C}}(U)$ then

$$\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega \in \Omega^{r+s}_{\mathbb{C}}(U)$$

Differential (p, q)-form

Differential (p, q)-form

Let M be a complex manifold of dimension n. We define the complex vector bundle of rank $\binom{n}{p}\binom{n}{q}$ over M as

$$\bigwedge^{p,q} T^*_{\mathbb{R}}M := \bigwedge^{p}\left((T^*_{\mathbb{R}}M)^{1,0}\right) \otimes_{\mathbb{C}} \bigwedge^{q}\left((T^*_{\mathbb{R}}M)^{0,1}\right)$$

whose fiber is $\bigwedge^{p,q} T^*_{w,\mathbb{R}}M$. The smooth sections of this vector bundle are called the *differential forms of type* (p, q) on M. The space of all (p, q)-forms on M is denoted by $\Omega^{p,q}(M)$.

Since $T^*_{\mathbb{C}}M = (T^*_{\mathbb{R}}M)^{1,0} \oplus (T^*_{\mathbb{R}}M)^{0,1}$ implies that

$$\bigwedge^{k}(T^*_{\mathbb{C}}M) \cong \bigoplus_{p+q=k} \bigwedge^{p}((T^*_{\mathbb{R}}M)^{1,0}) \otimes_{\mathbb{C}} \bigwedge^{q}((T^*_{\mathbb{R}}M)^{0,1}) = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*_{\mathbb{R}}M$$

We have

$$\Omega^k_{\mathbb{C}}(M) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

Differential (p, q)-form

Differential (p, q)-form

Let M be a complex manifold of dimension n. We define the complex vector bundle of rank $\binom{n}{p}\binom{n}{q}$ over M as

$$\bigwedge^{p,q} T^*_{\mathbb{R}}M := \bigwedge^{p}\left((T^*_{\mathbb{R}}M)^{1,0}\right) \otimes_{\mathbb{C}} \bigwedge^{q}\left((T^*_{\mathbb{R}}M)^{0,1}\right)$$

whose fiber is $\bigwedge^{p,q} T^*_{w,\mathbb{R}}M$. The smooth sections of this vector bundle are called the *differential forms of type* (p, q) on M. The space of all (p, q)-forms on M is denoted by $\Omega^{p,q}(M)$.

Since $T^*_{\mathbb{C}}M = (T^*_{\mathbb{R}}M)^{1,0} \oplus (T^*_{\mathbb{R}}M)^{0,1}$ implies that

$$\bigwedge^{k}(T^*_{\mathbb{C}}M) \cong \bigoplus_{p+q=k} \bigwedge^{p}((T^*_{\mathbb{R}}M)^{1,0}) \otimes_{\mathbb{C}} \bigwedge^{q}((T^*_{\mathbb{R}}M)^{0,1}) = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*_{\mathbb{R}}M$$

We have

$$\Omega^k_{\mathbb{C}}(M) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

Differential (p, q)-form

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then we define

$$\mathrm{d}z_\alpha := \mathrm{d}z_{\alpha_1} \wedge \ldots \wedge \mathrm{d}z_{\alpha_p}$$
$$\mathrm{d}\overline{z}_\beta := \mathrm{d}\overline{z}_{\beta_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{\beta_q}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_q)$ are multi-indices with $1 \leq \alpha_j, \beta_k \leq n$.

Proposition

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then $\omega \in \Omega^{p,q}(U)$ can be written uniquely as

$$\omega = \sum_{|\alpha|=p, |\beta|=q} f_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

where $f_{\alpha\beta}$ is a complex-valued smooth function on U, i.e. $f_{\alpha\beta} \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is smooth for all α, β.

Differential (p, q)-form

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then we define

$$\mathrm{d}z_\alpha := \mathrm{d}z_{\alpha_1} \wedge \ldots \wedge \mathrm{d}z_{\alpha_p}$$
$$\mathrm{d}\overline{z}_\beta := \mathrm{d}\overline{z}_{\beta_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{\beta_q}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_q)$ are multi-indices with $1 \leq \alpha_j, \beta_k \leq n$.

Proposition

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then $\omega \in \Omega^{p,q}(U)$ can be written uniquely as

$$\omega = \sum_{|\alpha|=p, |\beta|=q} f_{\alpha\beta} \, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

where $f_{\alpha\beta}$ is a complex-valued smooth function on U, i.e. $f_{\alpha\beta} \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is smooth for all α, β.

Exterior derivative

Differential of a k-form

$\mathrm{d} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$ is the complex linear extension of the usual exterior differential.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then for any $f \in \Omega^0_{\mathbb{C}}(U) = C^\infty(U)$ we have

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\,\mathrm{d}y_j = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\,\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\,\mathrm{d}\overline{z}_j$$

In general, if $\omega = \sum_{|\alpha|+|\beta|=k} f_{\alpha\beta}\,\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^k(U)$, we have

$$\begin{aligned} \mathrm{d} : \Omega^k(U) &\to \Omega^{k+1}(U) \\ \omega &\mapsto \sum_{|\alpha|+|\beta|=k} \mathrm{d}f_{\alpha\beta} \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \end{aligned}$$

Exterior derivative

Differential of a k-form

$\mathrm{d} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$ is the complex linear extension of the usual exterior differential.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then for any $f \in \Omega^0_{\mathbb{C}}(U) = C^\infty(U)$ we have

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\,\mathrm{d}y_j = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\,\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\,\mathrm{d}\overline{z}_j$$

In general, if $\omega = \sum_{|\alpha|+|\beta|=k} f_{\alpha\beta}\,\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^k(U)$, we have

$$\begin{aligned} \mathrm{d} : \Omega^k(U) &\to \Omega^{k+1}(U) \\ \omega &\mapsto \sum_{|\alpha|+|\beta|=k} \mathrm{d}f_{\alpha\beta} \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \end{aligned}$$

Exterior derivative

Differential of a k-form

$\mathrm{d} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$ is the complex linear extension of the usual exterior differential.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then for any $f \in \Omega^0_{\mathbb{C}}(U) = C^\infty(U)$ we have

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\,\mathrm{d}y_j = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\,\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\,\mathrm{d}\overline{z}_j$$

In general, if $\omega = \sum_{|\alpha|+|\beta|=k} f_{\alpha\beta}\,\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^k(U)$, we have

$$\begin{aligned} \mathrm{d} : \Omega^k(U) &\to \Omega^{k+1}(U) \\ \omega &\mapsto \sum_{|\alpha|+|\beta|=k} \mathrm{d}f_{\alpha\beta} \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \end{aligned}$$

Exterior derivative

Note that for $p+q=k$ we have natural projection operators

$$\Pi^{p,q} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{p,q}(M)$$

Differential of a (p,q)-form

We define $\partial := \Pi^{p+1,q} \circ \mathrm{d}$ and $\overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}$ as

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

Let $(U,\phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then given $\omega = \sum_{\alpha,\beta} f_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^{p,q}(U)$, we have

$$\partial\omega = \sum_{j=1}^{n} \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial z_j}\, \mathrm{d}z_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

$$\overline{\partial}\omega = \sum_{j=1}^{n} \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \overline{z}_j}\, \mathrm{d}\overline{z}_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

Exterior derivative

Note that for $p+q=k$ we have natural projection operators

$$\Pi^{p,q} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{p,q}(M)$$

Differential of a (p,q)-form

We define $\partial := \Pi^{p+1,q} \circ \mathrm{d}$ and $\overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}$ as

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

Let $(U,\phi) = (U, z_1, \dots, z_n)$ be a coordinate chart on M, then given $\omega = \sum_{\alpha,\beta} f_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^{p,q}(U)$, we have

$$\partial\omega = \sum_{j=1}^{n} \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial z_j}\, \mathrm{d}z_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

$$\overline{\partial}\omega = \sum_{j=1}^{n} \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \overline{z}_j}\, \mathrm{d}\overline{z}_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

Exterior derivative

Note that for $p+q=k$ we have natural projection operators

$$\Pi^{p,q} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{p,q}(M)$$

Differential of a (p,q)-form

We define $\partial := \Pi^{p+1,q} \circ \mathrm{d}$ and $\overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}$ as

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

Let $(U,\phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then given $\omega = \sum_{\alpha,\beta} f_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta \in \Omega^{p,q}(U)$, we have

$$\partial\omega = \sum_{j=1}^{n} \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial z_j}\, \mathrm{d}z_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

$$\overline{\partial}\omega = \sum_{j=1}^{n} \sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \overline{z}_j}\, \mathrm{d}\overline{z}_j \wedge \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\beta$$

Exterior derivative

Proposition

The differential operators ∂ and $\overline{\partial}$ satisfy the following properties:

1. $\mathrm{d} = \partial + \overline{\partial}$
2. $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$
3. Leibniz's rule, i.e.

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$
$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$$

for $\omega \in \Omega^{p,q}(M)$ and $\eta \in \Omega^{r,s}(M)$.

Exterior derivative

Proposition

The differential operators ∂ and $\overline{\partial}$ satisfy the following properties:

1. $\mathrm{d} = \partial + \overline{\partial}$
2. $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$
3. Leibniz's rule, i.e.

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$
$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$$

for $\omega \in \Omega^{p,q}(M)$ and $\eta \in \Omega^{r,s}(M)$.

Exterior derivative

Proposition

The differential operators ∂ and $\overline{\partial}$ satisfy the following properties:

1. $\mathrm{d} = \partial + \overline{\partial}$
2. $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$
3. Leibniz's rule, i.e.

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q}\omega \wedge \partial\eta$$
$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$$

for $\omega \in \Omega^{p,q}(M)$ and $\eta \in \Omega^{r,s}(M)$.

Dolbeault cohomology

$\overline{\partial}$-closed forms

$\overline{\partial}$-closed forms

Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$ is called $\overline{\partial}$-closed if $\overline{\partial}\omega = 0$. The space of all $\overline{\partial}$-closed (p,q)-forms on M is denoted by $\mathcal{Z}^{p,q}(M)$.

Let $(U,\phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then we can write the elements of $\omega \in \mathcal{Z}^{p,0}(U)$ is terms of local coordinates as:

$$\omega = \sum_{|\alpha|=p} f_\alpha \, dz_\alpha \quad \text{such that } \frac{\partial f_\alpha}{\partial \overline{z}_j} = 0 \quad \text{for all } \alpha, j$$

That is, $\mathcal{Z}^{p,0}(M)$ is the space of $(p,0)$-forms whose coefficients are complex-valued holomorphic functions on M.

Holomorphic p-form

We define $\mathcal{Z}^{p,0}(M)$ to be the space of *holomorphic p-forms* on M,and denote it by $\mathcal{O}^p(M)$. In particular, $\mathcal{Z}^{0,0}(M) = \mathcal{O}(M)$, the space of complex-valued functions holomorphic on M.

$\overline{\partial}$-closed forms

$\overline{\partial}$-closed forms

Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$ is called $\overline{\partial}$-closed if $\overline{\partial}\omega = 0$. The space of all $\overline{\partial}$-closed (p,q)-forms on M is denoted by $\mathcal{Z}^{p,q}(M)$.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then we can write the elements of $\omega \in \mathcal{Z}^{p,0}(U)$ is terms of local coordinates as:

$$\omega = \sum_{|\alpha|=p} f_\alpha \, dz_\alpha \quad \text{such that} \quad \frac{\partial f_\alpha}{\partial \overline{z}_j} = 0 \quad \text{for all } \alpha, j$$

That is, $\mathcal{Z}^{p,0}(M)$ is the space of $(p,0)$-forms whose coefficients are complex-valued holomorphic functions on M.

Holomorphic p-form

We define $\mathcal{Z}^{p,0}(M)$ to be the space of *holomorphic p-forms* on M,and denote it by $\mathcal{O}^p(M)$. In particular, $\mathcal{Z}^{0,0}(M) = \mathcal{O}(M)$, the space of complex-valued functions holomorphic on M.

$\overline{\partial}$-closed forms

$\overline{\partial}$-closed forms

Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$ is called $\overline{\partial}$-closed if $\overline{\partial}\omega = 0$. The space of all $\overline{\partial}$-closed (p,q)-forms on M is denoted by $\mathcal{Z}^{p,q}(M)$.

Let $(U, \phi) = (U, z_1, \ldots, z_n)$ be a coordinate chart on M, then we can write the elements of $\omega \in \mathcal{Z}^{p,0}(U)$ is terms of local coordinates as:

$$\omega = \sum_{|\alpha|=p} f_\alpha \, dz_\alpha \quad \text{such that } \frac{\partial f_\alpha}{\partial \overline{z}_j} = 0 \quad \text{for all } \alpha, j$$

That is, $\mathcal{Z}^{p,0}(M)$ is the space of $(p,0)$-forms whose coefficients are complex-valued holomorphic functions on M.

Holomorphic p-form

We define $\mathcal{Z}^{p,0}(M)$ to be the space of *holomorphic p-forms* on M,and denote it by $\mathcal{O}^p(M)$. In particular, $\mathcal{Z}^{0,0}(M) = \mathcal{O}(M)$, the space of complex-valued functions holomorphic on M.

$\overline{\partial}$-exact forms

$\overline{\partial}$-exact forms

Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$, for $q > 0$, is called $\overline{\partial}$-exact if $\omega = \overline{\partial}\eta$ for some differential form $\eta \in \Omega^{p,q-1}(M)$. The space of all $\overline{\partial}$-exact (p,q)-forms on M is denoted by $\mathcal{B}^{p,q}(M)$.

The trivial form $\omega \equiv 0$ is the only $(p,0)$-form which is $\overline{\partial}$-exact for any value of $p = 0, 1, \ldots, n$. That is, $\mathcal{B}^{p,0}(M)$ consists only of zero.

Proposition

On a complex manifold M, every $\overline{\partial}$-exact form is $\overline{\partial}$-closed.

Proof: Let M be an complex manifold and $\omega \in \mathcal{B}^{p,q}(M)$ such that $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{p,q-1}(M)$. We know that $\overline{\partial}\omega = \overline{\partial}(\overline{\partial}\eta) = 0$ hence $\omega \in \mathcal{Z}^{p,q}(M)$ for all $q \geq 1$. For $q = 0$, the statement is trivially true.

$\overline{\partial}$-exact forms

$\overline{\partial}$-exact forms

Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$, for $q > 0$, is called $\overline{\partial}$-exact if $\omega = \overline{\partial}\eta$ for some differential form $\eta \in \Omega^{p,q-1}(M)$. The space of all $\overline{\partial}$-exact (p,q)-forms on M is denoted by $\mathcal{B}^{p,q}(M)$.

The trivial form $\omega \equiv 0$ is the only $(p,0)$-form which is $\overline{\partial}$-exact for any value of $p = 0, 1, \ldots, n$. That is, $\mathcal{B}^{p,0}(M)$ consists only of zero.

Proposition

On a complex manifold M, every $\overline{\partial}$-exact form is $\overline{\partial}$-closed.

Proof: Let M be an complex manifold and $\omega \in \mathcal{B}^{p,q}(M)$ such that $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{p,q-1}(M)$. We know that $\overline{\partial}\omega = \overline{\partial}(\overline{\partial}\eta) = 0$ hence $\omega \in \mathcal{Z}^{p,q}(M)$ for all $q \geq 1$. For $q = 0$, the statement is trivially true.

$\overline{\partial}$-exact forms

$\overline{\partial}$-exact forms

Let M be a complex manifold. Then a differential form $\omega \in \Omega^{p,q}(M)$, for $q > 0$, is called $\overline{\partial}$-exact if $\omega = \overline{\partial}\eta$ for some differential form $\eta \in \Omega^{p,q-1}(M)$. The space of all $\overline{\partial}$-exact (p,q)-forms on M is denoted by $\mathcal{B}^{p,q}(M)$.

The trivial form $\omega \equiv 0$ is the only $(p,0)$-form which is $\overline{\partial}$-exact for any value of $p = 0, 1, \ldots, n$. That is, $\mathcal{B}^{p,0}(M)$ consists only of zero.

Proposition

On a complex manifold M, every $\overline{\partial}$-exact form is $\overline{\partial}$-closed.

Proof: Let M be an complex manifold and $\omega \in \mathcal{B}^{p,q}(M)$ such that $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{p,q-1}(M)$. We know that $\overline{\partial}\omega = \overline{\partial}(\overline{\partial}\eta) = 0$ hence $\omega \in \mathcal{Z}^{p,q}(M)$ for all $q \geq 1$. For $q = 0$, the statement is trivially true.

Dolbeault cohomology

Dolbeault cohomology of a complex manifold

The $(p,q)^{th}$ *Dolbeault cohomology group* of a complex manifold M is the quotient group

$$H_{\overline{\partial}}^{p,q}(M) := \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$

Hence, the Dolbeault cohomology of a complex manifold measures the extent to which $\overline{\partial}$-closed forms are not $\overline{\partial}$-exact on that manifold.

Proposition

If M is a complex manifold of dimension n then

1. $H_{\overline{\partial}}^{p,0}(M) = \mathcal{Z}^{p,0}(M) = \mathcal{O}^{p}(M)$
2. $H_{\overline{\partial}}^{p,q}(M) = 0$ for $q > n$

Dolbeault cohomology

Dolbeault cohomology of a complex manifold

The $(p,q)^{th}$ *Dolbeault cohomology group* of a complex manifold M is the quotient group

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$

Hence, the Dolbeault cohomology of a complex manifold measures the extent to which $\overline{\partial}$-closed forms are not $\overline{\partial}$-exact on that manifold.

Proposition

If M is a complex manifold of dimension n then

1. $H^{p,0}_{\overline{\partial}}(M) = \mathcal{Z}^{p,0}(M) = \mathcal{O}^p(M)$
2. $H^{p,q}_{\overline{\partial}}(M) = 0$ for $q > n$

Dolbeault cohomology

Dolbeault cohomology of a complex manifold

The $(p,q)^{th}$ *Dolbeault cohomology group* of a complex manifold M is the quotient group

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\mathcal{Z}^{p,q}(M)}{\mathcal{B}^{p,q}(M)}$$

Hence, the Dolbeault cohomology of a complex manifold measures the extent to which $\overline{\partial}$-closed forms are not $\overline{\partial}$-exact on that manifold.

Proposition

If M is a complex manifold of dimension n then

1. $H^{p,0}_{\overline{\partial}}(M) = \mathcal{Z}^{p,0}(M) = \mathcal{O}^p(M)$
2. $H^{p,q}_{\overline{\partial}}(M) = 0$ for $q > n$

$H_{\overline{\partial}}^{p,q}(\mathbb{C}) = 0$ for $q \geq 1$

$\overline{\partial}$-Poincaré lemma in one variable

If U is any open subset of $\mathbb{C}$ and $f \in C^\infty(U)$, then there exists $g \in C^\infty(U)$ such that $\dfrac{\partial g}{\partial \overline{z}} = f$.

If we consider $\omega = f\, \mathrm{d}\overline{z} \in \Omega^{0,1}(U) = \mathcal{Z}^{0,1}(U)$ for some open set $U \subset \mathbb{C}$, then the lemma implies that there exists $g \in \Omega^{0,0}(U)$ such that $\omega = \overline{\partial} g$. In particular, $H_{\overline{\partial}}^{0,1}(U) = 0$ for $U \subset \mathbb{C}$.

Similarly, for any $p \geq 0$ we will get $H_{\overline{\partial}}^{p,1}(U) = 0$ for $U \subset \mathbb{C}$.

Corollary

If U is any open subset of $\mathbb{C}$, then $H_{\overline{\partial}}^{p,q}(U) = 0$ for $q \geq 1$.

$H^{p,q}_{\overline{\partial}}(\mathbb{C}) = 0$ for $q \geq 1$

$\overline{\partial}$-Poincaré lemma in one variable

If U is any open subset of $\mathbb{C}$ and $f \in C^\infty(U)$, then there exists $g \in C^\infty(U)$ such that $\dfrac{\partial g}{\partial \overline{z}} = f$.

If we consider $\omega = f\,\mathrm{d}\overline{z} \in \Omega^{0,1}(U) = \mathcal{Z}^{0,1}(U)$ for some open set $U \subset \mathbb{C}$, then the lemma implies that there exists $g \in \Omega^{0,0}(U)$ such that $\omega = \overline{\partial} g$. In particular, $H^{0,1}_{\overline{\partial}}(U) = 0$ for $U \subset \mathbb{C}$.

Similarly, for any $p \geq 0$ we will get $H^{p,1}_{\overline{\partial}}(U) = 0$ for $U \subset \mathbb{C}$.

Corollary

If U is any open subset of $\mathbb{C}$, then $H^{p,q}_{\overline{\partial}}(U) = 0$ for $q \geq 1$.

$H^{p,q}_{\overline{\partial}}(\mathbb{C}) = 0$ for $q \geq 1$

$\overline{\partial}$-Poincaré lemma in one variable

If U is any open subset of $\mathbb{C}$ and $f \in C^\infty(U)$, then there exists $g \in C^\infty(U)$ such that $\dfrac{\partial g}{\partial \overline{z}} = f$.

If we consider $\omega = f\, d\overline{z} \in \Omega^{0,1}(U) = \mathcal{Z}^{0,1}(U)$ for some open set $U \subset \mathbb{C}$, then the lemma implies that there exists $g \in \Omega^{0,0}(U)$ such that $\omega = \overline{\partial} g$. In particular, $H^{0,1}_{\overline{\partial}}(U) = 0$ for $U \subset \mathbb{C}$.

Similarly, for any $p \geq 0$ we will get $H^{p,1}_{\overline{\partial}}(U) = 0$ for $U \subset \mathbb{C}$.

Corollary

If U is any open subset of $\mathbb{C}$, then $H^{p,q}_{\overline{\partial}}(U) = 0$ for $q \geq 1$.

Relationship with single variable complex analysis

To prove the $\overline{\partial}$-Poincaré lemma in one variable we use the *generalized Cauchy integral formula* for any point $z \in U$:

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} f(w)\frac{\mathrm{d}w}{w-z} + \frac{1}{2\pi i}\iint_{U} \frac{\partial f(w)}{\partial \overline{w}}\frac{\mathrm{d}w\wedge \mathrm{d}\overline{w}}{w-z}$$

where U be a region in $\mathbb{C}$ bounded by a simple closed rectifiable curve γ, and f be complex-valued smooth function in some open neighborhood V of $\overline{U}$.

If $f \in \mathcal{O}(U)$ then we get the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} f(w)\frac{\mathrm{d}w}{w-z}$$

Using this we can prove that

Proposition

If $U \subset \mathbb{C}$ is simply connected domain and $f : U \to \mathbb{C}$ is holomorphic, then f has a primitive in U.

Relationship with single variable complex analysis

To prove the $\overline{\partial}$-Poincaré lemma in one variable we use the *generalized Cauchy integral formula* for any point $z \in U$:

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} f(w)\frac{dw}{w-z} + \frac{1}{2\pi i}\iint_{U} \frac{\partial f(w)}{\partial \overline{w}}\frac{dw \wedge d\overline{w}}{w-z}$$

where U be a region in $\mathbb{C}$ bounded by a simple closed rectifiable curve γ, and f be complex-valued smooth function in some open neighborhood V of $\overline{U}$.
If $f \in \mathcal{O}(U)$ then we get the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} f(w)\frac{dw}{w-z}$$

Using this we can prove that

Proposition

If $U \subset \mathbb{C}$ is simply connected domain and $f : U \to \mathbb{C}$ is holomorphic, then f has a primitive in U.

$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q \geq 1$

$\overline{\partial}$-Poincaré lemma for $\mathbb{C}^n$

Let Δ be an open polydisc in the space $\mathbb{C}^n$, not necessarily having a compact closure, and $\omega \in \Omega^{p,q}(\Delta)$. If $q > 0$ and $\overline{\partial}\omega = 0$, then there is $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega = \overline{\partial}\eta$.

Since the open polydisc need not be bounded, we can put $\Delta = \mathbb{C}^n$ to get

Corollary

$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q \geq 1$.

Due to the lack of purely topological or intrinsic analytical description of the domains in $\mathbb{C}^n$ for $n \geq 2$ on which approximation theorems (like Runge's theorem) hold, we can't prove this lemma for general domains.

$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q \geq 1$

$\overline{\partial}$-Poincaré lemma for $\mathbb{C}^n$

Let Δ be an open polydisc in the space $\mathbb{C}^n$, not necessarily having a compact closure, and $\omega \in \Omega^{p,q}(\Delta)$. If $q > 0$ and $\overline{\partial}\omega = 0$, then there is $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega = \overline{\partial}\eta$.

Since the open polydisc need not be bounded, we can put $\Delta = \mathbb{C}^n$ to get

Corollary

$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q \geq 1$.

Due to the lack of purely topological or intrinsic analytical description of the domains in $\mathbb{C}^n$ for $n \geq 2$ on which approximation theorems (like Runge's theorem) hold, we can't prove this lemma for general domains.

$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q \geq 1$

$\overline{\partial}$-Poincaré lemma for $\mathbb{C}^n$

Let Δ be an open polydisc in the space $\mathbb{C}^n$, not necessarily having a compact closure, and $\omega \in \Omega^{p,q}(\Delta)$. If $q > 0$ and $\overline{\partial}\omega = 0$, then there is $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega = \overline{\partial}\eta$.

Since the open polydisc need not be bounded, we can put $\Delta = \mathbb{C}^n$ to get

Corollary

$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q \geq 1$.

Due to the lack of purely topological or intrinsic analytical description of the domains in $\mathbb{C}^n$ for $n \geq 2$ on which approximation theorems (like Runge's theorem) hold, we can't prove this lemma for general domains.

Relationship with Poincaré lemma

Last semester we proved that:

Poincaré lemma

Let U be a star-convex open set in $\mathbb{R}^n$. If $k \geq 1$, then every closed k-form on U is exact.

Unlike the Poincaré lemma, there isn't a simple topological condition on the domain which will ensure that the $\overline{\partial}$-closed forms are also $\overline{\partial}$-exact.

For $n = 1$ Poincaré lemma is equivalent to the Fundamental Theorem of Calculus, i.e. the existence of antiderviative of smooth functions defined on open sets in $\mathbb{R}$.

Relationship with Poincaré lemma

Last semester we proved that:

Poincaré lemma

Let U be a star-convex open set in $\mathbb{R}^n$. If $k \geq 1$, then every closed k-form on U is exact.

Unlike the Poincaré lemma, there isn't a simple topological condition on the domain which will ensure that the $\overline{\partial}$-closed forms are also $\overline{\partial}$-exact.

For $n = 1$ Poincaré lemma is equivalent to the Fundamental Theorem of Calculus, i.e. the existence of antiderviative of smooth functions defined on open sets in $\mathbb{R}$.

Dolbeault-Čech isomorphism

Exact sequence of sheaves

Using the invariance of Dolbeault cohomology for biholomorphic manifolds, we get that

$\overline{\partial}$-Poincaré lemma for M

If M be a complex manifold, then for all $w \in M$ there exists an open neighborhood U such that every $\overline{\partial}$-closed (p,q)-form on U is $\overline{\partial}$-exact for $q \geq 1$.

Recall that the smooth sections of the exterior power of a vector bundle, i.e. smooth maps of manifolds, form a sheaf. In particular, $\Omega^{p,q}$ is the sheaf of (p,q)-forms on M. Also, since $\overline{\partial}: \Omega^{p,q} \to \Omega^{p,q+1}$ is a map of sheaves, $\ker(\overline{\partial}) = \mathcal{Z}^{p,q}$ is a sheaf.

Corollary

The following is an exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

Exact sequence of sheaves

Using the invariance of Dolbeault cohomology for biholomorphic manifolds, we get that

$\overline{\partial}$-Poincaré lemma for M

If M be a complex manifold, then for all $w \in M$ there exists an open neighborhood U such that every $\overline{\partial}$-closed (p, q)-form on U is $\overline{\partial}$-exact for $q \geq 1$.

Recall that the smooth sections of the exterior power of a vector bundle, i.e. smooth maps of manifolds, form a sheaf. In particular, $\Omega^{p,q}$ is the sheaf of (p, q)-forms on M. Also, since $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$ is a map of sheaves, $\ker(\overline{\partial}) = \mathcal{Z}^{p,q}$ is a sheaf.

Corollary

The following is an exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

Exact sequence of sheaves

Using the invariance of Dolbeault cohomology for biholomorphic manifolds, we get that

$\overline{\partial}$-Poincaré lemma for M

If M be a complex manifold, then for all $w \in M$ there exists an open neighborhood U such that every $\overline{\partial}$-closed (p, q)-form on U is $\overline{\partial}$-exact for $q \geq 1$.

Recall that the smooth sections of the exterior power of a vector bundle, i.e. smooth maps of manifolds, form a sheaf. In particular, $\Omega^{p,q}$ is the sheaf of (p, q)-forms on M. Also, since $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$ is a map of sheaves, $\ker(\overline{\partial}) = \mathcal{Z}^{p,q}$ is a sheaf.

Corollary

The following is an exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

Dolbeault theorem

Dolbeault-Čech isomorphism

Let M be a complex manifold. Then for each $p, q \geq 0$ there exists a group isomorphism

$$H^{p,q}_{\bar{\partial}}(M) \cong \check{\mathrm{H}}^q(M, \mathcal{O}^p)$$

Proof outline: For $q = 0$, we know that both $H^{p,0}_{\bar{\partial}}(M)$ and $\check{\mathrm{H}}^0(M, \mathcal{O}^p)$ are isomorphic to the group of holomorphic p-forms on M. That is

$$\boxed{H^{p,0}_{\bar{\partial}}(M) \cong \check{\mathrm{H}}^0(M, \mathcal{O}^p)}$$

Dolbeault theorem

Dolbeault-Čech isomorphism

Let M be a complex manifold. Then for each $p, q \geq 0$ there exists a group isomorphism

$$H_{\overline{\partial}}^{p,q}(M) \cong \check{\mathrm{H}}^{q}(M, \mathcal{O}^{p})$$

Proof outline: For $q = 0$, we know that both $H_{\overline{\partial}}^{p,0}(M)$ and $\check{\mathrm{H}}^{0}(M, \mathcal{O}^{p})$ are isomorphic to the group of holomorphic p-forms on M. That is

$$\boxed{H_{\overline{\partial}}^{p,0}(M) \cong \check{\mathrm{H}}^{0}(M, \mathcal{O}^{p})}$$

Short exact sequence of differential forms

Now let's restrict our attention to $q \geq 1$. Consider the following long exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

In particular, we have a family of short exact sequence of sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^p & \hookrightarrow & \Omega^{p,0} & \xrightarrow{\overline{\partial}} & \mathcal{Z}^{p,1} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{Z}^{p,1} & \hookrightarrow & \Omega^{p,1} & \xrightarrow{\overline{\partial}} & \mathcal{Z}^{p,2} & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & \mathcal{Z}^{p,\ell} & \hookrightarrow & \Omega^{p,\ell} & \xrightarrow{\overline{\partial}} & \mathcal{Z}^{p,\ell+1} & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Short exact sequence of differential forms

Now let's restrict our attention to $q \geq 1$. Consider the following long exact sequence of sheaves of differential forms

$$0 \longrightarrow \mathcal{O}^p \hookrightarrow \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \Omega^{p,2} \xrightarrow{\bar{\partial}} \cdots$$

In particular, we have a family of short exact sequence of sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^p & \hookrightarrow & \Omega^{p,0} & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,1} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{Z}^{p,1} & \hookrightarrow & \Omega^{p,1} & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,2} & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & \mathcal{Z}^{p,\ell} & \hookrightarrow & \Omega^{p,\ell} & \xrightarrow{\bar{\partial}} & \mathcal{Z}^{p,\ell+1} & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Exact sequence of Čech cohomology

Since complex manifolds are paracompact, each short exact sequence of sheaves will induce a long exact sequences of Čech cohomology

$$\begin{array}{ccccccccc}
\cdots \to & \check{\mathrm{H}}^q(M,\Omega^{p,0}) & \longrightarrow & \check{\mathrm{H}}^q(M,\mathcal{Z}^{p,1}) & \xrightarrow{\Delta} & \check{\mathrm{H}}^{q+1}(M,\mathcal{O}^p) & \longrightarrow \cdots \\
\cdots \to & \check{\mathrm{H}}^q(M,\Omega^{p,1}) & \longrightarrow & \check{\mathrm{H}}^q(M,\mathcal{Z}^{p,2}) & \xrightarrow{\Delta} & \check{\mathrm{H}}^{q+1}(M,\mathcal{Z}^{p,1}) & \to \cdots \\
& \vdots & & \vdots & & \vdots & \vdots \\
\cdots \to & \check{\mathrm{H}}^q(M,\Omega^{p,\ell}) & \to & \check{\mathrm{H}}^q(M,\mathcal{Z}^{p,\ell+1}) & \xrightarrow{\Delta} & \check{\mathrm{H}}^{q+1}(M,\mathcal{Z}^{p,\ell}) & \to \cdots \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

Fine sheaves

Now let's study one of these long exact sequence of Čech cohomology. Firstly, note that

$$\check{\mathrm{H}}^0(M, \Omega^{p,\ell}) \cong \Omega^{p,\ell}(M) \text{ and } \check{\mathrm{H}}^0(M, \mathcal{Z}^{p,\ell}) \cong \mathcal{Z}^{p,\ell}(M)$$

Also, for $p, \ell \geq 0$, $\Omega^{p,\ell}$ are smooth sections of vector bundles and hence are fine sheaves. Therefore, we have $\check{\mathrm{H}}^q(M, \Omega^{p,\ell}) = 0$ for all $\ell \geq 1$. Hence for any $\ell \geq 0$ we get the exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0 \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell+1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \Delta \\ \cdots \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^3(M, \mathcal{Z}^{p,\ell}) \xleftarrow{\Delta} \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell+1}) \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell}) \end{array}$$

Fine sheaves

Now let's study one of these long exact sequence of Čech cohomology. Firstly, note that

$$\check{\mathrm{H}}^0(M, \Omega^{p,\ell}) \cong \Omega^{p,\ell}(M) \text{ and } \check{\mathrm{H}}^0(M, \mathcal{Z}^{p,\ell}) \cong \mathcal{Z}^{p,\ell}(M)$$

Also, for $p, \ell \geq 0$, $\Omega^{p,\ell}$ are smooth sections of vector bundles and hence are fine sheaves. Therefore, we have $\check{\mathrm{H}}^q(M, \Omega^{p,\ell}) = 0$ for all $\ell \geq 1$. Hence for any $\ell \geq 0$ we get the exact sequence

$$\begin{array}{c}
0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0 \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell+1}) \\
\downarrow \Delta \\
\cdots \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^3(M, \mathcal{Z}^{p,\ell}) \xleftarrow{\Delta} \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell+1}) \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell})
\end{array}$$

Fine sheaves

Now let's study one of these long exact sequence of Čech cohomology. Firstly, note that

$$\check{\mathrm{H}}^0(M, \Omega^{p,\ell}) \cong \Omega^{p,\ell}(M) \text{ and } \check{\mathrm{H}}^0(M, \mathcal{Z}^{p,\ell}) \cong \mathcal{Z}^{p,\ell}(M)$$

Also, for $p, \ell \geq 0$, $\Omega^{p,\ell}$ are smooth sections of vector bundles and hence are fine sheaves. Therefore, we have $\check{\mathrm{H}}^q(M, \Omega^{p,\ell}) = 0$ for all $\ell \geq 1$. Hence for any $\ell \geq 0$ we get the exact sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0 \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell+1})$$

$$\downarrow \Delta$$

$$\cdots \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^3(M, \mathcal{Z}^{p,\ell}) \xleftarrow{\Delta} \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell+1}) \longleftarrow 0 \longleftarrow \check{\mathrm{H}}^2(M, \mathcal{Z}^{p,\ell})$$

First isomorphism theorem

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

Since this sequence is exact, the map

$$\Delta : \mathcal{Z}^{p,\ell+1}(M) \to \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell})$$

is a surjective group homomorphism. Hence by the *first isomorphism theorem* we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong \frac{\mathcal{Z}^{p,\ell+1}(M)}{\ker(\Delta)} \quad \text{for all } \ell \geq 0$$

First isomorphism theorem

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

Since this sequence is exact, the map

$$\Delta : \mathcal{Z}^{p,\ell+1}(M) \to \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell})$$

is a surjective group homomorphism. Hence by the *first isomorphism theorem* we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong \frac{\mathcal{Z}^{p,\ell+1}(M)}{\ker(\Delta)} \quad \text{for all } \ell \geq 0$$

First isomorphism theorem

Now consider the following part of the above sequence

$$0 \longrightarrow \mathcal{Z}^{p,\ell}(M) \hookrightarrow \Omega^{p,\ell}(M) \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,\ell+1}(M) \xrightarrow{\Delta} \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

Since this sequence is exact, the map

$$\Delta : \mathcal{Z}^{p,\ell+1}(M) \to \check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell})$$

is a surjective group homomorphism. Hence by the *first isomorphism theorem* we get

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong \frac{\mathcal{Z}^{p,\ell+1}(M)}{\ker(\Delta)} \quad \text{for all } \ell \geq 0$$

$H^{p,1}_{\overline{\partial}}(M) \cong \check{\mathrm{H}}^1(M, \mathcal{O}^p)$

By the exactness of the sequence, we also get that

$$\begin{aligned}\ker(\Delta) &= \mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \mathcal{Z}^{p,\ell+1}(M)\}\\ &= \mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \Omega^{p,\ell+1}(M)\}\\ &= \mathcal{B}^{p,\ell+1}(M)\end{aligned}$$

Hence, we have

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong H^{p,\ell+1}_{\overline{\partial}}(M) \quad \text{for all } \ell \geq 0 \tag{1}$$

Note that $\mathcal{Z}^{p,0} = \mathcal{O}^p$, hence from (1) we get

$$\boxed{\check{\mathrm{H}}^1(M, \mathcal{O}^p) \cong H^{p,1}_{\overline{\partial}}(M)}$$

$H^{p,1}_{\overline{\partial}}(M) \cong \check{\mathrm{H}}^1(M, \mathcal{O}^p)$

By the exactness of the sequence, we also get that

$$\begin{aligned}\ker(\Delta) &= \operatorname{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \mathcal{Z}^{p,\ell+1}(M)\} \\ &= \operatorname{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \Omega^{p,\ell+1}(M)\} \\ &= \mathcal{B}^{p,\ell+1}(M)\end{aligned}$$

Hence, we have

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong H^{p,\ell+1}_{\overline{\partial}}(M) \quad \text{for all } \ell \geq 0 \tag{1}$$

Note that $\mathcal{Z}^{p,0} = \mathcal{O}^p$, hence from (1) we get

$$\boxed{\check{\mathrm{H}}^1(M, \mathcal{O}^p) \cong H^{p,1}_{\overline{\partial}}(M)}$$

$H^{p,1}_{\overline{\partial}}(M) \cong \check{\mathrm{H}}^1(M, \mathcal{O}^p)$

By the exactness of the sequence, we also get that

$$\begin{aligned}\ker(\Delta) &= \mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \mathcal{Z}^{p,\ell+1}(M)\} \\ &= \mathrm{im}\{\overline{\partial} : \Omega^{p,\ell}(M) \to \Omega^{p,\ell+1}(M)\} \\ &= \mathcal{B}^{p,\ell+1}(M)\end{aligned}$$

Hence, we have

$$\check{\mathrm{H}}^1(M, \mathcal{Z}^{p,\ell}) \cong H^{p,\ell+1}_{\overline{\partial}}(M) \quad \text{for all } \ell \geq 0 \tag{1}$$

Note that $\mathcal{Z}^{p,0} = \mathcal{O}^p$, hence from (1) we get

$$\boxed{\check{\mathrm{H}}^1(M, \mathcal{O}^p) \cong H^{p,1}_{\overline{\partial}}(M)}$$

$H^{p,q}_{\bar\partial}(M) \cong \check{H}^q(M, \mathcal{O}^p)$ for all $q \geq 2$

Next we consider the remaining parts of the long exact sequence, i.e. for $q \geq 1$ and $\ell \geq 0$ we have

$$0 \longrightarrow \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \cong \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \quad \text{for all } q \geq 1, \ell \geq 0 \qquad (2)$$

Again substituting $\mathcal{Z}^{p,0} = \mathcal{O}^p$ and restricting our attention to $q \geq 2$, we apply (2) recursively to get

$$\check{H}^q(M, \mathcal{O}^p) \cong \check{H}^{q-1}(M, \mathcal{Z}^{p,1}) \cong \cdots \cong \check{H}^1(M, \mathcal{Z}^{p,q-1})$$

Then using (1) we get

$$\boxed{\check{H}^q(M, \mathcal{O}^p) \cong H^{p,q}_{\bar\partial}(M)} \quad \text{for all } q \geq 2$$

Hence completing the proof.

$H^{p,q}_{\bar\partial}(M) \cong \check{H}^q(M, \mathcal{O}^p)$ for all $q \geq 2$

Next we consider the remaining parts of the long exact sequence, i.e. for $q \geq 1$ and $\ell \geq 0$ we have

$$0 \longrightarrow \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \cong \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \quad \text{for all } q \geq 1, \ell \geq 0 \tag{2}$$

Again substituting $\mathcal{Z}^{p,0} = \mathcal{O}^p$ and restricting our attention to $q \geq 2$, we apply (2) recursively to get

$$\check{H}^q(M, \mathcal{O}^p) \cong \check{H}^{q-1}(M, \mathcal{Z}^{p,1}) \cong \cdots \cong \check{H}^1(M, \mathcal{Z}^{p,q-1})$$

Then using (1) we get

$$\boxed{\check{H}^q(M, \mathcal{O}^p) \cong H^{p,q}_{\bar\partial}(M)} \quad \text{for all } q \geq 2$$

Hence completing the proof.

$H^{p,q}_{\bar\partial}(M) \cong \check{H}^q(M, \mathcal{O}^p)$ for all $q \geq 2$

Next we consider the remaining parts of the long exact sequence, i.e. for $q \geq 1$ and $\ell \geq 0$ we have

$$0 \longrightarrow \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \cong \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \quad \text{for all } q \geq 1, \ell \geq 0 \qquad (2)$$

Again substituting $\mathcal{Z}^{p,0} = \mathcal{O}^p$ and restricting our attention to $q \geq 2$, we apply (2) recursively to get

$$\check{H}^q(M, \mathcal{O}^p) \cong \check{H}^{q-1}(M, \mathcal{Z}^{p,1}) \cong \cdots \cong \check{H}^1(M, \mathcal{Z}^{p,q-1})$$

Then using (1) we get

$$\boxed{\check{H}^q(M, \mathcal{O}^p) \cong H^{p,q}_{\bar\partial}(M)} \quad \text{for all } q \geq 2$$

Hence completing the proof.

$H^{p,q}_{\bar\partial}(M) \cong \check{H}^q(M, \mathcal{O}^p)$ for all $q \geq 2$

Next we consider the remaining parts of the long exact sequence, i.e. for $q \geq 1$ and $\ell \geq 0$ we have

$$0 \longrightarrow \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \xrightarrow{\Delta} \check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \longrightarrow 0$$

The group homomorphism Δ is an isomorphism since this is an exact sequence of abelian groups

$$\check{H}^{q+1}(M, \mathcal{Z}^{p,\ell}) \cong \check{H}^q(M, \mathcal{Z}^{p,\ell+1}) \quad \text{for all } q \geq 1, \ell \geq 0 \qquad (2)$$

Again substituting $\mathcal{Z}^{p,0} = \mathcal{O}^p$ and restricting our attention to $q \geq 2$, we apply (2) recursively to get

$$\check{H}^q(M, \mathcal{O}^p) \cong \check{H}^{q-1}\left(M, \mathcal{Z}^{p,1}\right) \cong \cdots \cong \check{H}^1\left(M, \mathcal{Z}^{p,q-1}\right)$$

Then using (1) we get

$$\boxed{\check{H}^q(M, \mathcal{O}^p) \cong H^{p,q}_{\bar\partial}(M)} \quad \text{for all } q \geq 2$$

Hence completing the proof.

Exponential sheaf sequence

Proposition

$\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Proof: Consider the *exponential sheaf sequence* on $\mathbb{C}^n$

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

It induces a long exact sequence of cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

Next, we note that for all $q > 0$

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = 0 \Rightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) = 0$$

Since $\mathbb{C}^n$ is contractible and Čech cohomology of constant sheaves on smooth manifolds is a homotopy invariant, we get

$$\check{\mathrm{H}}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0 \text{ for } q > 0$$

Using these and the exact sequence we get $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Exponential sheaf sequence

Proposition

$\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Proof: Consider the *exponential sheaf sequence* on $\mathbb{C}^n$

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

It induces a long exact sequence of cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

Next, we note that for all $q > 0$

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = 0 \Rightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) = 0$$

Since $\mathbb{C}^n$ is contractible and Čech cohomology of constant sheaves on smooth manifolds is a homotopy invariant, we get

$$\check{\mathrm{H}}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0 \text{ for } q > 0$$

Using these and the exact sequence we get $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Exponential sheaf sequence

Proposition

$\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Proof: Consider the *exponential sheaf sequence* on $\mathbb{C}^n$

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

It induces a long exact sequence of cohomology

$$\cdots \longrightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{H}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{H}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

Next, we note that for all $q > 0$

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = 0 \Rightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}) = 0$$

Since $\mathbb{C}^n$ is contractible and Čech cohomology of constant sheaves on smooth manifolds is a homotopy invariant, we get

$$\check{H}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0 \text{ for } q > 0$$

Using these and the exact sequence we get $\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Exponential sheaf sequence

Proposition

$\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Proof: Consider the *exponential sheaf sequence* on $\mathbb{C}^n$

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

It induces a long exact sequence of cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

Next, we note that for all $q > 0$

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = 0 \Rightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) = 0$$

Since $\mathbb{C}^n$ is contractible and Čech cohomology of constant sheaves on smooth manifolds is a homotopy invariant, we get

$$\check{\mathrm{H}}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0 \text{ for } q > 0$$

Using these and the exact sequence we get $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Exponential sheaf sequence

Proposition

$\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Proof: Consider the *exponential sheaf sequence* on $\mathbb{C}^n$

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

It induces a long exact sequence of cohomology

$$\cdots \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{\mathrm{H}}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

Next, we note that for all $q > 0$

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = 0 \Rightarrow \check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}) = 0$$

Since $\mathbb{C}^n$ is contractible and Čech cohomology of constant sheaves on smooth manifolds is a homotopy invariant, we get

$$\check{\mathrm{H}}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0 \text{ for } q > 0$$

Using these and the exact sequence we get $\check{\mathrm{H}}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Exponential sheaf sequence

Proposition

$\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Proof: Consider the *exponential sheaf sequence* on $\mathbb{C}^n$

$$0 \longrightarrow \underline{\mathbb{Z}} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

It induces a long exact sequence of cohomology

$$\cdots \longrightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}) \longrightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}^*) \xrightarrow{\Delta} \check{H}^{q+1}(\mathbb{C}^n, \underline{\mathbb{Z}}) \longrightarrow \check{H}^{q+1}(\mathbb{C}^n, \mathcal{O}) \longrightarrow \cdots$$

Next, we note that for all $q > 0$

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = 0 \Rightarrow \check{H}^q(\mathbb{C}^n, \mathcal{O}) = 0$$

Since $\mathbb{C}^n$ is contractible and Čech cohomology of constant sheaves on smooth manifolds is a homotopy invariant, we get

$$\check{H}^q(\mathbb{C}^n, \underline{\mathbb{Z}}) = 0 \text{ for } q > 0$$

Using these and the exact sequence we get $\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$.

Analytic hypersurface

Analytic hypersurface

Let M be a n-dimensional complex manifold. An *analytic hypersurface* of M is a subset $H \subset M$ such that for every point $w \in M$ there exists an open neighborhood $w \in U \subset M$ and $f \in \mathcal{O}(U)$ such that $U \cap H = \{z \in U : f(z) = 0\}$.

In general, analytic hypersurface cannot be given by global function.

Cousin problem

Any analytic hypersurface in $\mathbb{C}^n$ is the zero locus of an entire function $f : \mathbb{C}^n \to \mathbb{C}$.

Proof outline: Since $\mathcal{O}_w$ is a UFD, we can choose an open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{C}^n$ and functions $h_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$U_\alpha \cap H = \{z \in U : h_\alpha(z) = 0\}$$

where h_α is not divisible by the square of any non-unit. Then use $\check{\mathrm{H}}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$ to get the desired global function.

Analytic hypersurface

Analytic hypersurface

Let M be a n-dimensional complex manifold. An *analytic hypersurface* of M is a subset $H \subset M$ such that for every point $w \in M$ there exists an open neighborhood $w \in U \subset M$ and $f \in \mathcal{O}(U)$ such that $U \cap H = \{z \in U : f(z) = 0\}$.

In general, analytic hypersurface cannot be given by global function.

Cousin problem

Any analytic hypersurface in $\mathbb{C}^n$ is the zero locus of an entire function $f : \mathbb{C}^n \to \mathbb{C}$.

Proof outline: Since $\mathcal{O}_w$ is a UFD, we can choose an open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{C}^n$ and functions $h_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$U_\alpha \cap H = \{z \in U : h_\alpha(z) = 0\}$$

where h_α is not divisible by the square of any non-unit. Then use $\check{\mathrm{H}}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$ to get the desired global function.

Analytic hypersurface

Analytic hypersurface

Let M be a n-dimensional complex manifold. An *analytic hypersurface* of M is a subset $H \subset M$ such that for every point $w \in M$ there exists an open neighborhood $w \in U \subset M$ and $f \in \mathcal{O}(U)$ such that $U \cap H = \{z \in U : f(z) = 0\}$.

In general, analytic hypersurface cannot be given by global function.

Cousin problem

Any analytic hypersurface in $\mathbb{C}^n$ is the zero locus of an entire function $f : \mathbb{C}^n \to \mathbb{C}$.

Proof outline: Since $\mathcal{O}_w$ is a UFD, we can choose an open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{C}^n$ and functions $h_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$U_\alpha \cap H = \{z \in U : h_\alpha(z) = 0\}$$

where h_α is not divisible by the square of any non-unit. Then use $\check{H}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$ to get the desired global function.

Analytic hypersurface

Analytic hypersurface

Let M be a n-dimensional complex manifold. An *analytic hypersurface* of M is a subset $H \subset M$ such that for every point $w \in M$ there exists an open neighborhood $w \in U \subset M$ and $f \in \mathcal{O}(U)$ such that $U \cap H = \{z \in U : f(z) = 0\}$.

In general, analytic hypersurface cannot be given by global function.

Cousin problem

Any analytic hypersurface in $\mathbb{C}^n$ is the zero locus of an entire function $f : \mathbb{C}^n \to \mathbb{C}$.

Proof outline: Since $\mathcal{O}_w$ is a UFD, we can choose an open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathbb{C}^n$ and functions $h_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$U_\alpha \cap H = \{z \in U : h_\alpha(z) = 0\}$$

where h_α is not divisible by the square of any non-unit. Then use $\check{\mathrm{H}}^1(\mathbb{C}^n, \mathcal{O}^*) = 0$ to get the desired global function.

References I

- Griffiths, P. and Harris, J. (1994). *Principles of Algebraic Geometry*. John Wiley & Sons, New York.
- Gunning, R. C. and Rossi, H. (1965). *Analytic Functions of Several Complex Variables*, volume 368 of AMS Chelsea Publishing. American Mathematical Society, Providence.
- Huybrechts, D. (2005). *Complex Geometry*. Universitext. Springer-Verlag, Berlin Heidelberg.
- Krantz, S. G. (1992). Function Theory of Several Complex Variables, volume 340 of AMS Chelsea Publishing. American Mathematical Society, Providence.
- Taylor, J. L. (2002). *Several Complex Variables with Connections to Algebraic Geometry and Lie Groups*, volume 46 of Graduate Studies in Mathematics. American Mathematical Society, Providence.

References II

- Warner, F. W. (1983). *Foundations of Differentiable Manifolds and Lie Groups*, volume 94 of Graduate Texts in Mathematics. Springer-Verlag, New York.
- Voisin, C. (2002). Hodge Theory and Complex Algebraic Geometry, I, volume 76 of Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge.
- Wells, R. O. (2008). Differential Analysis on Complex Manifolds, volume 65 of Graduate Texts in Mathematics. Springer-Verlag, New York.
- Zinger, A. (2015). Notes on Smooth Manifolds and Vector Bundles. `http://www.math.stonybrook.edu/~azinger/research/WarnerDG.pdf`. Last visited on 09 April 2019.