What is Abel's Theorem Anyway?

Steven L. Kleiman

Abstract. Supplementing other treatments, this article discusses the history and meaning of four theorems that have been accepted as Abel's Theorem. The discussion explains Abel's own proofs, and pays due attention to the hyperelliptic case. Section 1 explains how each theorem came to be called Abel's Theorem. Section 2 treats Abelian integrals, Clebsch's geometric reformulation, and Abel's Elementary Function Theorem. Section 3 treats Abel's Equivalence Theorem, the genus, and adjoints. Section 4 treats the Riemann–Roch Theorem and Abel's Relations Theorem. Section 5 explains the various forms of Abel's Addition Theorem and Abel's proofs of them. Section 6 discusses the Abel map, and uses it to prove the Addition Theorem in its most elaborate form. Section 7 discusses the Picard and Albanese varieties, and explains Abel's version of the genus. Section 8 sums it all up, and concludes that only the Addition Theorem can rightfully be called *Abel's Theorem*.

1 Why Four Theorems?

Did Abel prove just *one* theorem? The name *Abel's Theorem* has always referred to a theorem Abel proved about sums of integrals of algebraic functions – never mind, he was the first to prove the insolvability of the quintic by radicals, and the first to investigate the convergence of Newton's binomial formula with an arbitrary complex exponent (see [68], pp. 754, 964).

O.A. Laudal et al. (eds.), *The Legacy of Niels Henrik Abel*

However, four different theorems have been touted as the one and only Abel's Theorem. Why? Because Abel published three papers on the subject.

To make it easier to refer to the four, let us call them the Elementary Function Theorem, the Equivalence Theorem, the Relations Theorem, and the Addition Theorem. Their statements are explained informally later in this section, and more formally in Sects. 2, 3, 4, and 5.

Abel's first paper [1] treats the Addition Theorem in some detail, and proves versions of the other three theorems as preliminary results along the way. Abel submitted it in 1826, but it did not appear until 1841.

Meanwhile, virtually two years after Abel submitted the first paper, he feared it was lost forever, and so he wrote the second paper [2] (see [109], pp. 453, 469). In it, he summarized the Addition Theorem informally, then he treated it in a major special case, the case of hyperelliptic integrals.

The following January, Abel wrote the third paper [3]. In it, he proved only the Elementary Function Theorem, but in full generality. It is just a preliminary result, but Abel didn't say so. At the time, he was slowly dying of tuberculosis, and writing the paper was a strenuous effort (see [109], p. 482).

Abel's first paper is fairly long: sixty-one handwritten pages. Abel was proud of its quality, and hopeful it would secure his career. He finished it in Paris, and showed it to Cauchy. Alas, Cauchy merely glanced at it. Shortly afterwords, on October 30, Abel submitted it in person to the Royal Academy of Sciences, a part of the French Institute. Fourier, the secretary, read the introduction aloud, and the academy appointed Cauchy and Legendre as referees. (See [87], p. 56; see [12], p. 309; see [82], p. 150; see [109], p. 411.)

The treatment itself is actually elementary – "really only a wonderful exercise in the integral calculus," as Forsyth [47], p. 580, put it. However, the treatment is poorly organized and hard to read. Moreover, Gray [53], p. 365, observed: "it is occasionally wrong, which is more serious, and it operates at such a level of generality that it was and is not clear that matters are always as Abel described." And Kline [68], p. 655, explained that Abel's "paper was very difficult to understand, partly because he tried to prove what we would today call an existence theorem by actually computing the result." Also, Cooke [37], p. 411, felt that "the paper owes its great length to Abel's insecurity about the foundations of analysis, based on the bitter experience of errors he had fallen into by using insufficiently grounded methods."

So the paper was reworked, "in a shortened and simplified form," in 1881 by Rowe [97], in part on the basis of Boole's 1857 work in the same direction. However, in 1894, Brill and Noether [17], Fn., pp. 213–214, said that Rowe "did not cover all the difficult parts." Then, in 1900, Forsyth [47], pp. 579–601, gave an exposition, which was "adopted" from Rowe's, and further shortened and simplified by using the theory of Riemann surfaces.

At any rate, the details in Abel's paper have been less important than the concepts when others have developed the paper's mathematical content.

The first paper was not published until 1841. So it did not appear in the 1839 edition of Abel's collected works, which was "prepared at the request" of King Oscar I of Sweden and Norway, and paid for by the state through the Church Department, which handled education. The editor, Holmboe, had to ask the department for help in obtaining a copy of the paper, and a formal diplomatic request was sent to the French authorities. There was no reply, but the paper was published at last. (See [82], p. 258, and [109], pp. 548–549.)

Abel's original manuscript, in his handwriting, is preserved in Italy, except for pp. 21–24, which remain lost. The manuscript has had an interesting, but long and sad history. It was related by Ore [82], pp. 246–261, summarized by Stubhaug [109], pp. 549–551, and recently updated by Del Centina [41].

Del Centina edited the book [42] containing his update. The book also contains facsimiles of the manuscript, of the four missing pages in Libri's handwriting, and of Legendre's cover letter. (G. Libri was a professor of mathematics and member of the academy, which charged him with overseeing the printing in 1841; see [41], p. 91.) In addition, the book contains some other articles about Abel's life and work and about Florence's Moreniana Library, which houses fifty-three pages of the manuscript and Legendre's letter.

Legendre's letter is undated; likely, it was written in June 1829. Indeed, on June 4, Legendre learned of Abel's death. On June 29, Cauchy submitted the referees' report, which Legendre cosigned. The report was weak: it said most of the results had already appeared in print, but owing to Abel's great achievements and tragic death, publication was being recommended. The recommendation was accepted. (See [82], pp. 247 and 233.)

In the letter, Legendre wrote (see the transcription in [41] on p. 96) that he was the first to look at the manuscript, but found the script illegible and the symbols poorly formed. So he asked Cauchy to referee it. However, Cauchy was distracted by other matters, and forgot about the paper until March 1829 when a German scientist expressed his astonishment that it had not yet been refereed. Cauchy then found the manuscript, but the two of them were concerned since Abel had already published part of his work in Crelle's journal and would probably continue with the rest. They then learned of Abel's unfortunate death, which changed the situation.

But the facsimile shows perfectly legible script and well-formed symbols!

The German scientist was Jacobi. He was astonished by a footnote in Abel's second paper [2], p. 445. In the introduction, Abel said he would prove the Addition Theorem in full generality in a subsequent issue of the journal, but here he would treat a special case, which includes elliptic functions. (At the time, the term "elliptic function" referred to the integral, not, as later and now, its inverse.) In the footnote, Abel added sadly that he had presented a memoir on the general case to the Royal Academy toward the end of 1826. Legendre wrote to Jacobi about the paper in February 1929, and Jacobi replied, asking how come Abel's great discovery could have been ignored for so long. (See [41], p. 89, and [82], p. 246; also see [111], p. 402, about Steiner's minor role.)

In [2], Abel treated the hyperelliptic case. It is considerably simpler; indeed, the paper is one-fifth as long as the Paris memoir [1]. Yet, this case has the flavor of the general case. Moreover, "since the longer paper didn't appear until 1841," as Cooke [37], p. 411, pointed out, "it was primarily the hyperelliptic case that introduced Abel's new methods to the world."

Abel's treatment in [2] is computationally involved too. Indeed, eight pages of its twelve are devoted to obtaining, in the hyperelliptic case, a relatively explicit form of the key Elementary Function Theorem. Nevertheless, the degree of abstraction was strikingly novel. Indeed, in a letter to Gauss on 2 January 1829, Bessel wrote (as quoted in [82], p. 213): "I have been very surprised about Abel's theorem, which makes it possible to discover the properties of integrals without executing them. I believe an entirely new phase of the integral calculus has been brought to light. I wish that Abel would pursue it further in many applications. It appears unintelligible to me that such a theorem has been presented to the Academy in Paris, and yet remains unknown."

Abel's treatment was analyzed carefully both historically and mathematically by Brill and Noether [17], pp. 209–210, by Houzel [58], pp. 73–74, and by Cooke [37], pp. 413–418. In particular, Brill and Noether felt they found the "bridge that leads from Euler to Abel in a certain transformation that Abel applied." However, Cooke [37], p. 405, argued that a somewhat different route is more plausible.

In the introduction to [2], Abel put the **Addition Theorem** roughly this way: *the integrals of arbitrary algebraic differentials form a very large class of transcendental functions with an addition formula; if an arbitrary sum of functions arising from the same differential cannot be given by a single function of the same sort, as is the case for elliptic functions, at least the sum can be given by a sum of a specific number of them, plus certain algebraic and transcendental terms*.

The preceding statement is basically what Cooke [37], p. 400, called the "commonly-accepted informal statement of Abel's theorem." In fact, the Addition Theorem was accepted as Abel's Theorem by Gray [52], p. 142, and [53], p. 366, by Shafarevich [107], pp. 416–417, by Kline [68], p. 654, by Ore [83], p. 15, by Klein [67], p. 104, by Baker [8], p. 209, and by Weierstrass [111], pp. 403–404. (The half-page statement of the theorem in [68] is strikingly similar to that in [8], although [8] is not cited.)

However, contrary to what Abel announced in [2], in his third paper [3], he proved only the key, but preliminary, **Elementary Function Theorem.** It provides a sufficient condition for this sum of integrals to be given by the algebraic and transcendental terms alone, so by an elementary function. The condition is not imposed on the common integrand, but on the variables of the integrals, their upper limits; they must form the full set of roots of a certain resultant equation with variable coefficients.

The paper is very short: two pages. The statement fills the first page; the proof, the second. The mathematics is more elegant and more conceptual than in [2]. There is no introduction, but in the very last sentence, Abel said he would develop a number of illuminating applications later.

The Elementary Function Theorem too has been commonly accepted as Abel's Theorem by historians and mathematicians. Some have also explained that Abel went on to derive his Addition Theorem in full generality. Notably, Dieudonné [43], pp. 19–20, and Wirtinger [120], p. 159, did so.

Other authors have simply gone on to derive the Addition Theorem, without clearly crediting it to Abel. For example, see the treatments by Hensel and Landsberg [57], by Goursat [51], pp. 244–250 and 665–685, by Bliss [14], pp. 119–132, and by Griffiths [55], pp. 327–345. Goursat, however, treated the Addition Theorem only in the hyperelliptic case.

Still other authors have also explained, at most, how to recover the older addition theorems for circular functions and elliptic functions. For example, see the treatments by Clebsch and Gordan [33], Ch. 2, by Clebsch and Lindemann [34], Ch. 6, by Cremona [40], p. 161, by Enriques and Chisini [44], p. 141, by Markusevich [76], pp. 37–40, and by Laudal [72], p. 68.

Some treatments require more discussion here. First, Rowe [97], p. 713 and p. 721, accepted the Elementary Function Theorem as Abel's Theorem, and described it as "the principal theorem of Abel's memoir [1]." Correspondingly, Rowe cited Equation (37) in [1], but nothing in [3]. Nevertheless, Rowe devoted twelve pages, a third of [97], to Abel's derivation of the Addition Theorem. Later, Forsyth [47], pp. 579–601, did about the same thing.

Brill and Noether [17], p. 212, referred to the Elementary Function Theorem as the *celebrated* (berühmt) Abel's Theorem. However, they then called the Addition Theorem the *main result* (Hauptergebnis) in [1]. Thus they felt that Abel's main result was not commonly accepted as Abel's Theorem.

By contrast, Ore [82], p. 219, and [83], p. 17, felt that Abel had, in [3], reproved the "main theorem from his Paris memoir [1]." Thus he felt that the Elementary Function Theorem is Abel's main result. Yet Ore [83], p. 15, accepted the Addition Theorem as Abel's Theorem.

Under the heading *Abel's Addition Theorem*, Birkhoff and Merzbach [11], pp. 188–190, gave a translation of the introduction to Abel's first paper [1], which includes informal statements of the Relations Theorem and the Addition Theorem. Then they said: "We reproduce below Picard's [*sic*] simplified proof of Abel's Addition [*sic*] Theorem." Thus they ended with a statement labeled *Abel's Theorem*, but which is, in fact, a statement of Abel's Elementary Functions Theorem.

Birkhoff and Merzbach's version of Picard's statement and proof is nearly identical to Coolidge's statement and proof in [38], p. 213. They cited both Picard and Coolidge, whereas Coolidge just cited Abel's third paper [3]. But Coolidge gave a valid explanation: "We can do no better than to reproduce his [Abel's] original form, with slight changes of wording and notation."

Coolidge did cite Picard on a related matter. Both he, p. 213, and Birkhoff and Merzbach, pp. 176–177, quoted Picard's famous pronouncement on Abel's Theorem: "Perhaps never in the history of science has so important a proposition been obtained using such simple considerations."

Birkhoff and Merzbach cited the second edition (1905) of Picard's book [90], pp. 437–439, whereas Coolidge cited the third edition (1926, p. 464). In the former, Picard cited only Abel's Paris paper [1], and then proved a version of Abel's Elementary Function Theorem. Picard's treatment is rather different from Abel's, but also rather different from the one "reproduced" by Birkhoff and Merzbach. Later, on p. 426, Picard arrived at Abel's Addition Theorem, although only for integrals of the first kind; he did not credit it to Abel, but did say that it follows from Abel's Theorem.

For an integral of the first kind (one finite everywhere), the elementary function reduces to a constant. So two sums of these integrals have the same value (modulo periods) if the variables of each form the set of roots of the same resultant equation, but with different choices of coefficients. Two such sets are called (linearly, or rationally) *equivalent*. And Abel's **Equivalence Theorem** says this: *if two sets of variables are equivalent, then the corresponding sums have the same value for every integral of the first kind.*

Abel, however, did not deal with integrals of the first kind as such, although he certainly did seek conditions guaranteeing the constancy of the sum of integrals. According to Brill and Noether [17], p. 275, Riemann was the first to formulate and prove the Equivalence Theorem in full generality essentially as above. Riemann [92], pp. 123–124, called it Abel's Addition Theorem, and referred to Jacobi's proof of it in the hyperelliptic case [64], § 8. Also, he said that, for an arbitrary Abelian integral, a similar argument yields the Elementary Function Theorem, although he did not use the name.

The converse of the Equivalence Theorem holds too, although Abel did not recognize it. It and the direct assertion were combined and proved under the heading of *Abel's Theorem* by Weyl on p. 149 of his celebrated book [119] on Riemann surfaces, although Weyl did mention that Abel proved only the direct assertion. In a footnote, Weyl explained further that Abel's theorem was "developed in splendid simplicity in the short note [3]," and "is more general, in that it concerns not only integrals of the first kind ... The converse of Abel's theorem for integrals of the first kind may be read between the lines in Riemann; it was stated explicitly (without completely adequate proof) by Clebsch [29], p. 198." But Weyl failed to mention Weierstrass.

After the appearance in 1913 of Weyl's book, it has become common in the theory of Riemann surfaces – as Patterson [86], p. 10, pointed out – to do as Weyl did, and accept the full necessary and sufficient criterion as Abel's Theorem, even though Weyl noted doing so is historically misleading.

However, Severi called the full criterion the *Abel–Riemann Theorem* on p. 271 of his 1921 book [104], which was based on his 1907–8 lectures in Padua, but the name "Abel–Riemann Theorem" never caught on. Severi also said the result can be found somewhere unspecified in Weierstrass's work, and indeed Weierstrass [111], pp. 407–419, did prove it all.

The Relations Theorem is the fourth theorem accepted as Abel's Theorem. Notably, it was accepted by Houzel [58], p. 72, and by Bell [9], p. 322. Later Bell [10], pp. 501–502, accepted the Addition Theorem, citing Baker [8], p. 209, as his authority.

Abel stated the Relations Theorem informally in the introduction to his Paris paper [1], p. 146. Then he said that this theorem yields the Addition Theorem, which he also stated informally. Unfortunately, he did not make comprehensive formal statements of his results, let alone use the heading of "Theorem," in [1], as he did in his later two papers. So there is some uncertainty about precisely how to formulate these two theorems.

Moreover, Abel included two assertions in this statement of the Addition Theorem, which he omitted from his later statement in the introduction to [2]. The first assertion concerns the variables of the given sum of integrals and those of the reduced sum: the latter must be algebraic functions of the former. The second assertion is that, in the given sum, the integrals can be multiplied by arbitrary rational numbers.

The **Relations Theorem** lies a step beyond the Elementary Function Theorem, but they sound similar. Both assert that the same sum of integrals is given by an elementary function. However, in the Elementary Function Theorem, the integration variables are assumed to form the set of roots of a single resultant equation whose coefficients are polynomials in certain variables; moreover, these variables are the arguments of the elementary function. In the Relations Theorem, the integration variables are assumed to satisfy a specific number p of algebraic relations. In fact, these relations make the last p variables into algebraic functions of the first variables; moreover, the first variables are also the arguments of the elementary function.

The situation is a bit more tangled. Abel began the technical discussion in [1] with a short conceptual proof of the Elementary Function Theorem; it is similar to his proofs in [3] and [4]. Unfortunately, on p. 149 of [1], he said this theorem is what he had announced on p. 146. It isn't! There is no mention in the Elementary Function Theorem of a specific number of relations among the integration variables; Abel stressed the matter in the introduction, and devoted pp. 170–180 to it. Fortunately, neither Houzel nor Bell were mislead; both indicated the importance of the number.

Legendre, it seems, never appreciated the full extent of Abel's Addition Theorem. But, he did come to appreciate its importance for hyperelliptic integrals, to which he devoted the third supplement to his *Traité des fonctions elliptiques* in two volumes, 1825–26. (See [82], pp. 204, 208, 213, 219–220, 233–234, and [68], p. 421.) In the supplement, Legendre introduced the term "ultra-elliptic" to refer to the more general integrals.

Legendre sent a copy of the supplement to Crelle for review on March 24, 1832, and at Crelle's request, Jacobi reviewed it. He [63], p. 415, translated "ultra-elliptiques" by *hyperelliptischen*, and the prefix "hyper" has stuck, even in French. Moreover, since Abel had introduced the integrals, Jacobi suggested calling them *Abelian transcendents* (Abelschen Transcendenten). However, by 1847, he [65], p. 151, had begun calling them *Abelian integrals* (Abelschen Integralen) as well. Furthermore, Jacobi was inspired to give in [64], dated 12 July 1832, the first of several treatments of Abel's Addition Theorem in the hyperelliptic case, which he called simply *Abel's Theorem.*

In his cover letter, Legendre [75] praised Abel's Addition Theorem, calling it, in the immortal words of Horace's Ode 3, XXX.1 (see [9], p. 307), a monument more lasting than bronze (monumentum aere perennius). In turn, Jacobi [63], p. 415, said, "Surely, as the noblest monument to this extraordinary genius, it is preferable that this theorem acquire the name of Abel's Theorem." And ever after, it has been called **Abel's Theorem!**

2 Abel's Elementary Function Theorem

An *Abelian integral* is simply an integral of the form

$$\psi x := \int R(x, y)\, dx$$

where x is an independent variable, R is a *rational function*, and $y = y(x)$ is an (integral) *algebraic function*. That is, R is the quotient of two polynomials. And y is the implicit multivalued function defined by an irreducible equation of the form

$$f(x, y) := y^n + f_1(x)y^{n-1} + \cdots + f_n(x) = 0 \tag{2.1}$$

where the $f_i(x)$ are polynomials in x.

The notation here and below is not always the same as Abel's, but is essentially self-consistent, whereas Abel's changed from paper to paper.

For example, $y := \sqrt{x}$ is defined by $f(x, y) := y^2 - x$. More generally, let $\varphi(x)$ be a squarefree polynomial. Then $y := \sqrt{\varphi(x)}$ is defined by the irreducible polynomial $f(x, y) := y^2 - \varphi(x)$. Now, any polynomial in x and y can be reduced to one of the form $\varphi_1(x)y + \varphi_2(x)$; just replace each even power y^{2r} by $\varphi(x)^r$, and each odd power y^{2r+1} by $\varphi(x)^r y$.

Hence ψx can be written in the form

$$\psi x = \int \frac{\pi_1(x)\, dx}{\pi_2(x)} + \int \frac{\rho_1(x)\, dx}{\rho_2(x)\sqrt{\varphi(x)}} \tag{2.2}$$

where the $\pi_i(x)$ and $\rho_i(x)$ are polynomials. The first summand can be integrated by the method of partial fractions, and the integral expressed as a sum of rational functions and logarithms, so as an elementary function.

Let d be the degree of $\varphi(x)$. If d is 1 or 2, then, via a rational change of variables, the second summand can be transformed into one like the first, and then integrated. Indeed, if $\varphi(x) = ax + b$, then

$$\int \frac{\rho_1(x)\, dx}{\rho_2(x)\sqrt{\varphi(x)}} = \int \frac{2\rho_1\big((y^2 - b)/a\big)\, dy}{a\rho_2\big((y^2 - b)/a\big)} .$$

Suppose $\varphi(x) = ax^2 + bx + c$. Then by completing the square and replacing x by $u := \alpha x + \beta$ and y by $v := \gamma y$ for suitable constants $\alpha,\ \beta,\ \gamma$, the second summand can be transformed into one with $\varphi(x) = 1 - x^2$. Finally, setting

$$x = \frac{2t}{1 + t^2} \quad \text{and} \quad y = \frac{1 - t^2}{1 + t^2} ,$$

transforms the second summand in (2.2) into one like the first.

If $d \geq 3$, then the second summand in (2.2) usually cannot be expressed as an elementary function. It is called an *elliptic* integral when d is 3 or 4, as such integrals arise in rectifying the ellipse. The second summand was termed a *hyperelliptic* integral when $d \geq 5$ in 1832 by Legendre and Jacobi, since such integrals share many formal properties with the elliptic integrals.

Usually, the Abelian integral ψx is interpreted, following the dictates of rigor, as a contour integral with fixed lower limit on the Riemann surface of f. Furthermore, Dieudonné [43], p. 19, felt Abel himself meant just such an integral. And, indeed, Abel [1], p. 149, spoke of "integrating between certain limits." But it seems more likely, as Cooke [37], pp. 400, 412–413, suggested, that Abel really meant an indefinite complex integral of the sort used by Euler. Indeed, when Abel [1], p. 145, summarized his work, he spoke of a function with a given sort of derivative.

Abel began his study of advanced mathematics by reading Euler's three standard texts (*Introductio, Institutiones calculi differentialis,* and *Institutiones calculi integralis*). He read them together with his mathematics teacher and mentor, Holmboe, at the Christiania Cathedral School, a preparatory school. Furthermore, Abel discovered his theorem before leaving for Paris; indeed, it is proved in full generality in the posthumous fragment [4]. (See [12], p. 2; see [111], pp. 401–402; see [17], pp. 213, 274; and see [37], p. 407.)

When Abel was in Paris, Cauchy had just begun to develop his theory of integration, and he published a preliminary version in his own journal. Abel wrote to Holmboe that he read the issues diligently, but they seem not to have had much influence on his work on his theorem. (See [109], p. 409; see [82], p. 147; and see [37], p. 412.)

Furthermore, as Cooke [37], p. 412, observed, "absolutely no knowledge of contour integrals or the residue theorem is even remotely hinted at in the present paper [2], even though their implicit presence is obvious to a modern mathematician." On the contrary, as Cooke noted on p. 414, Abel "used a principle that we recognize nowadays as a special case of the residue theorem." In addition, Cooke noted on p. 416 that Abel used a formula "that seems to foreshadow Cauchy's formulas for the derivatives of an analytic function."

Abel used this principle and this formula in [1] too, but not in [3], nor in [4]. Indeed, Abel used them to obtain fairly explicit, though intricate, forms of the Elementary Function Theorem in [1] and [2]. However, he did not use them in his strikingly short proofs of the more conceptual forms of the theorem in [1] again, and in [3] and [4].

There is no geometry in Abel's work, even though it might seem he considered himself a geometer (see [72], p. 68). Indeed, on his roundabout trip to Paris, Abel registered at an inn in Predazzo as "professore della geometria" (see [109], p. 391). And he began his Paris paper [1] with these words: "The transcendental functions considered up to now by geometers [*sic*] are very few in number." Thus, under geometry, Abel included, as did others, the analysis of functions arising as arc lengths, areas, and the like.

Yet, Abel's work might have benefited if he had known more about the singular points and the points at infinity of an algebraic curve in the projective plane. Moreover, projective geometry was a subject of considerable interest in France and

Germany when Abel visited them (see [68], pp. 834–860). Also, he was a good friend of the geometer Steiner (see [82], p. 96).

However, in 1864 Clebsch [29] reformulated Abel's theorem geometrically, and derived results about contacts between various curves and surfaces. In the introduction, Clebsch said that such applications had not been sought earlier, because Riemann's 1857 theory of Abelian integrals was difficult to understand despite the more recent efforts of younger mathematicians. Moreover, according to Brill and Noether [17], p. 320, Clebsch wrote Roch in August 1864, saying that, even after great effort, he understood little of Riemann's paper [92].

Clebsch studied in Königsberg from 1850 to 1854, learning function theory from Richlot and geometry from Hesse, both of whom had studied under Jacobi. Hesse encouraged Clebsch to pursue the algebraic geometry of Cayley, Salmon, and Sylvester. Clebsch's later melding of their work with that of Abel and Riemann led to a sea change in algebraic geometry, which turned toward the study of birational invariants. Clebsch learned about Riemann's great conceptual theory in 1863 from Gordan. However, Riemann's approach was analytic and topological, not really algebra-geometric.

Gordan studied in Berlin, where he followed Kummer's lectures on number theory in 1855, in Königsberg, where he was influenced by Jacobi's school, and in Breslau, where he completed his dissertation in 1862. He then visited Göttingen to pursue his interest in Riemann's theory. Unfortunately, Riemann's health was failing, and he spent that winter and spring in Italy. However, he left behind several students who understood his work, including Neumann, Prym, and Roch.

At the time, Clebsch was a professor in Karlsruhe, about to move to Giessen, and he invited Gordan to go there with him. In 1868, Clebsch succeeded Riemann in Göttingen. Alas, Clebsch died of diphtheria at 39 in November 1872, and Gordan left Giessen the next year to become a professor in Erlangen. (See [18]; see [46]; see [48], p. 448; see [52], p. 141; see [53], pp. 367–368; and see [68], pp. 934–937.)

Clebsch [29] reformulated Abel's setup much as follows. View the equation $f(x, y) = 0$ as defining a (projective) plane curve C. Let $g(x, y) = 0$ define a second curve D. Say C and D intersect in μ points, coalescing allowed, with abscissas $x_1, \ldots, x_\mu$. View $x_1, \ldots, x_\mu$ as (a complete set of conjugate algebraic) functions of the coefficients $a_1, \ldots, a_r$ of $g(x, y)$. See Fig. 1.

In Abel's two-page third paper [3], the Elementary Function Theorem is formulated in its conceptual form basically as follows.

Theorem 2.1 (Elementary Function). *There exist constants $k_1, \ldots, k_m$ and rational functions $u, v_1, \ldots, v_m$ of $a_1, \ldots, a_r$ such that*

$$\psi x_1 + \cdots + \psi x_\mu = u + k_1 \log v_1 + \cdots + k_m \log v_m \ . \tag{2.3}$$

For example, let us work out the case $y := \varphi(x)$ where $\varphi(x)$ is a polynomial (compare with Abel [4], pp. 60–64); here $f(x, y) := y - \varphi(x)$. Expand $R(x, \varphi(x))$ in partial fractions in x. Since the assertion is linear in R, we may assume $R = (x - b)^q$.

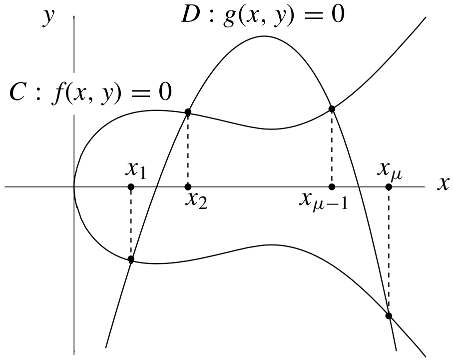

Fig. 1. Clebsch's geometric reformulation (courtesy of J.-M. Økland)

If $q \neq -1$, then $\psi x_1 + \cdots + \psi x_\mu$ is equal to

$$u := \frac{(x_1 - b)^{q+1}}{(q+1)} + \cdots + \frac{(x_\mu - b)^{q+1}}{(q+1)} .$$

However, u is a symmetric rational function in $x_1, \ldots, x_\mu$. And they are all the roots of $g(x, \varphi(x))$. Therefore, u is a rational function in the coefficients of $g(x, \varphi(x))$. But they are linear combinations of the coefficients $a_1, \ldots, a_r$ of $g(x, y)$. So the sum is a rational function in $a_1, \ldots, a_r$, as asserted. If $q = -1$, then $\psi x_1 + \cdots + \psi x_\mu$ is equal to $\log(x_1 - b) + \cdots + \log(x_\mu - b)$, so to $\log v$ where

$$v := (x_1 - b) \cdots (x_\mu - b) .$$

But v is, similarly, a rational function in $a_1, \ldots, a_r$, as asserted.

The key here is a basic theorem about a polynomial equation: *any symmetric rational function in its roots can be expressed as a rational function in its coefficients*. This theorem is also the key to the general case (see [107], p. 417, for example). Indeed, Dieudonné [43], p. 20, described Abel's proof as "remarkable" since it yields such a "general" result, yet is "hardly more than an exercise in the theory of the symmetric functions of the roots of a polynomial." When Abel used this theory in [3], p. 516, he just said it was "known," and gave no reference.

In fact, this theorem on symmetric functions was proved, for the first time, in 1771 by Vandermonde, although, a hundred years earlier, Newton had proved it for various sums and products of the roots (see [68], p. 600). Moreover, Brill and Noether [17], pp. 209–211, suggested it was just such advances in the theory of equations that allowed Abel to generalize Euler's addition formula of 1756–57 for elliptic integrals. Cooke [37], p. 400, added: "Although Brill and Noether do not go into details, it seems clear that a still-primitive understanding of complex numbers had retarded the full understanding of the significance of symmetric polynomials in Euler's day."

Similarly, Kline [68], p. 645, suggested Abel may have gotten some ideas from Gauss, especially from his *Disquisitiones Arithmeticae* of 1801. Also, as Cox, Little, and O'Shea [39], p. 314, noted, in 1816 Gauss published another proof of this theorem on symmetric functions (which introduced the lex ordering) as part of his second proof of the fundamental theorem of algebra.

On the other hand, on p. 56 in [59], Houzel gives a rather different impression of the history of this key theorem on symmetric functions. He says that Newton discovered the general case, but published only a particular case; the general case was published, for the first time, by Waring in 1762, at a period when it waws "common knowledge." If so, then Euler must have known it when he worked on his addition formula. Why then was euler unable to generalize it? It would be very nice to know the full story.

Of course, the method of partial fractions also plays a key role here. But, the method was well known, already when Euler worked, having been introduced in 1702 independently by Leibniz and James Bernoulli. And both used logarithms of complex numbers then too. (See [68], p. 407.)

3 Abel's Equivalence Theorem

Abel proved only the Elementary Function Theorem, Theorem 1, in his third paper [3]. But, in his other three papers, [1], [2], and [4], he addressed two further questions. First, when is the sum $\psi x_1 + \cdots + \psi x_\mu$ in (2.3) constant? Second, how many of the x_i can be varied independently?

Remarkably, both questions lead to the same number p, the (geometric, or true) genus of $C : f(x, y) = 0$. Furthermore, the first question leads to the Equivalence Theorem, and the second, to the Relations Theorem and from there to the Addition Theorem. The first question is discussed in this section; the second, in the next, Sect. 4.

Suppose the integral ψx is of the *first kind,* that is, everywhere finite, even at infinity – in other words, bounded. (The division of Abelian integrals into three kinds was introduced in 1793 by Legendre for elliptic integrals, and in 1857 by Riemann in general; see [68], p. 421 and p. 663.) It follows, as explained next, that, when we vary the coefficients $a_1, \ldots, a_r$ of the auxiliary equation $g(x, y) = 0$, the function

$$\sigma(a_1, \ldots, a_r) := \psi x_1 + \cdots + \psi x_\mu \ .$$

remains constant.

Indeed (compare with [76], p. 38), since ψx_i remains bounded as x_i varies, $\sigma(a_1, \ldots, a_r)$ remains bounded as the a_j vary. Now, owing to Theorem 1, each partial derivative $\partial\sigma/\partial a_j$ is a rational function. So $\partial\sigma/\partial a_j$ too is bounded. Hence it is constant. Therefore, σ is linear and bounded, so constant.

Fix $a_1, \ldots, a_r$. Say $D : g(x, y) = 0$ meets $C : f(x, y) = 0$ in the points

$$z_1 = (x_1, y_1), \ldots, z_\mu = (x_\mu, y_\mu) \ .$$

Make a second choice of the coefficients, say $a'_1, \ldots, a'_r$. Let $g'(x, y) = 0$ and

$$z'_1 = \left(x'_1, y'_1\right), \ldots, z'_\mu = \left(x'_\mu, y'_\mu\right)$$

be the corresponding items. Then the constancy simply means the equation

$$\psi x_1 + \cdots + \psi x_\mu = \psi x'_1 + \cdots + \psi x'_\mu \tag{3.1}$$

holds (modulo periods) for all ψ of the first kind.

Possibly, some of the z's are equal to some of the z''s. If so, then some of the terms in (3.1) cancel. Also, the other z's and z''s are the zeros and poles of the restriction of g/g' to C. So the z's and z''s form sets (or point-groups) that are equivalent in the following sense.

In general, two sets of points of C, repetitions allowed, are called (linearly, or rationally) *equivalent* if, after all common points are removed, the remaining points are the zeros and poles of a meromorphic (or rational) function on C. Extra care must be taken if C is singular; one way to proceed is to work on the desingularization of C, the Riemann surface of f.

Consider two equivalent sets, say consisting of $z_1, \ldots, z_\nu$ and $z'_1, \ldots, z'_\nu$. Pass a curve D of suitably high degree through $z_1, \ldots, z_\nu$. Say D intersects C in $z_{\nu+1}, \ldots, z_\mu$ as well. As the two sets are equivalent, there is a second curve D' of the same degree so that D' intersects C in $z'_1, \ldots, z'_\mu, z_{\mu+1}, \ldots, z_\nu$. (In basically this context, equivalent sets are said to be "co-residual"; see [8], p. 136 and [38], p. 206.) Hence, the preceding discussion yields Abel's half of the full Equivalence Theorem.

Theorem 3.2 (Equivalence). *Consider two sets of points of C with abscissas $x_1, \ldots, x_\nu$ and $x'_1, \ldots, x'_\nu$. The sets are equivalent if and only if the equation*

$$\psi x_1 + \cdots + \psi x_\nu = \psi x'_1 + \cdots + \psi x'_\nu ,$$

holds for all ψ of the first kind.

As mentioned in Sect. 1, the other half of the theorem was stated explicitly for the first time by Clebsch [29], p. 198, in 1864. In fact, there Clebsch assumed C is nonsingular. However, as Brill and Noether [17], pp. 326–327, pointed out, Clebsch shortly afterward published two papers in which he allowed C to have nodes and cusps.

A necessary and sufficient condition that the integral $\psi x := \int R(x, y)\, dx$ be of the first kind is that the integrand be a differential of the form

$$R(x, y)\, dx = \frac{h(x, y)}{\partial f/\partial y}\, dx \tag{3.2}$$

where h satisfies two conditions. First, h must vanish suitably wherever $\partial f/\partial y$ vanishes on C, and h must behave similarly at infinity; such an h is called an *adjoint* of C, and more is said about adjoints below. Second, h must be a *special* adjoint;

that is, the bound $\deg h \leq \deg f - 3$ must hold in addition. (See [8], pp. 121–122; see [38], p. 206; and see [68], p. 936.) Such differentials too are said to be *of the first kind*.

For example, suppose $f(x, y) := y^2 - \varphi(x)$ where $\varphi(x)$ is a squarefree polynomial of degree d. Set

$$p := \begin{cases} (d-1)/2, & \text{if } d \text{ is odd ;} \\ (d-2)/2, & \text{if } d \text{ is even .} \end{cases} \tag{3.3}$$

Then every differential of the first kind is a linear combination of the following p basic ones (see [76], p. 33, for example):

$$\frac{dx}{\sqrt{\varphi(x)}}, \frac{x\,dx}{\sqrt{\varphi(x)}}, \ldots, \frac{x^{p-1}\,dx}{\sqrt{\varphi(x)}} . \tag{3.4}$$

Abel, though, was not interested in integrals of the first kind, although some commentators, such as Sylow [110], p. 298, felt he should have been. Rather, Abel was interested in the constancy of the sum $\psi x_1 + \cdots + \psi x_\mu$. In the case $f := y^2 - \varphi(x)$, his Theorem III on p. 450 in [2] provides an explicit expression for the right side of (2.3). Then, on p. 451, he said it is easy to see that this expression is constant when ψx is any linear combination of the integrals of the differentials listed in (3.4). Abel did not claim to have found every integral for which the sum is constant, but likely he felt he had.

For any $C : f = 0$, Abel investigated this constancy in [1], pp. 159–170, but his results are not definitive. "At the cost of rather complicated computations," as Dieudonné [43], p. 20, put it, Abel concluded constancy obtains when the integrand is of the form (3.2) and the coefficients of $h(x, y)$ satisfy certain linear relations (see also [17], p. 216). Dieudonné added that the form (3.2) is "central in Riemann's theory."

Abel denoted the number of linearly independent h by γ, and gave several formulas for γ. They are complicated, but notably depend only on the degree d of C and on certain numbers associated to the singularities. He worked with the singularities by expressing y as a fractional power series in x, a method introduced by Newton in 1671 and treated by De Gua in 1740, Euler in 1748–1749, and Cramer in 1750 (see [68], pp. 552–554, and [17], pp. 133, 135, 139, 141, 216).

Notably, Abel found that

$$\gamma \leq \pi \text{ where } \pi := (d-1)(d-2)/2 . \tag{3.5}$$

At one time, π was called the "virtual" genus of C, but now π is normally called its "arithmetic" genus.

Abel found that equality holds in (3.5) when the coefficients of f satisfy no special relation. In fact, equality holds whenever C is nonsingular everywhere, including at infinity. Notably, however, if C is hyperelliptic, then C is nonsingular at finite distance, but singular at infinity.

For any $C : f = 0$, the number of linearly independent integrals of the first kind is equal to the genus p, where by the *genus* is meant half the first Betti number of the associated Riemann surface. This result was proved in essence in 1857 by Riemann himself [92], p. 137, (or [11], p. 198, in English translation). In fact, he used p to denote the genus; this use of the letter p soon became common, and has remained so. Notably, however, Weierstrass [111] used ϱ instead to denote this quantity, and used no term to refer to it.

Whether p is viewed topologically, as half the Betti number, or analytically, as the number of integrals, p is plainly a *birational invariant*. In other words, p is invariant under a map from C to another plane curve if the map is given by rational functions in x and y and has an inverse given similarly. Such a transformation is just a kind of change of variables in the integrals, and it induces an analytic isomorphism between the associated Riemann surfaces.

The term "genus" is not due to Riemann. Rather, it was introduced in 1865 by Clebsch [30], p. 43, in order to indicate his aim of using this invariant p as a means for classifying curves. And indeed, he proved in § 1 of [30] that every curve of genus 0 is birationally equivalent to a line and in [32] that every curve of genus 1 is birationally equivalent to a nonsingular plane cubic.

Earlier, plane curves $C : f(x, y) = 0$ were classified according to their *degree* (or order), that is, the degree of f. This method was introduced in 1637 by Descartes, and used by Wallis, Newton, James Bernoulli and others. (See [68], pp. 308–334 and 547–554.)

Also in 1865, Clebsch [31], p. 98, gave an important algebra-geometric formula for p, namely,

$$p = (d-1)(d-2)/2 - \delta - \kappa \tag{3.6}$$

where d is the degree of C and where δ and κ are the numbers of nodes and cusps, provided C has no higher singularities. Clebsch then related this formula to the Plücker formulas.

When C is nonsingular, $p = \gamma$ by (3.6) and the statement after (3.5). When C is singular, p and γ can differ. Indeed, Clebsch and Gordan [33], p. 49, observed that the sum $\psi x_1 + \cdots + \psi x_\mu$ can remain constant for certain integrals ψx of the third kind when C has nodes and cusps. (See also [17], pp. 216–217.) The full story involves the restrictions placed on the variables a_i, the coefficients of $g(x, y)$, on which the sum depends. The situation was not understood fully until after 1950; see Sect. 7.

The term "genus" was accepted right away by most all, except writers in English; they used the term "deficiency" instead. The notion of deficiency had already been introduced in 1729 by Maclaurin: he proved an irreducible plane curve of degree d has at most $(d-1)(d-2)/2$ nodes and cusps, and he considered the difference between this maximum number and the actual number, that is, the right side of (3.6) (see [68], p. 552). This difference was termed the deficiency and denoted D, not p, by Cayley [27], p. 1, also in 1865, although Cayley cited Cramer's book of 1750, not Maclaurin's of 1720. The term deficiency was used for about fifty years.

Cayley [27], p. 1, also cited Clebsch's paper [30], not because the term genus was introduced there, but because there "it was first explicitly stated" that, if a curve of degree d has $(d-1)(d-2)/2$ nodes and cusps, then it is birational to the line, and so can be traced globally by a single algebraic parameter. Cayley noted that this fact follows from Riemann's general theory, and then gave it a simple direct algebra-geometric proof. Cayley termed such curves *unicursal.*

For example, let $C : f(x, y) = 0$ be the *lemniscate*. It was introduced by James Bernoulli in 1694 in connection with his study of bending rods. Here

$$f(x, y) := \left(x^2 + y^2\right)^2 - a^2 \left(x^2 - y^2\right)$$

where a is a constant, and C is a figure eight. (See [68], p. 412.) Hence C has a node at the origin, but it also has two more imaginary nodes at infinity. So (3.6) yields $p = 0$; whence, C is unicursal. Therefore, any Abelian integral $\psi x := \int R(x, y)\, dx$ is given by an elementary function in x.

However, Bernoulli tried to find the arc length s of the lemniscate C in terms of the polar radius r, and was led to this formula:

$$s = \int_0^r \frac{a^2\, dr}{\sqrt{a^4 - r^4}} \,.$$

He "surmised that this integral ... could not be integrated in terms of the elementary functions," as Kline [68], p. 412, put it, and continued, "Seventeenth-century attempts to rectify the ellipse, whose arc length is important for astronomy, ... [and] the problem of finding the period of a simple pendulum led" to similar elliptic integrals.

The birational invariance of the genus was proved algebra-geometrically in 1866 by Clebsch and Gordan [33], pp. 54–60. They begin with (3.6), and proceeded to determine the degree and the numbers of nodes and cusps on the transformed curve. Klein, who [66], p. 2, called Clebsch "one of my principal teachers," nevertheless, on p. 5, criticized the proof for involving "a long elimination, without affording the true geometrical insight into [the] meaning" of the genus.

On the other hand, as Kline [68], pp. 939–940, explained, Clebsch was no longer "satisfied merely to show the significance of Riemann's work for curves. He sought now to establish the theory of Abelian integrals on the basis of the algebraic theory of curves ... One must appreciate that at this time Weierstrass's more rigorous theory of Abelian integrals was not known and Riemann's foundation – his proof of existence based on Dirichlet's principle – was not only strange but not well established."

Clebsch and Gordan's proof was not the first algebra-geometric proof of the invariance of the genus, as is sometimes said (for example, on p. 939 in [68]). Indeed, Cayley [27], p. 3, found one the year before, in 1865. However, Clebsch and Gordan's work inspired further advances; Cayley's didn't really.

The term *adjoint* (adjungirte) was introduced in 1874 by Brill and Noether on p. 272 of their great paper [16]. (See [52], p. 147; see [17], p. 335; and see [34], p. 429. In [58], p. 110, the term is attributed to Clebsch, but not documented.) The notion is implicit in Abel's work, in Riemann's, and in Clebsch's, but it is explicit and central in Brill and Noether's; so they had to give it a name.

Brill and Noether's paper [16] is the next milestone in algebraic geometry after Clebsch and Gordan's book [33]. Indeed, as Gray [52], p. 148, wrote, "It should be stressed that with this work the algebraic geometers not only caught up with Riemann, they surpassed him in generality and rigour. It is from this point that they began to make new discoveries … "

Brill studied under Clebsch in Karlsruhe and Giessen, graduating in 1864 and passing his *Habilitation* in 1867. He taught as *Dozent* in Giessen until 1869, then as professor in Darmstadt until 1875, in Munich until 1884, and in Tübingen until 1918. (See [91].)

Noether (Emmy's father, Max) earned his doctorate in 1868 under the physicist Kirchhoff in Heidelberg without a dissertation. He learned about Riemann's theory of algebraic functions from Kirchhoff and by reading Riemann and Clebsch–Gordan. Lüroth was in Heidelberg; he had studied with Clebsch in Giessen, and encouraged Noether to go there. There Noether joined the circle of young mathematicians around Clebsch and Gordan in lectures, in seminars, on strolls, and over coffee. Noether passed his *Habilitation* in 1870 back in Heidelberg; he taught there until 1874, when he moved to Erlangen permanently. (See [15], p. 213; see [25], p. 161; and see [69], p. 125.)

A curve $C : f(x, y) = 0$ is said to have *multiplicity* m at a point z, and z is said to be an m*-fold* point of C, if the Taylor expansion of f at z begins with a form of degree m. This form has m linear factors, and they define the tangent lines to C at z. If $m > 1$ and if the lines are distinct, then z is an *ordinary* multiple point. For example, a *node* is an ordinary double point.

When Brill and Noether [16], p. 272, defined adjoints, they assumed every singularity (multiple point) is ordinary. In § 7, they explained how to make do in general by reducing the singularities to ordinary ones through a sequence of quadratic transformations of the ambient plane, a method Noether had developed in 1871. "Actually he merely indicated a proof which was perfected and modified by many writers," as Kline [68], p. 941, pointed out.

It is also possible to transform the curve into one only with nodes by using birational transformations that do not extend to the ambient plane. Notably, Kronecker developed a method of doing so, and, as Kline [68], p. 941, observed, "he communicated this method verbally to Riemann and Weierstrass in 1858, lectured on it from 1870 on, and published it in 1881." Many others worked on the matter as well, and Bliss devoted his 1923 AMS Presidential Address [13] to an account of this work.

There are occasions, however, when it is neither customary nor desirable to reduce the singularities of the given curve. For example, a hyperelliptic curve C is given by an equation of the form $y^2 = \varphi(x)$ where $\varphi(x)$ is a squarefree polynomial, and an equation of any other form would mask the hyperellipticity. Yet, if say $d := \deg \varphi$, then C has a point z of multiplicity $d - 2$ at the end of the y-axis, and if $d \geq 4$, then this singularity is not ordinary as the line at infinity is the only tangent to C at z.

Let $C : f = 0$ and $\Gamma : h = 0$ be curves. Then Γ and h are called *adjoints* of C under the following conditions. First, Γ must be allowed to be reducible and to have multiple components, including the line at infinity, so that the various h of given degree or less, plus 0, form a vector space. Now, suppose every multiple point z of C is ordinary. If z is an m-fold point of C, then z must be an $(m-1)$-fold point of Γ. This definition is basically that given by Brill and Noether in [16], p. 272.

The definition of adjoint was extended in 1884 by Noether [79], p. 337, to curves C with arbitrary singularities. He proceeded by induction on the number of quadratic transformations required to reduce the singularities of C to ordinary multiple points. Namely, Γ is an *adjoint* of C if the first transform Γ' of Γ is an adjoint of the first transform of C. However, care must be taken in forming Γ'. Let the quadratic transform be centered at the point z of multiplicity m on C and μ on Γ. Then Γ' must include the exceptional line corresponding to z with multiplicity $\mu - m + 1$.

The first rigorous general theory of adjoints was developed by Gorenstein in his 1950 Harvard thesis (see [94], p. 191), which was supervised by Zariski (see [85], p. 119) and published in 1952 in [50]. In effect, on p. 431, Gorenstein termed Γ an *adjoint* of C if the restriction $h|C$ belongs to the conductor, in the field of meromorphic (or rational) functions on C, of the subring generated by x and y, and if a corresponding condition holds at infinity. He proved, on pp. 434–435, that a differential is of the first kind if and only if it is of the form (3.2) where h is an adjoint in his sense. His sense and Noether's are shown to be equivalent, for instance, in Casas's text [20], p. 152.

4 Abel's Relations Theorem

In Abel's Paris paper [1], he first studied these two questions. Given an Abelian integral ψx, when is the sum $\psi x_1 + \cdots + \psi x_\mu$ equal to an elementary function? When does the sum remain constant? His answers are discussed above in Sects. 2 and 3.

On p. 170, Abel began his study of a third question. How many of the x_i in (2.3) can be varied independently? That is, in the intersection of the curves $C : f(x, y) = 0$ and $D : g(x, y) = 0$, how many of the common points can be assigned at will on C by making suitable choices of the coefficients of $g(x, y)$? The answer is, as Abel put it on p. 172, "rather remarkable."

The answer is also rather subtle. Indeed, in 1898, Scott [99], p. 261, wrote: "The theory of the intersections of curves has probably led its investigators into more errors than any other modern theory. Even the history of the central question, the so-called Cramer paradox, is usually given incorrectly, with the omission of all reference to Maclaurin. This [omission] is all the more surprising, inasmuch as Cramer himself ascribes [the paradox] to Maclaurin."

The Cramer paradox is this. A curve $C : f(x, y) = 0$ of degree d is determined by $(d^2 + 3d)/2$ points. Indeed, each point places one linear relation on the coefficients of f; moreover, f has $(d + 2)(d + 1)/2$ coefficients, but one can be normalized to be 1 since C is also defined by the vanishing of cf for any nonzero scalar c. Now, a second curve $D : g(x, y) = 0$ of degree d meets C in d^2 points, in general. So these d^2 points do not determine a unique curve of degree d through them, yet

$$d^2 \geq \left(d^2 + 3d\right)/2 \text{ if } d \geq 3 .$$

How can this be?

According to Scott [99], pp. 262–263, Maclaurin, in 1720, made "no attempt to explain" the paradox. But in 1750, independently, both Cramer and Euler explained it. Namely, the d^2 relations are not independent. In fact, according to Brill and Noether [17], pp. 289–290, Euler recognized that precisely $(d^2 + 3d)/2 - 1$ are independent, and in 1818, Lamé gave the reason why: the d^2 points also lie on the curve defined by the vanishing of any linear combination $af + bg$; so the relations determine all but one of the available coefficients. For example, through eight arbitrary (general) points on a cubic, there pass infinitely many other cubics, but they all pass through the same ninth point; it is not arbitrary, but depends on the eight.

What did Abel do? Once again, his work is simpler in [2], where he treated the hyperelliptic case. So let $f(x, y) := y^2 - \varphi(x)$ where $\varphi(x)$ is a nonzero squarefree polynomial of degree d, but any $d \geq 1$ works here. Now, as noted before, $g(x, y)$ can be reduced to the form

$$g(x, y) = \theta_1(x)y - \theta_2(x)$$

by replacing each even power y^{2r} by $\varphi(x)^r$, and each odd power y^{2r+1} by $\varphi(x)^r y$. Observe that this reduction preserves the common zeros of f and g, which are the common points of $C : f = 0$ and $D : g = 0$ at finite distance. Assume θ_1 and θ_2 are not identically zero.

Abel allowed for the possibility that, as the coefficients of $g(x, y)$ are varied, $\theta_2(x)$ and $\varphi(x)$ retain a fixed common factor $\varphi_1(x)$. Say

$$\theta_2(x) = \theta(x)\varphi_1(x) \text{ and } \varphi(x) = \varphi_1(x)\varphi_2(x) .$$

Mathematically, $\varphi_1(x)$ is needed at times, see below. Abel's recognition of this need was one milestone on the route to his work from Euler's; for an interesting historical discussion of $\varphi_1(x)$, see Cooke's [37], pp. 408–409, 413.

Eliminating y between $f(x, y)$ and $g(x, y)$ gives the "resultant" polynomial

$$F(x) := \theta(x)^2\varphi_1(x) - \theta_1(x)^2\varphi_2(x) .$$

Let its roots be $x_1, \ldots, x_\mu$. They are the abscissas of the variable points of intersection of C and D. Let $y_1, \ldots, y_\mu$ be the corresponding ordinates. So y_i is determined by the equation $g(x_i, y_i) = 0$, or

$$\theta_1(x_i)y_i - \theta(x_i)\varphi_1(x_i) = 0 . \tag{4.1}$$

Note that, if x' is a root of $\varphi_1(x)$, then $(x', 0)$ is a fixed point of intersection. Conversely, every fixed point at finite distance is of this form. Say $x_1, \ldots, x_\alpha$ can be assigned at will by choosing the coefficients of $g(x, y)$, and say $x_{\alpha+1}, \ldots, x_\mu$ are determined by $(x_1, y_1), \ldots, (x_\alpha, y_\alpha)$.

Equation (4.1) may be viewed as linear relation among the coefficients of $\theta_1(x)$ and $\theta(x)$. However, Abel did not consider (4.1) explicitly in [2]. Rather, he formed the equivalent equation

$$\theta(x_i)\sqrt{\varphi_1(x_i)} = \varepsilon_i\theta_1(x_i)\sqrt{\varphi_2(x_i)} \tag{4.2}$$

where $\varepsilon_i = \pm 1$. Since $F(x_i) = 0$, (4.2) holds; it is equivalent to (4.1) since $f(x_i, y_i) = 0$. The value of ε_i depends on the choice of signs in the square roots, but remains constant as x_i varies, as Abel noted on p. 449.

Normalize $g(x, y)$ by assuming $\theta_1(x)$ is monic. Let m and n be the degrees of θ_1 and θ. Then g has $m + n + 1$ free coefficients. They yield the coefficients of F. Hence $\alpha \le m + n + 1$. In fact,

$$\alpha = m + n + 1\ , \tag{4.3}$$

and this equality is "easy to see from the shape of (4.2)," as Abel said on p. 453, but without further explanation.

Here's an explanation, which can be rephrased in terms of (4.2). Suppose $\alpha < m+n+1$. Then $(x_1, y_1), \ldots, (x_\alpha, y_\alpha)$ fail to determine the values of the coefficients of g. Make two different choices of these values, obtaining g_1 and g_2 say. Then g_1 and g_2 vanish at (x_i, y_i) for $i = 1, \ldots, \alpha$, so for $i = 1, \ldots, \mu$ since the first α points determine them all. Let (x_0, y_0) be another point of C. Set $c_i := g_i(x_0, y_0)$ and $g' := c_1g_2 - c_2g_1$. Then g' vanishes at (x_i, y_i) for $i = 0, 1, \ldots, \mu$. But then eliminating y between $f(x, y)$ and $g'(x, y)$ gives a resultant polynomial of degree μ with $\mu + 1$ roots, a contradiction.

Let d_1 and d_2 be the degrees of φ_1 and φ_2. Note $\mu = \deg F$. Hence

$$\mu = \max\{\,2n + d_1,\ 2m + d_2\,\} \text{ and } d_1 + d_2 = d\ . \tag{4.4}$$

Therefore, (4.3) yields

$$2\mu \ge 2n + d_1 + 2m + d_2 = 2\alpha - 2 + d\ ;$$

furthermore, the inequality is strict if d is odd. Hence (3.3) yields

$$\mu - \alpha \ge p\ ; \tag{4.5}$$

furthermore, $\mu - \alpha = p$ if and only if computing $\Delta := (2m + d_2) - (2n + d_1)$ gives $\Delta = \pm 1$ when $d = 2p + 1$, and gives $\Delta = 0$ when $d = 2p + 2$.

For example, given $\alpha \ge 0$, write $\alpha + p = q(2p + 2) + d_2$ with $0 \le d_2 \le 2p + 1$. Set $m := q(p + 1)$. Define n by (4.3). Define d_1 and μ by (4.4). Then $\mu - \alpha = p$.

On p. 454, Abel arrived at Inequality (4.5). He asserted that equality can be achieved, but did not say how. Of course, equality does not hold for every m and n, despite what is said in [58], p. 74. Moreover, when equality holds, if μ is odd and d

is even, then d_1 must be odd; in particular, then φ_1 cannot be constant, despite what is said in [37], p. 409.

Abel noted that the minimum value p of $\mu - \alpha$ is equal to the number p of differentials in (3.4), which he found to have constant sum $\psi x_1 + \cdots + \psi x_\mu$. This equality appears to be a coincidence in Abel's work, but likely he felt something deeper is involved since he called attention to it.

Earlier, Abel [1], pp. 170–180, studied a similar situation with an arbitrary curve $C : f(x, y) = 0$. His goal was the same, "to find the minimum ... of $\mu - \alpha$," as he said on p. 172. And he tried to relate this minimum to the number γ of independent integrals with constant sum $\psi x_1 + \cdots + \psi x_\mu$; this γ was discussed above in Sect. 3.

In [1], Abel's computations are more involved, and his results are less conclusive. On p. 180, he arrived at the equation

$$\mu - \alpha = \gamma - \iota$$

where $\iota \geq 0$. Moreover, he observed that ι vanishes in general. Brill and Noether [17], p. 219, gave a concrete example where $\iota > 0$.

In fact, Abel did not write ι, but $A + B$. The nature of A and B was clarified a bit by Sylow [110], p. 298, and a bit more by Brill and Noether [17], pp. 218–219; notably, A reflects the behavior at the singularities of C at finite distance, and B does so at infinity. However, the situation is subtle; in particular, care is required in the matters of which D to take and of how to let D vary; see Sect. 7.

The situation has been completely clarified over the course of time through the efforts of many. First, in 1857, Riemann [92], §5, studied the number, $\alpha + 1$ say, of independent meromorphic functions with poles only at μ given points, repetition allowed, on a Riemann surface of genus p; he found $\mu - \alpha \leq p$. Then, in 1864, Roch [93] refined Riemann's inequality into the equation

$$\mu - \alpha = p - i \tag{4.6}$$

where i is the number of independent differentials of the first kind vanishing at the μ points. The number i is called the *index of specialty*.

In 1874, Brill and Noether [16] gave the first algebra-geometric treatment of (4.6), whose assertion they named the *Riemann–Roch Theorem*. "They took from Clebsch the idea that it was to be studied geometrically, that is, in terms of a linear family of adjoint curves," as Gray [54], p. 814, observed. Moreover, later they themselves [17], p. 358, pointed out that their theory served to replace Abel's Theorem as a tool in algebraic geometry.

The idea is essentially this (but see [43], p. 39; see [38], pp. 205–207; see [25], p. 165; and see [49], Ch. 8). Under the relation of (linear) equivalence, which was discussed in Sect. 3, the sets of points of C form equivalence classes, called *complete linear systems*, or series. Each has a *degree*, or order, μ and a *dimension* α. By definition, μ is the number of points that vary from set to set. And α is the maximum number of points that can be assigned at will on C; that is, α general points lie in one and only one set in the system.

Plainly $\alpha + 1$ is equal to the number of independent functions with poles only at the μ variable points in any given set in the system. Furthermore, $i - 1$ is equal to the dimension of the system cut out by the special adjoint curves through those μ points. And $p - 1$ is equal to the dimension of the system cut out by all the special adjoint curves; it is called the *canonical system*. Thus (4.6) has a purely algebra-geometric interpretation directly on C, and this interpretation is compatible with birational transformation.

For instance, consider the canonical system. As just observed, for it, $\alpha = p - 1$. Also, plainly, $i - 1 = 0$. Hence, $\mu = 2p - 2$. And, indeed, $2p - 2$ is the number of zeros of a differential of the first kind.

Hence, if $\mu \geq 2p - 1$, then $i = 0$. Indeed, as just observed, every special adjoint curve cuts C in $2p - 2$ variable points; so no such curve can pass through the μ variable points in a set of the given system. So then $\mu - \alpha = p$.

Remarkably, up to a suitable fixed set of points on C, every complete linear system is cut by the adjoint curves of suitably high degree passing through that fixed set. Conversely, consider all the adjoint curves of given degree through a fixed set of points on C; up to a possibly larger fixed set, these curves cut out a complete linear system on C.

Already Abel recognized the need for fixed points. Of course, he worked with abscissas, roughly as follows. Eliminating y between $f(x, y)$ and $g(x, y)$, he [1], p. 147, obtained their resultant polynomial $r(x)$, and then factored it,

$$r(x) = F_0(x) \cdot F(x)\,,$$

so that, as the coefficients of $g(x, y)$ vary, the roots of $F_0(x)$ remain fixed and those of $F(x)$ vary. (See also [17], p. 214.)

Let the roots of $F(x)$ be $x_1, \ldots, x_\mu$. Let $y_1, \ldots, y_\mu$ be the ordinates of the corresponding points of intersection of C and D. Say $(x_1, y_1), \ldots, (x_\alpha, y_\alpha)$ can be assigned at will, and $(x_{\alpha+1}, y_{\alpha+1}), \ldots, (x_\mu, y_\mu)$ are determined by them. Abel [1], pp. 170–171, considered the quotient

$$F^{(1)}(x) := \frac{F(x)}{(x - x_1)\cdots(x - x_\alpha)}\,,$$

whose roots are precisely $x_{\alpha+1}, \ldots, x_\mu$.

Abel observed that the coefficients of $F^{(1)}(x)$ are rational functions in

$$x_1, y_1; \ldots ; x_\alpha, y_\alpha\,. \tag{4.7}$$

Indeed, these quantities provide α linear relations among the coefficients of $g(x, y)$. These relations can be solved owing to the choice of α. Therefore, the coefficients of $g(x, y)$ are rational functions in the quantities in (4.7). Hence so are the coefficients of $r(x)$, since they are polynomials in those of $f(x, y)$ and $g(x, y)$. Finally, the coefficients of $F^{(1)}(x)$ are polynomials in those of $r(x)$ and $F_0(x)$ and in $x_1, \ldots, x_\alpha$.

Since y is an algebraic function of x, the $\mu - \alpha$ relations

$$F^{(1)}(x_{\alpha+1}) = 0, \ldots, F^{(1)}(x_\mu) = 0 \tag{4.8}$$

make $x_{\alpha+1}, \ldots, x_\mu$ into algebraic functions of $x_1, \ldots, x_\alpha$. Hence $y_{\alpha+1}, \ldots, y_\mu$ are algebraic functions of them as well. Abel viewed the relations in (4.8) as distinct conditions on the individual variables $x_{\alpha+1}, \ldots, x_\mu$.

On p. 170, Abel also combined these considerations with the more conceptual form of the Elementary Function Theorem, which he stated on p. 149. Specifically, he observed that the right-hand side of (2.3) becomes a linear combination of an algebraic function and of logarithms of algebraic functions of $x_1, \ldots, x_\alpha$.

Strictly speaking, the roots of $F_0(x)$ yield terms on the left-hand side of (2.3). However, since these terms remain constant, they can be moved to the right-hand side, and incorporated in u. Abel did not include these terms in his statement on p. 149.

The preceding considerations yield the following theorem, which appears to be what Abel was aiming for.

Theorem 4.3 (Relations). *Let ψx be an Abelian integral, and let $\mu \geq 2p - 1$ where p is the genus. Set $\alpha := \mu - p$. Then, for some m,*

$$\psi x_1 + \cdots + \psi x_\mu = u + k_1 \log v_1 + \cdots + k_m \log v_m \tag{4.9}$$

where $k_1, \ldots, k_m$ are constants and where $u, v_1, \ldots, v_m$ are algebraic functions of $x_1, \ldots, x_\alpha$, provided that $x_{\alpha+1}, \ldots, x_\mu$ satisfy precisely p relations of the form (4.8) *that arise from a system of all adjoint curves of the same degree through some fixed set of points.*

5 Abel's Addition Theorem

The final step is to prove the Addition Theorem. However, its statement appears in different forms in different places. The simplest form is an immediate consequence of the Relations Theorem. The most elaborate form is explained in this section, but its proof calls for the development of an additional tool, the Abel map, which is the subject of Sects. 6 and 7.

On pp. 170 and 185 in [1], Abel gave the Addition Theorem in its simplest form. It asserts that the sum of an arbitrary number α of Abelian integrals ψx_i can be reduced (modulo periods) to an elementary function v of the x_i diminished by a sum of p such integrals, where, as always, p is the genus:

$$\psi x_1 + \cdots + \psi x_\alpha = v - (\psi x_{\alpha+1} + \cdots + \psi x_\mu) \text{ where } \mu := \alpha + p\,. \tag{5.1}$$

Furthermore, $x_{\alpha+1}, \ldots, x_\mu$ are algebraic functions of $x_1, \ldots, x_\alpha$; these same functions work for any integral ψx associated to the same algebraic function $y(x)$, but v depends on the choice of ψx.

To derive this assertion, apply the Relations Theorem as follows. As usual, denote the curve associated to ψx by C. Form the system of adjoint curves of high enough degree, so that it cuts out on C a complete linear system of degree μ' with $\mu' \geq \mu$ and $\mu' \geq 2p - 1$. The Relations Theorem now provides (4.9), but with μ' in place of μ.

Fix $\mu' - \mu$ general points on C, say with abscissas $x_{\mu+1}, \dots, x_{\mu'}$, and require the adjoint curves to pass through them. In effect, we have reordered the variables $x_1, \dots, x_{\mu'}$ so that $x_{\alpha+1}, \dots, x_\mu$ are algebraic functions of $x_1, \dots, x_\alpha$ and $x_{\mu+1}, \dots, x_{\mu'}$. Then we fix the values of the latter so that, of the p relations of algebraic dependence, none become trivial. Thus $x_{\alpha+1}, \dots, x_\mu$ become algebraic functions of $x_1, \dots, x_\alpha$. Hence, so do the functions $u, v_1, \dots, v_m$ in (4.9); thus they form the desired v. Finally, the sum $\psi x_{\mu+1} + \cdots + \psi x_{\mu'}$ is constant; so it may be incorporated in v.

Let us now see how to tie in the hyperelliptic case as discussed in Sect. 4. Using its notation, assume $\mu = \alpha + p$. The Addition Theorem is trivial when $\alpha \le p$: just fix any $p - \alpha$ points on C, and use their abscissas as the needed additional x's on the right. So assume $\alpha \ge p + 1$. Finally, assume $d \ge 2$, so that $d = \deg C$.

Let E be the line at infinity. Then $(d-3)E$ is a special adjoint curve since, as noted in Sect. 3, the constant 1 is an adjoint polynomial. Hence the sum $D + (d-3)E$ is an adjoint curve. Consider all the adjoint curves that have the same degree as $D + (d-3)E$ and that pass through the fixed points of the system of sets cut out on C by $D + (d-3)E$ as D varies. These fixed points serve as the $\mu' - \mu$ points considered above.

These adjoints cut on C a complete linear system. It includes the system cut by the various D, up to the fixed set cut by $(d-3)E$. So both systems have the same number μ of variable points. Hence both have the same number α of points that can be assigned at will; indeed, the smaller one has α by assumption, and the larger has α by the Riemann–Roch (4.6), since $\mu \ge 2p+1$, so $i = 0$. Therefore, the two systems have the same sets of variable points. Thus Abel's procedure in the hyperelliptic case in [2] is, indeed, a special case of his general procedure in [1], at least when the latter is interpreted in terms of the theory of adjoints.

Aesthetically, (5.1) would be more pleasing without the minus sign on the right-hand side, and Abel [1], p. 186, showed how to replace it easily with a plus sign. At the same time, he showed how to allow minus signs on the left. Namely, given any $x'_1, \dots, x'_{\alpha'}$, (5.1) yields

$$\psi x'_1 + \cdots + \psi x'_{\alpha'} + \psi x_{\alpha+1} + \cdots + \psi x_\mu = v' - \left(\psi x''_1 + \cdots + \psi x''_p\right) .$$

Subtract this formula from (5.1), and set $V := v - v'$. The result is

$$\psi x_1 + \cdots + \psi x_\alpha - \psi x'_1 - \cdots - \psi x'_{\alpha'} = V + \psi x''_1 + \cdots + \psi x''_p , \tag{5.2}$$

which is essentially Abel's Formula (112), p. 186.

In the hyperelliptic case, Abel's main result is Théorème VIII on p. 454 in [2]. Its Formula (39) is a lot like (5.2), but is essentially this:

$$\psi x_1 + \cdots + \psi x_\alpha - \psi x'_1 - \cdots - \psi x'_{\alpha'} = V + \varepsilon_1 \psi x''_1 + \cdots + \varepsilon_p \psi x''_p \tag{5.3}$$

where $\varepsilon_k = \pm 1$ for $k = 1, \dots, p$. Abel's use of ε_k is interesting. Cooke [37], p. 401, pointed out that Euler had ignored signs in the elliptic case, $p = 1$.

On p. 455, Abel noted that the value of ε_k is determined by the equation

$$\theta\left(x_k''\right)\sqrt{\varphi_1\left(x_k''\right)} = -\varepsilon_k\theta_1\left(x_k''\right)\sqrt{\varphi_2\left(x_k''\right)}\,.$$

This equation is similar to (4.2), but has a minus sign because, when (5.3) is derived, $\psi x_k''$ arises on the left-hand side, and is moved to the right.

On p. 455, Abel also noted that the coefficients of the polynomials $\theta(x)$ and $\theta_1(x)$ are determined by the system of $\alpha + \alpha'$ equations

$$\theta(x_i)\sqrt{\varphi_1(x_i)} = \theta_1(x_i)\sqrt{\varphi_2(x_i)} \text{ and } \theta\left(x_j'\right)\sqrt{\varphi_1\left(x_j'\right)} = -\theta_1\left(x_j'\right)\sqrt{\varphi_2\left(x_j'\right)}$$

where $i = 1, \ldots, \alpha$ and $j = 1, \ldots, \alpha'$. Thus the curve D is determined.

So, in the intersection $C \cap D$, the point with abscissa x_k'' has ordinate y_k'' given by (4.1) with $x_k'',\ y_k''$ in place of $x_i,\ y_i$. But Abel did not use this equation. Rather, it would seem he set

$$y_k'' := \sqrt{\varphi_1\left(x_k''\right)}\sqrt{\varphi_2\left(x_k''\right)}\,.$$

And choosing the wrong signs for the square roots leads to the ordinate $-y_k''$. In fact, Abel did not use ordinates in [2]. (And he wrote $y_1, \ldots,$ not $x_1'', \ldots$.)

But why does the ordinate $-y_k''$ yield as integral $-\psi x_k''$ up to an elementary function? Abel didn't say. However, $-y_k''$ does so because of the Elementary Function Theorem; just apply it to the system of vertical lines $D : x - a = 0$.

With (5.1) in mind and the algebraic function $y(x)$ fixed, Abel stressed two fundamental properties of the number p: first, p depends on $y(x)$, but is independent of ψx and α; second, p is minimal.

The first property, independence, means that, given any integrand, the sum of α integrals, for any α, can be reduced to a sum of p integrals, plus an elementary function. This property plays a key role in Abel's derivation above of (5.2) from (5.1), as he himself noted on p. 186. Of course, if the sum of $p + 1$ integrals can be reduced to a sum of p, then the sum of any number α can be reduced to p too by a simple induction on α.

The second property, minimality, means there exists some integrand such that the sum of p integrals cannot be reduced to a sum of $p - 1$ integrals plus an elementary function. In fact, there exists one such of the first kind.

This minimality shows "clearly why Euler's original statement could not be generalized," as Dieudonné [43], p. 20, put it. Indeed, Cooke [37], pp. 392–393, explained:"In 1751 Euler was given the duty of reading the collected works of Fagnano, which led him to discover an addition theorem for elliptic integrals." Let us see why Euler's Addition Theorem cannot be generalized.

To be sure, from 1714 to 1720, Fagnano found ad hoc algebraic relations among the lengths of cords and arcs of lemniscates, ellipses, and hyperbolas. Earlier, in 1698, John Bernoulli (James's brother) noted there are algebraic relations among the arguments of sums and differences of logarithms and inverse trigonometric functions;

he asked if this property is shared by other functions that arise as integrals. (See [68], pp. 413–418, and [58], pp. 6–8.)

Fagnano's work led Euler in 1757 to discover the addition formula

$$\int_0^{x_1} \frac{dx}{\sqrt{1-x^4}} \pm \int_0^{x_2} \frac{dx}{\sqrt{1-x^4}} = \int_0^{x_3} \frac{dx}{\sqrt{1-x^4}}$$

where the variables $x_1,\ x_2,\ x_3$ must satisfy the symmetric relation

$$x_1^4x_2^4x_3^4 + 2x_1^4x_2^2x_3^2 + 2x_1^2x_2^4x_3^2 + 2x_1^2x_2^2x_3^4 \\ + x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2 = 0\ .$$

(Also see [43], pp. 18–19, [107], pp. 412–413, [58], p. 8, [76], pp. 1–4, 28–40.)

In 1759, Euler generalized this addition formula to some other elliptic integrals; he expressed the sum or difference of two integrals as a third plus an elementary function. Moreover, as Kline [68], p. 420, observed, "He expressed regret that his methods were not extensible to higher roots than the square root or to radicands of higher than fourth degree." In fact, no methods work for these integrals, as Abel showed: for them, $p \geq 2$, and p is the minimal number of integrals in the reduced sum. Moreover, as an example, Abel devoted the last part of his paper [1], pp. 188–211, to the case $y^n = \varphi(x)$ where $\varphi(x)$ is a polynomial.

The importance of minimality was recognized soon after the appearance of Abel's second and third papers, [2] and [3], in 1828 and 1829. Indeed, between 1838 and 1841, when Abel's [1] first paper finally appeared, Jürgensen, Broch, and Minding treated the minimality in various cases; their work was compared with Abel's by Brill and Noether [17], pp. 225–231. On p. 217, they noted that "Abel's competitors (Mitbewerber)" had more difficulty than he did because they did not know his normal form (3.2) for the integrands.

The minimality of p is related to another significant property of (5.1): the uniqueness of $x_{\alpha+1}, \ldots, x_\mu$ when $\alpha \geq p$; that is, if $x'_{\alpha+1}, \ldots, x'_\mu$ work too, then they are equal to $x_{\alpha+1}, \ldots, x_\mu$ up to order, given the v. Abel did not notice this uniqueness. However, it and the minimality are implicit in Riemann's paper [92], and are explicit in Weierstrass's paper [111], pp. 419, 421; a version of their treatments is given in the next section.

In its most elaborate form, Abel's Addition Theorem asserts that any linear combination of Abelian integrals is equal to the sum of p integrals, plus an elementary function; the combining coefficients can be any rational numbers – positive or negative, integers or fractions. Abel [1], pp. 187–188, derived this assertion in short order, but without adequate justification.

Basically, Abel suggested that, if the combining coefficients are integers, then it suffices, in (5.2), to take additional x_i and x'_i, and let them coalesce; however, he did not explain why the x''_i remain determinate. Furthermore, when he introduced a common denominator n, he claimed, unconvincingly, that it is also possible to introduce arbitrary positive integers as combining coefficients on the right. Then he took all these coefficients equal to n, and finally he divided both sides by n. This argument was repeated, with little change, by Rowe [97], p. 743. Its spirit reappears in Sect. 6.

Nevertheless, the division of Abelian integrals is a subject with a substantial history (see [68], pp. 417–419, 756; see [58], pp. 8, 23–26, 84; and see [43], p. 21). In 1718, Fagnano showed how to find the midpoint of any arc of the lemniscate. In 1825, Legendre treated the division of any elliptic integral of the first kind by any odd number n; he claimed there is an equation of degree n^2 for the new upper limit, but did not prove it. In 1827 and 1829, Abel made a systematic study of this matter, among others, in two one-hundred-page papers on elliptic functions; of course, he wrote both after his Paris paper.

In 1832, the night before his fatal duel, Galois wrote an account of his research, which has been preserved. In particular, he wrote that, for any genus p, the division by any n of the sum of p Abelian integrals leads to an equation of degree n^{2p}. This statement was proved by Clebsch [29], § 15, in 1864. Starting in 1834 with Jacobi, there has been a lot of work done over the years on two related matters: on the behavior of theta functions when their arguments are divided by n, and on arbitrary isogenies of Abelian varieties.

Here is a formal statement of the Addition Theorem in its most elaborate form. Its (5.4) is essentially Abel's Formula (119) in [1], p. 188. A proof is discussed at the end of Sect. 6.

Theorem 5.4 (Addition). *Let $y(x)$ be an algebraic function, p its genus, and ψx an associated Abelian integral. Let α be a positive integer, and $h_1, \dots, h_\alpha$ rational numbers. Then*

$$h_1\psi x_1 + \cdots + h_\alpha \psi x_\alpha = v + \psi x'_1 + \cdots + \psi x'_p \tag{5.4}$$

where v is an elementary function of $x_1, \dots, x_\alpha$, and where $x'_1, \dots, x'_p$ are algebraic functions of them.

More precisely, v is a $\mathbb{C}$-linear combination of an algebraic function of $x_1, \dots, x_\alpha$ and of logarithms of algebraic functions; if ψx is of the first kind, then v is constant. Moreover, $x'_1, \dots, x'_p$ work for every choice of ψx; also, if $h_1 = 1, \dots, h_\alpha = 1$ and $\alpha \geq p$, then $x'_1, \dots, x'_p$ are unique, given the v.

Lastly, p is minimal: given algebraic functions $x'_1, \dots, x'_{p-1}$ of $x_1, \dots, x_p$, there exists an integral of the first kind ψx such that, for any constant v,

$$\psi x_1 + \cdots + \psi x_p \neq v + \psi x'_1 + \cdots + \psi x'_{p-1}\ .$$

6 The Abel Map

Theorem 4 is proved at the end of this section through an elaboration of the ideas of Riemann [92] and Weierstrass [111]. The proof involves a new tool, the Abel map, and so it is discussed first. The map combines the Equivalence Theorem and the Riemann–Roch Theorem in a powerful sophisticated geometric form.

To begin, note that the definition of ψx involves an inherent ambiguity. It is a fancy sort of constant of integration. It arises in part from the choice of the starting point of integration, but in part from the choice of the path of integration. For example, notice that

$$\int_1^x \frac{1}{x}\,dx = \log x + 2m\pi\sqrt{-1} \tag{6.1}$$

if the path is chosen to wind m times around the origin in the complex plane.

Euler "drew attention to the fact that the function $\int_0^y dx/\sqrt{1-x^4}$ has in the real domain a 'modulus of multi-valuedness' similar to the inverse trigonometric functions," as Shafarevich [107], p. 414, said crediting Slavutin. Abel [1], p. 149, noted the presence of the ambiguity in connection with the sum of integrals in (2.3), but he simply dismissed it as unimportant so long as the variables are kept within suitable limits. In particular, he did not observe that the ambiguity is related to the genus p.

The ambiguity was discussed by Cauchy in 1846 (see [68], pp. 640–641). However, it was treated definitively by Riemann in 1857 and 1865 (see [68], pp. 662–665; see [107], pp. 417–421; see [58], p. 97; and see [43], pp. 85–87). Riemann proved every integral of the first kind has $2p$ *periods*, which are numbers like the number $2\pi\sqrt{-1}$ in (6.1). These periods generate, by $\mathbb{Z}$-linear combination, all possible changes in the value of the integral arising from changes in the path of integration, with fixed end points. (Note, however, that $\int_1^x \frac{1}{x}\,dx$ is of the third kind and genus 0.)

In effect, Riemann did the following. He fixed a basis $\psi_1 x, \dots, \psi_p x$ of the integrals of the first kind. Then, inside the vector space $\mathbb{C}^p$, he formed the discrete subgroup, or lattice, generated by the corresponding $2p$ p-vectors of periods. He proved the lattice has rank $2p$, and he formed the quotient

$$J := \mathbb{C}^p/(\text{period lattice}). \tag{6.2}$$

By 1920, for example in Severi's book [104], p. 272, J was called the *Jacobian* (or Jacobian variety, or Jacobi variety), and still is. (In 1907, Enriques and Severi [45], spoke of the "Jacobi surface" of a curve of genus 2.)

Let C' be the Riemann surface of $y(x)$; so C' is the desingularization of the plane curve C associated to $y(x)$. Let $C^{(\alpha)}$ be the α-fold *symmetric product* of C'; by definition, $C^{(\alpha)}$ is the quotient of the direct product C^α of α copies of C' under permutation of the factors. So $C^{(\alpha)}$ parameterizes the (unordered) sets of α points of C', repetitions allowed. Let us continue to abuse notation, in the spirit of Abel's work, by using $x_1, \dots, x_\alpha$ to refer both to points of C' and to the abscissas of their images in C.

In effect, Riemann introduced and studied the following map:

$$\Psi_\alpha\colon C^{(\alpha)} \to J \text{ given by } \Psi_\alpha\{x_1, \dots, x_\alpha\} = \left(\sum_{i=1}^{\alpha} \psi_1 x_i, \dots, \sum_{i=1}^{\alpha} \psi_p x_i\right).$$

This map is rather important, but for a long time, it had no name.

In 1948, Weil [113, (c)], p. 150, gave Ψ_α a nondescript name, the "canonical map," and this name has often been used since then. Since about 1970 (see for example [35], p. 284), Griffiths's school has used the name "Abel–Jacobi map" for a major generalization of Ψ_α, introduced in 1952 by Weil [115], p. 94. Maybe the best name for Ψ_α is just the *Abel map*; it is historically correct and conveniently short.

The fibers of Ψ_α are the complete linear systems owing to the Equivalence Theorem. It now follows from the Riemann–Roch Theorem that Ψ_α is surjective for $\alpha \geq p$ (since $\dim \mathrm{Im}(\Psi_\alpha) = p$ and $C^{(\alpha)}$ is compact) and it also follows that Ψ_α is injective on a saturated dense open subset U_α for $\alpha \leq p$. In fact, U_α is *Zariski* open; that is, its complement is defined by polynomial equations. In particular, Ψ_p restricts to an isomorphism between dense open subsets of $C^{(p)}$ and J.

A priori, J is a complex analytic manifold, and Ψ_α is a complex analytic map; indeed, they are constructed by means of integrals. A posteriori, J and Ψ_α are algebraic; they are given by polynomials! This result was established through the efforts of many mathematicians over the course of a century (see [58], pp. 110–111, and [43], pp. 85–87). This result is remarkable! And it is essential in the proof below of Theorem 4, as it guarantees that the functions $x'_1, \ldots, x'_p$ are, in fact, algebraic.

Furthermore, J is an *Abelian variety,* or compact algebraic group variety: not only do its points form a group, but the operations of adding and of inverting are given by polynomial maps. This abstract definition was given in 1948 by Weil [113], (c) p. 110, and it is now the standard definition in algebraic geometry. In fact, an Abelian variety is not simply compact, but is projective; this result was proved in 1957 by Weil [116].

Of course, J is an Abelian (commutative) group by construction. However, every Abelian variety is an Abelian group. This result was discovered in 1889 by Picard [88], p. 222, in the case where the variety is a surface (see [74], p. 357). The general case was proved in 1948 by Weil [113], (c) p. 107. However, Weil said Chevalley had proved the result earlier in an unpublished work using differential forms and the adjoint group. Weil gave a different proof, which is rather elementary and purely algebraic.

It would be reasonable to guess this result gave rise to the name "Abelian variety," but it did not. The name stems rather from another important result: an Abelian variety is parameterized globally by Abelian functions with common periods. This result was proved in 1889 for surfaces and in 1895 in general by Picard [88], p. 222, and [89], p. 250; his proof was completed at certain points of analysis in 1903 by Painlevé [84]. In the general case, Picard had to assume to begin with that the Abelian variety is an Abelian group. (See [74], pp. 355, 369; see [121], p. 104; and see [60], p. 256.)

An *Abelian function* is, by definition, a meromorphic function in p complex variables with $2p$ independent periods (see also [76], pp. 41–44). The general theory of Abelian functions was begun by Riemann [92] in 1857, and developed by many others, particularly by Weierstrass in 1869, 1876, and 1879, by Frobenius in 1884, and by Poincaré in 1897 and 1902.

Riemann [92] was initially interested in inverting Abelian integrals of the first kind, that is, in finding an expression for the inverse of the Abel map $\Psi_p\colon C^{(p)} \to J$ on the dense open subset $\Psi_p U_p$ of J. The inverse can be expressed using the coordinate functions on $C^{(p)}$, so in terms of functions on $\Psi_p U_p$. Since $J := \mathbb{C}^p/\text{periods}$, these functions can be lifted to an open subset of $\mathbb{C}^p$, and then continued to meromorphic functions on $\mathbb{C}^p$ with $2p$ periods. These special meromorphic functions, Riemann named "Abelian functions."

Later, in 1859, Riemann studied arbitrary meromorphic functions in p variables. He proved, according to Hermite, that they have at most $2p$ independent periods, which satisfy bilinear relations, known today as "Riemann's bilinear relations," like those satisfied by the Abelian functions coming from a curve. In 1869, Weierstrass observed that not every Abelian function arises from a curve; in other words, not every Abelian variety is a Jacobian.

Consider the set of all the Abelian functions whose periods include a given lattice of rank p. Plainly, this set is a field K. In fact, K has transcendence degree p over $\mathbb{C}$; in other words, any $p+1$ Abelian functions satisfy a polynomial equation, but some p satisfy none. This statement is implicit in Riemann's work. Weierstrass formulated it explicitly, but did not offer a proof. The first complete proof was published in 1902 by Poincaré.

It follows that K is the field of rational functions on a p-dimensional projective algebraic variety, which is parameterized on a Zariski open set by p of them. There are many such varieties! Among them, there is a unique Abelian variety. The latter result was proved in 1919 by Lefschetz [73], p. 82. At the time, all the varieties with field K were called "Abelian varieties." However, Lefschetz [73], p. 83, said he would use the name only for the distinguished variety.

Some authors – including Severi, Siegel [108], p. 98, and Markusevich [76], p. 148 – never accepted the more restrictive definition of Abelian variety. Rather, they called the distinguished variety a "Picard variety." In this usage, for example see Severi's paper [102], p. 381, the pth symmetric product $C^{(p)}$ is an Abelian variety since the Abel map $\Psi_p\colon C^{(p)} \to J$ is birational; that is, pullback identifies the function fields. So beware!

Originally, the notion of Abelian variety was even broader: it included any variety whose functions form a subfield of finite index in a given field of Abelian functions, in other words, a variety that is parameterized on a classical open set by Abelian functions (see [98], p. 213). For example, an Abelian surface is just a hyperelliptic surface (see [101], p. 436, and [98], p. 213). The index was termed the "rank" by Enriques and Severi in 1909 according to Scorza [98]; however, on p. 134, he said he would use the term "Abelian variety" only when the rank is 1.

In 1832, Jacobi [64], p. 10, posed the famous problem that inspired Riemann; it is known today as the *Jacobi Inversion Problem.* In Jacobi's own words (as translated in [11], p. 210), he asked: "what, in the general case, are those functions whose inverses are Abelian transcendents, and what does Abel's theorem show about them?" On p. 11, Jacobi answered the question for hyperelliptic curves of genus 2. He did so "after long, fruitless efforts, to discover the natural generalization of 'inversion' of elliptic integrals," as Dieudonné [43], p. 20, put it. Did Abel consider inversion in general? Bjerknes [12], p. 216, and Shafarevich [107], p. 419, suggested that Abel did at least have it in mind. At any rate, it is ultimately due to this work of Jacobi's that J is called the Jacobian.

The inversion of elliptic integrals was introduced in 1827 first by Abel and then, a couple of months later, by Jacobi, but Jacobi did not credit Abel. (Thirty years earlier, Gauss inverted them, but his work appeared only much later, after his death in 1855.) Nevertheless, here too Abel inspired Jacobi; this conclusion is universally accepted today; indeed, it is inescapable given all the evidence (see [12], pp. 183–216; see [82], pp. 180–190; see [58], pp. 17–18; see [43], p. 20; and see [109], pp. 452–454, 538).

The properties of the Abel map $\Psi_\alpha\colon C^{(\alpha)} \to J$ yield Theorem 4 as follows. First, suppose p is not minimal. Then there exist algebraic functions $x'_1, \dots, x'_{p-1}$ of $x_1, \dots, x_p$ such that, for each $\psi_i x$ in the basis of integrals of the first kind, there exists a constant v_i such that, modulo periods,

$$\psi_i x_1 + \cdots + \psi_i x_p = v_i + \psi_i x'_1 + \cdots + \psi_i x'_{p-1} .$$

Hence $\Psi_p C^{(p)} \subseteq (v_1, \dots, v_p) + \Psi_{p-1} C^{(p-1)} \subseteq J$. But $\Psi_p C^{(p)} = J$ as Ψ_p is surjective. Hence $\Psi_{p-1} C^{(p-1)} = J$. But this equation is false, since $C^{(p-1)}$ is of complex dimension $p-1$ and J is of complex dimension p.

Next, to prove the asserted uniqueness, suppose that there exist constants v_i and algebraic functions $x'_1, \dots, x'_p$ and $x''_1, \dots, x''_p$ of $x_1, \dots, x_\alpha$ such that

$$\psi_i x_1 + \cdots + \psi_i x_\alpha = v_i + \psi_i x'_1 + \cdots + \psi_i x'_p$$
$$\psi_i x_1 + \cdots + \psi_i x_\alpha = v_i + \psi_i x''_1 + \cdots + \psi_i x''_p$$

modulo periods for $i = 1, \dots, p$. Then

$$\Psi_p\{x'_1, \dots, x'_p\} = \Psi_p\{x''_1, \dots, x''_p\} \text{ in } J .$$

Form $U := \Psi_\alpha^{-1}\big((v_1, \dots, v_p) + \Psi_p U_p\big)$. Then U is open and dense in $C^{(\alpha)}$ since Ψ_α is surjective as $\alpha \geq p$. Let $\{x_1, \dots, x_\alpha\}$ vary in U. Then $\{x'_1, \dots, x'_p\}$ and $\{x''_1, \dots, x''_p\}$ belong to U_p. Hence these two points of U_p are equal since Ψ_p is injective on U_p. Thus the asserted uniqueness is proved.

Finally, to prove (5.4), let n be the least common denominator of $h_1, \dots, h_\alpha$, and say $h_1 = n_1/n, \dots, h_\alpha = n_\alpha/n$. Form the map Φ_α from the direct product C^α to J,

$$\Phi_\alpha\colon C^\alpha \to J, \text{ given by } \Phi_\alpha(x_1, \dots, x_\alpha) = n_1\Psi_1 x_1 + \cdots + n_\alpha \Psi_1 x_\alpha . \qquad (6.3)$$

Fix $(v_1, \dots, v_p) \in J$ so that the open set $(v_1, \dots, v_p) + \Psi_p U_p$ meets $\Phi_\alpha C^\alpha$. Then, since Ψ_p is an isomorphism on U_p, the equation

$$\Phi_\alpha(x_1, \dots, x_\alpha) = (v_1, \dots, v_p) + \Psi_p \left\{x'_1, \dots, x'_p\right\} \qquad (6.4)$$

defines $x'_1, \dots, x'_p$ as a full set of conjugate algebraic functions of $x_1, \dots, x_\alpha$ as the latter vary as a tuple in the open set $\Phi_\alpha^{-1}((v_1, \dots, v_p) + \Psi_p U_p) \subset C^\alpha$.

If $n > 1$, then the sum $h_1\psi x_1 + \cdots + h_\alpha \psi x_\alpha$ is ambiguous, even modulo periods. For example, $\frac{1}{2}\int_1^x \frac{1}{x}\,dx$ might mean either $\frac{1}{2}\log x$ or $\frac{1}{2}\log x + \pi\sqrt{-1}$; the two are distinct modulo the period $2\pi\sqrt{-1}$, but their doubles are equal. Similarly, $\frac{1}{n}\int_1^x \frac{1}{x}\,dx$ has n distinct interpretations. Likewise, since an integral of the first kind ψx has $2p$ periods, $\frac{1}{n}\psi x$ has n^{2p} distinct interpretations. Thus multiplication by n defines a covering map $\nu_n\colon J \to J$ of degree n^{2p}; it is a group homomorphism, and its kernel is $(\mathbb{Z}/n)^{2p}$.

Hence $\nu_n(\Psi_p U_p)$ is open. Therefore, after $x_1, \ldots, x_\alpha$ are further restricted, there exist algebraic functions $x_1'', \ldots, x_p''$ of them (in fact, there are n^{2p} different choices, and they form a full set of conjugates) such that

$$n\Psi_p\left\{x_1'', \ldots, x_p''\right\} = \Psi_p\left\{x_1', \ldots, x_p'\right\}\ .$$

Therefore, (6.3) and (6.4) yield

$$n_1\Psi_1 x_1 + \cdots + n_\alpha \Psi_1 x_\alpha = (v_1, \ldots, v_p) + n\Psi_p\left\{x_1'', \ldots, x_p''\right\}\ . \tag{6.5}$$

Notice, (6.5) yields (5.4) for all integrals ψx of the first kind; however, this case is recovered below.

To derive (5.4) in full generality, restrict $x_1, \ldots, x_\alpha$ further so that there exist algebraic functions $x_1''', \ldots, x_p'''$ of them such that

$$\Psi_p\left\{x_1''', \ldots, x_p'''\right\} = -\Psi_p\left\{x_1'', \ldots, x_p''\right\}\ . \tag{6.6}$$

Put $q := n_1 + \cdots + n_\alpha + np$. Form the two sets

$$\left\{x_1''', \ldots, x_p''', x_1'', \ldots, x_p''\right\} \in C^{(2p)} \text{ and}$$
$$\left\{n_1x_1, \ldots, n_\alpha x_\alpha, nx_1''', \ldots, nx_p'''\right\} \in C^{(q)}$$

where n_1x_1 stands for x_1 repeated n_1 times, and so forth.

Then, as $x_1, \ldots, x_\alpha$ vary, the first set above varies in the fiber of Ψ_{2p} over 0. And owing to (6.5) and (6.6), the second set varies in the fiber of Ψ_q over $(v_1, \ldots, v_p)$. So each set varies in a linear system. It now follows from the Elementary Function Theorem that, for any integral ψx,

$$\psi x_1''' + \cdots + \psi x_p''' + \psi x_1'' + \cdots + \psi x_p'' = v'$$
$$n_1\psi x_1 + \cdots + n_\alpha\psi x_\alpha + n\psi x_1''' + \cdots + n\psi x_p''' = v''$$

where v' and v'' are linear combinations of an algebraic function of $x_1, \ldots, x_\alpha$ and of logarithms of algebraic functions. Set $v := v''/n - v'$. Then (5.4) follows. The proof of Theorem 4 is now complete.

In this proof, the integrals of the first kind predominate. Those of the second and third kind only enter briefly at the end, and these integrals are handled by the Elementary Function Theorem. Nevertheless, they had a rich history in the nineteenth and twentieth centuries, and it would be a worthwhile endeavor to write an account of it.

7 Abel's Version of the Genus

In the preceding section, Abel's Addition Theorem was proved in its most elaborate form, Theorem 4. In the proof, the Jacobian J and the Abel map Ψ_α played a major role, and their algebraicity was crucial. Historically, they also played a substantial role in the development of the theory of algebraic surfaces from about 1870 to 1920; often implicit, this role involved both their direct application to suitable curves and families of curves and their indirect application through their generalization to the Picard and Albanese varieties. However, it was the needs of arithmetic in the 1920s and 1930s that provided the impetus to develop in the 1940s and 1950s a purely algebra-geometric theory. And this theory has led finally to a full understanding of Abel's version of the genus.

Mathematically, J and Ψ_α arise from the symmetric product $C^{(\alpha)}$, which is algebraic by construction. The fibers of Ψ_α are the complete linear systems of degree α; so they are algebraic subvarieties. Why not construct J algebra-geometrically as the quotient of $C^{(\alpha)}$ divided by the relation of linear equivalence? Why not define Ψ_α as the quotient map?

Of course, algebraic geometers have constructed J and Ψ_α in just this way for over a century. Furthermore, they have defined the group structure on J essentially as described in the next paragraph. This definition was attributed by Severi in [104], p. 272, and again in [105], p. 12, to Castelnuovo [21], although Castelnuovo did not mention J explicitly in this 1893 paper.

Fix a base point x_0; it corresponds to the fixed lower limit of integration. Given two linear systems of degree α, take a set from each. Their union determines a linear system of degree 2α. So it is of dimension at least $2\alpha - p$ by the Riemann–Roch Theorem. Assume $\alpha \geq p$. Then this system contains a set of the form $\{\alpha x_0, x_1, \ldots, x_\alpha\}$. Define the sum to be the linear system determined by $\{x_1, \ldots, x_\alpha\}$. It is easy to see that this sum is well defined, and compatible with the sum in J as defined by (6.2).

In 1905, Castelnuovo [22], pp. 478–481, showed how this algebra-geometric construction of J and Ψ_α generalizes naturally to yield an Abelian variety parameterizing the complete linear systems fibering an algebraic system of curves on an algebraic surface. Castelnuovo did not use the term "Abelian variety"; it was not in use at the time. Rather, he spoke of the *Picard variety* of the surface to honor Picard's discovery (see the preceding section) that a compact commutative group variety is parameterized by Abelian functions. Castelnuovo needed these functions to complete research of his own, of Enriques's, and of Severi's into the fundamental nature of irregular surfaces. (See [23], pp. 492–493; see [24], pp. 709–710; see [74], p. 369; and see [121], pp. 104, 162–163.)

In 1950, Weil [114], p. 439, subtly refined the use of the term "Picard variety," restricting it to mean the Abelian variety parameterizing complete linear systems of subvarieties of codimension 1 of a given variety of any dimension; the parameterization is part of the package. In his commentary [117], I, p. 572, Weil explained: "Historically speaking, it would have been justified to give it Castelnuovo's name, but it was a question of tampering as little as possible with common usage rather than rendering unto this master his due honor." Weil's refined usage quickly became standard.

Also, Weil [114], p. 438, discussed a related Abelian variety, and named it the *Albanese variety*, which is the name that has stuck. Weil also suggested the name "dual Picard variety," and used it in his paper [115] in 1952.

The Albanese variety is the target of a map Ψ whose source is the given variety X. Like the Abel map, Ψ is given by simple integrals of the first kind. More abstractly, Ψ is characterized by a universal property: any map from X to an Abelian variety factors through Ψ. For a curve, the Albanese variety and the Picard variety coincide; both are equal to the Jacobian. This statement is an abstract algebra-geometric version of the Equivalence Theorem.

However, the Albanese variety of a surface had already been introduced in 1913 by Severi [103]. He noted that it is distinct from the Picard variety, but each admits a homomorphism onto the other (the two are "isogenous," to use another common term Weil introduced in [113]; see [117], I, p. 570). For the next forty years, Severi and his students spoke of there being two Picard varieties associated to a surface (see [7], p. 1, for example). So "historically speaking, it would have been justified to give it" Severi's name.

Also, Weil offered rather weak mathematical justification for having given the variety Albanese's name: in [114], p. 438, he said only "it had been considered mainly by Albanese"; in [117], I, p. 571, he said roughly that Albanese had obtained a bit better understanding of the fibers of Ψ than Severi had.

Perhaps, Weil acted instead out of a sense of poetic justice; this idea comes from reading between the lines in [117], I, on pp. 553 and 562 as well. Furthermore, Lang [71] documents other cases where Weil did not simply ignore social conventions for giving credit, but took "license for obscuring and misrepresenting works and original ideas of others who opened up the field, and for poking fun at them."

On the one hand, perhaps Weil felt Severi had not given Castelnuovo enough credit, and on the other, perhaps Weil felt he owed Albanese a debt of gratitude for having established in São Paulo, Brazil, a mathematical library where Weil "profited greatly" from reading for the first time the works of Castelnuovo, Torelli, and others. Both Albanese and Weil were in São Paulo for about a year, the period of time after Albanese returned to Brazil from four years back in Italy and before he died on 8 June 1947, but there is no evidence the two men ever met (see [26], p. 2, and [118], pp. 188, 192).

Severi was not pleased by the name! (See [117], I, p. 571, and [106], p. 80.)

In 1946, Weil [112] carefully rebuilt the foundations of algebraic geometry, and in 1948, he [113] developed an algebra-geometric theory of the Riemann–Roch Theorem, Abelian varieties, and the Jacobian. This theory is the first valid in arbitrary characteristic, and it served as the basis for Weil's two great proofs in [113] of E. Artin's celebrated 1921 conjectures about the arithmetic of a curve defined by polynomial equations with coefficients in a finite field.

In fact, in 1882, Dedekind and Weber gave an abstract algebraic proof of the Riemann–Roch Theorem, and in 1929, F. K. Schmidt observed that their proof required little change to make it valid in arbitrary characteristic. In the 1930s, Hasse proved Artin's "Riemann Hypothesis" in genus 1 using an analogue of the theory of elliptic functions, and he and Deuring suggested a geometric way to generalize the proof to higher genus. This work inspired Weil. (See [43], pp. 65 and 82–83, and [71], pp. 50–52.)

Weil constructed the Jacobian J by patching together copies of the set U_p parameterizing linear systems of degree p and dimension 0 on the curve C. But Weil left open two questions: Is J defined over the same coefficient field as C? Is J embeddable in projective space?

These questions were answered in the affirmative in the 1950s by Chow, Matsusaka, and Weil himself. All three constructed both the Albanese and the Picard varieties of a variety of any dimension and in any characteristic. Matsusaka gave the first construction of the Picard variety. It is a lot like Castelnuovo's, but Matsusaka used the theory of Chow coordinates (developed by Chow and van der Waerden in 1938) to form the quotient, which, from the start, has the right coefficient field and is projective.

Chow gave a similar construction of the Jacobian, and then obtained the Albanese and Picard varieties by using his theory of the "image" and "trace" of the Jacobian of a "general curve," which is a 1-dimensional section of the ambient variety by a general linear space. Chow's work appeared after Matsusaka's, but his construction of the Jacobian was announced earlier by Weil [114], p. 438, in 1950, and the year before by Chow himself, according to a footnote in [28], p 453.

Weil gave a more complete and elegant treatment, based on the "see-saw principle," taken directly from Severi, and the theorems of the square and the cube, in a course at the University of Chicago. He also developed a suitable theory of descent of coefficient fields. His work became the core of Lang's book [70] of 1959. (See [116], p. 248, and [117], II, pp. 540–541.) This book also contains many historical notes describing the work of Chow, Matsusaka, and many others, along with references to the original papers.

In the 1950s, Rosenlicht published a remarkable series of papers, which grew out of his 1950 Harvard thesis. Like Gorenstein's thesis on adjoints, Rosenlicht's thesis was supervised by Zariski, who had studied Abelian functions and algebraic geometry with Castelnuovo, Enriques, and Severi in Rome from 1921 to 1927 (see [85], Ch. 3). Rosenlicht generalized the theory of the Abel map to a curve C with arbitrary singularities in arbitrary characteristic. His work leads to a full explanation of the final mystery in Abel's Paris paper [1]: the significance of the number γ, Abel's version of the genus.

In [94], Rosenlicht generalized the two basic notions: (linear) equivalence of sets of points, and differentials of the first kind. To discuss them, let $\mathcal{O}$ be the (semi-local) ring of rational functions on C that can be expressed as a fraction whose numerator and denominator are polynomials in the coordinate functions on C and whose denominator vanishes at no singularity of C.

If two point sets contain no singularity, then they are called *equivalent* if, after all common points are removed, the remaining points are the zeros and poles of a function in $\mathcal{O}$ whose reciprocal is also in $\mathcal{O}$.

Care must be taken with a point set that contains a singularity. It is best to interpret the set as a subscheme defined locally by a principal ideal. This idea was introduced by Cartier in his 1958 Paris thesis [19], p. 222. The theory was developed further by Grothendieck in his February 1962 Bourbaki talk [56], No. 232, § 4, and he called these subschemes *Cartier divisors*. This name is now common, but for the sake of simplicity, not used below.

A generalized differential of the first kind, or *Rosenlicht differential*, is a meromorphic differential ω on the desingularization C' of C possessing two properties: first, ω has no pole at any $z \in C'$ lying over a simple point of C; second, for each $h \in \mathcal{O}$, the sum $\sum_z \operatorname{res}_z h\omega$ vanishes as $z \in C'$ ranges over the points lying over the singularities of C.

Let π be the number of independent Rosenlicht differentials; it is known today as the *arithmetic genus* of C. For example, if C is plane a curve of degree d, then $\pi = (d-1)(d-2)/2$ by Rosenlicht's lemma on p. 186. For any C, his Theorem 7 asserts a generalized Riemann–Roch Formula: $\mu - \alpha = \pi - i$ where i is the number of independent Rosenlicht differentials that vanish at a given set of μ points and where α is the dimension of the system of all sets that are equivalent, in the generalized sense, to the given set.

Let $\overline{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in the function field of C, and put $\delta := \dim \overline{\mathcal{O}}/\mathcal{O}$. Rosenlicht's Theorem 8 asserts $\pi = p + \delta$. For example, if C has only nodes and cusps, then δ is the number of them; if also C is plane, then Rosenlicht's formula recovers Clebsch's formula, (3.6).

In [95], Rosenlicht generalized "Abel's theorem," that is, the Equivalence Theorem, Theorem 2. Namely, his Theorem 1 asserts that, over $\mathbb{C}$, two sets of points are equivalent if and only if, for every Rosenlicht differential, the two corresponding sums of integrals (from a fixed point) have the same value.

Rosenlicht then fixed a basis $\omega_1, \ldots, \omega_\pi$ of the Rosenlicht differentials. His Theorem 2 asserts that, inside the vector space $\mathbb{C}^\pi$, the corresponding period vectors form a discrete subgroup. Its rank is strictly less than 2π if C is singular. So the quotient P is a complex Lie group, but P is not compact, so is not an Abelian variety. Rosenlicht termed P the *generalized Jacobian,* and observed that it is the target of a generalized Abel map Φ_α.

Let ω_0 be a meromorphic differential. Suppose that, for any two equivalent sets of points, the two corresponding sums of integrals of ω_0 have the same value. Then ω_0 is a Rosenlicht differential. Indeed, suppose not. Then the following argument (inspired by [95], p. 507) yields a contradiction.

Given $\alpha \geq \pi + 1$, let $z_1, \ldots, z_\alpha$ vary independently in an open subset U of the smooth locus of C containing no pole of ω_0. Form the $\pi + 1$ by α matrix $M := \left[\omega_i/dz_j\right]$ of functions. Since $\omega_0, \ldots, \omega_\pi$ are independent, M is of rank $\pi + 1$ on an open subset of the self-product U^α. But M is equal to the Jacobian matrix of the following generalized Abel map:

$$\Lambda\colon U^\alpha \to \mathbb{C}^{\pi+1} \text{ given by}$$
$$\Lambda(z_0,\ldots,z_\alpha) = \left(\sum_{j=0}^{\alpha}\int^{z_j}\omega_0,\ldots,\sum_{j=0}^{\alpha}\int^{z_j}\omega_\pi\right)$$

Therefore, by the Implicit Function Theorem, the image ΛU^α contains a nonempty open subset V of $\mathbb{C}^{\pi+1}$.

By hypothesis, Λ remains constant as the set $\{z_1,\ldots,z_\alpha\}$ varies in a linear system. Hence, by the Riemann–Roch Theorem, the fibers of Λ are all of dimension at least $\alpha-\pi$. Consequently, no such V can exist, a contradiction.

For example, suppose C is plane. Then the curves D of given degree cut on C a complete linear system of equivalent sets by Rosenlicht's Corollary on p. 175 in [94]; whence, up to a fixed set of points, so do all the D that pass through that fixed set. Conversely, every complete linear system on C is of the latter form, because any set of points lies in some D of suitable degree.

By the observations above, the Rosenlicht differentials ω are characterized by the constancy of the sum

$$\int^{z_1}\omega+\cdots+\int^{z_\mu}\omega \tag{7.1}$$

where $\{z_1,\ldots,z_\mu\}=C\cap D$ as D ranges over the curves of given degree passing through a given set, for all sets and degrees.

For instance, if C has only nodes and cusps, then the Rosenlicht differentials ω of the third kind are just the differentials flagged by Clebsch and Gordan [33], p. 49, because, for these ω, the sum (7.1) remains constant.

Is the constancy of (7.1) what Abel considered? Did he define γ as the number of independent such ω? If so, then $\gamma=\pi$. But, as Brill and Noether [17], pp. 216 and 222, pointed out, Abel found instead that $\gamma=p$ if C is smooth at finite distance and has mild singularities at infinity, for instance, if C is hyperelliptic. Rowe [97] carried Abel's computations further; in more cases, he arrived at an expression for γ (actually, for $\mu-\alpha$), and in a supplement, Cayley proved that this expression is equal to p.

How can the constancy of (7.1) lead to different values of γ? Through different restrictions on D. Clebsch and Gordan, in effect, placed no restrictions on D. Abel [1], p. 147, however, required D, to begin with, to be defined by the vanishing of a polynomial $g(x,y)$ of the form:

$$g(x,y):=g_1(x)y^{n-1}+g_2(x)y^{n-2}+\cdots+g_n(x)$$

where n is the highest power of y in the polynomial $f(x,y)$ defining C as in (2.1). See the discussion of the hyperelliptic case in Sects. 4 and 5.

More precisely, $\gamma=p$ if the D are required to satisfy the adjunction conditions, which are discussed in Sect. 3, and $\gamma=\pi$ if not.

There is, however, a middle ground: require the D to satisfy only some of the adjunction conditions. This idea was introduced in an 1879 monograph by Lindemann, who aimed to prove a generalized Riemann–Roch theorem. However, the same year in [78], Noether detected a gap in Lindemann's transcendental proof, and replaced it by an algebraic proof, which also yields additional results. (See the footnote in [78], p. 507, and also [106], p. 9.)

In fact, there are a number of partial desingularizations $C^\dagger$ that lie between C and its full desingularization. Indeed, it follows from Theorems 4, 5, and 8 in Rosenlicht's paper [94] that there exists a $C^\dagger$ whose arithmetic genus γ is any given number between p and π.

Given a $C^\dagger$, both it and C have canonical systems associated to their respective Rosenlicht differentials. The system on $C^\dagger$ is obtained from the system on C by imposing the appropriate adjunction conditions, and the latter are defined by the conductor ideal associated to the two curves. This statement can be seen as part of Grothendieck's extensive duality theory, first described in his May 1957 Bourbaki talk [56], No. 149.

If $C^\dagger$ has bad (non-Gorenstein) singularities, then additional care must be taken with the notion of canonical system, and for that matter, of any linear system. From a technical point of view, it is best to work with "ω-pseudo-divisors" as introduced in [6]; they are defined by the invertible subsheaves of the torsion-free sheaf of Rosenlicht differentials, the dualizing sheaf ω. From a conceptual point of view, the theory of ω-pseudo-divisors can been seen to be close in spirit to Brill and Noether's theory in [16] and Noether's in [78].

For any curve C, plane or not, Rosenlicht proved the generalized Jacobian P and the Abel map Φ_α are algebraic. To do so, he [95], Thm. 7, gave a second construction of them, a purely algebraic one. It is modeled on Weil's construction for smooth curves via patching, and works in any characteristic.

In 1962, Oort [81] gave another algebraic construction by a "method of group extensions," which works in any characteristic. The generalized Jacobian P is an extension of the Jacobian J of the desingularized curve by a linear group. Oort assumed J exists. Then he constructed the extension in a way that shows how its nature depends on the nature of the singularities.

In his February 1962 Bourbaki talk [56], No. 232, Grothendieck outlined a sophisticated new construction, yielding a Picard variety associated to any given variety. When the variety is a curve, Grothendieck's Picard variety is just Rosenlicht's generalized Jacobian. (See also Grothendieck's letter [36], p. 197, of 9 August 1960 to Serre.) In spirit, the construction is a lot like Castelnuovo's and Matsusaka's, but it is a lot more refined. To replace Matsusaka's use of Chow coordinates, Grothendieck introduced a new parameter space, the "Hilbert scheme."

Grothendieck, however, had to appeal to the theory of Chow coordinates for a key finiteness result: in projective space, the subvarieties of given degree form a bounded family (see [56], No. 221, p. 7). Grothendieck's action was "scandalous," since, on p. 1, he proclaimed that Hilbert schemes were "destined to replace" Chow coordinates as a tool – a prophetic statement.

In 1966, Mumford [77], Lect. 14, saved the day. He introduced an important new concept, now known as "Castelnuovo–Mumford" regularity; it yields a simple direct proof of the needed finiteness. In addition, Mumford gave the details of Grothendieck's construction of the Picard variety, worked out in the case of surfaces.

Furthermore, Mumford showed how Grothendieck's work yields the first algebraic proof of a basic theorem, which asserts the completeness of the characteristic linear system of a suitably positive curve on a surface; in fact, this proof is the goal of [77]. In 1904, first Enriques and then Severi gave algebraic proofs, which were later found to be deficient. Meanwhile, in 1905, Castelnuovo used the theorem to find the dimension of the Picard variety. In 1910, Poincaré gave a rigorous analytic proof (see [121], pp. 98–104).

For any curve C, the algebraicity and other properties of P and Φ_α yield a new version of Theorem 4, the elaborate form of the Addition Theorem. In it, p is replaced by π, and integrals of the first kind are replaced by integrals of Rosenlicht differentials. The proof at the end of Sect. 6 carries over with almost no change; there is no need for P to be compact, nor for multiplication by n to define a map $\nu_n \colon P \to P$ of any particular (finite) degree.

It is interesting to compare the two versions of Theorem 4. The new one asserts that an arbitrary sum of integrals of the same Rosenlicht differential can be reduced to a sum of π such integrals, plus a constant. The old version asserts that the initial sum can be reduced to a sum of p such integrals, plus an elementary function v. If $p < \pi$, then v need not be constant.

In 1957, Rosenlicht [96] proved a sort of universal mapping property for generalized Jacobians. However, he did not begin with a singular curve and its Jacobian. Rather, he began with a map $\Upsilon \colon U \to G$ where U is a smooth open curve and G is a commutative algebraic group. Then he embedded U in a suitable compact singular curve C so that Υ extends to a map $C \to G$ that factors through the generalized Jacobian of C. Consequently, this universal mapping property does not amount to an abstract algebra-geometric version of the Equivalence Theorem for a given singular curve.

Nevertheless, Rosenlicht [96], p. 81, observed that his universal mapping property "can be used to prove theorems about algebraic groups themselves." Also, he noted that Serre too had proved this property independently and that Serre and Lang had applied it to class field theory. Then in 1959 Serre published a monograph [100], where he developed all this theory from scratch.

Generalized Jacobians appear in another major part of the theory of Abelian integrals, which concerns the way they vary and degenerate, or specialize, as the underlying curve does. The subject was traced by Severi [105], p. 14, back to some of Klein's work in 1874, in which he studied the limits of integrals of the first kind on a smooth plane quartic as the quartic varies, acquiring a double point. From 1890 to 1910, the theory was developed by Picard, Poincaré, and Lefschetz, and by Castelnuovo, Enriques, and Severi, and applied to the theory of algebraic surfaces. (See [24], pp. 725–727; see [121], Chaps. 6–7; see [43], pp. 35, 53; and see [60], pp. 253–255.)

In 1947, Severi published a monograph [105], about generalized Jacobians and their behavior under specialization for curves with at most nodes and cusps. In fact, according to Serre [100], p. 114, Severi was the first author to discuss the generalized Jacobian explicitly: he studied its analytic structure and its algebraic structure, but "without always properly distinguishing the one from the other."

In 1956, Igusa [62], Thm. 3, established the compatibility of specializing a curve with specializing its generalized Jacobian when the general curve is smooth and the special curve has at most one node. Moreover, in Sect. 1, he proved that such a specialization obtains in the important case now known as a "Lefschetz pencil": the curves are cut on a smooth surface by a general 1-parameter family of hyperplanes. Igusa worked in arbitrary characteristic, and was the first to do so; he explained on p. 171, that Néron had, in 1952, studied the total space of such a family of Jacobians, but Néron had not explicitly considered the special fiber.

Igusa's approach is like Castelnuovo's, Chow's, and Matsusaka's before him and Grothendieck's after him. However, Grothendieck went considerably further: not only did he prove compatibility with specialization for an arbitrary family of (irreducible) curves, in equicharacteristic or mixed, but he worked with a family of varieties of arbitrary dimension.

In particular, when a curve varies and degenerates, its arithmetic genus π, which is the dimension of its generalized Jacobian, remains constant, whereas its geometric genus p drops. So, for the purpose of studying the degeneration of Abelian integrals, Abel's version of the genus, π or γ, is better suited to the job than Riemann's version p.

Thus, though it once seemed odd that Abel had focused on the constancy of the sum of integrals, rather than on the boundedness of the summands, in fact the constancy turned out to be the deeper, subtler condition. Indeed, after the significance of the boundedness was explained by Riemann, it took a hundred years more before the significance of the constancy was fully understood. Once again, Abel's intuition has proven to be sound!

8 Conclusion

For various reasons, four different theorems have been commonly accepted by respected scholars as the celebrated theorem known as *Abel's Theorem*. In fact, some scholars have implicitly accepted two of the four as the same. But the four theorems are not simply versions of each other. Rather, they are mathematically distinct, and are the results of successive steps forward.

The Elementary Function Theorem is the result of the first step. It is the key to the other theorems. Abel proved it in relatively explicit forms in [1] and [2]; he also proved it, with far less work, in more elegant and more conceptual forms in [1], [3], and [4]. In [1], p. 150, he explained why he proved it twice: the conceptual method is not computationally effective.

However, in [3], Abel proved only the Elementary Function Theorem, and he did not identify it as a preliminary result. Moreover, it is the one theorem common to Abel's four papers. Doubtless, these reasons are ultimately why this theorem is so widely accepted as Abel's Theorem.

The Equivalence Theorem is the result of the second step. Combined with its converse, which Abel did not recognize, it has become a fundamental result in an important part of modern mathematics, the theory of Riemann surfaces. So the theorem needs a name for ready reference, and it is common and fitting to use Abel's name since he made the seminal contribution. Historical correctness being a secondary consideration, mathematicians will doubtless always refer to the full Equivalence Theorem as Abel's Theorem.

The Relations Theorem is the result of the third step. In the introduction to [1], Abel stated it informally in a way that makes it sound deceptively like a more refined form of the Elementary Function Theorem. Furthermore, he reinforced this false idea a few pages later. There appears to be no further reason for interest in this theorem; so there appears to be no strong reason to accept it as Abel's Theorem.

The Addition Theorem is the result of the fourth and last step. In the introductions to both [1] and [2], Abel makes it clear that this theorem is his goal. However, he stated the theorem in different forms.

In the introduction to [2], Abel stated the Addition Theorem in its simplest form, which is an immediate consequence of the Relations Theorem. However, he gave proofs only for hyperelliptic integrals.

In the introduction to [1], Abel stated the Addition Theorem in a more elaborate form, and in the course of the paper, he refined the statement even further. Although Abel's proof is, apparently, not fully valid, nevertheless, in its most elaborate form, the Addition Theorem can be proved using the Jacobian variety and the Abel map, which combine the full Equivalence Theorem and the Riemann–Roch Theorem in a powerful algebra-geometric form.

The Addition Theorem involves a numerical invariant, the genus. But the genus comes in two versions, Abel's and Riemann's. In fact, Abel's definition is nebulous, but it can be made clear and precise. Then the two genera may differ in value; if so, then Riemann's has the smaller value, ostensibly an advantage. With either genus, the Addition Theorem sounds the same, mutatis mutandis. However, Abel's genus is better suited for the study of the degeneration of Abelian integrals.

In 1832, Jacobi successfully advocated that, as a monument to Abel's genius, the Addition Theorem be called Abel's Theorem. Yet, Jacobi knew only the simplest form of the theorem; the more elaborate forms appear only in [1], which was, at the time, temporarily misplaced.

Thus, of the four theorems, only the Addition Theorem can rightfully be called **Abel's Theorem!**

Acknowledgments. The author is indebted to Beverly, his wife, and to Ragni Piene, the editor, for finding numerous obscure passages and typographical errors in previous drafts.

References

[1] N. H. Abel, *Mémoire sur une propriété générale d'une classe très étendue de fonctions transcendantes.* Mémoires présentés par divers savants. **VII** (1841), 176–264. = [5] **I**, XII, pp. 145–211. ▶ Introduction trans. in [11], pp. 188–189.

[2] N. H. Abel, *Remarques sur quelques propriétés générales d'une certaine sorte de fonctions transcendantes.* J. für Math. **3** (1828), 313–323. = [5] **I,** XXI, pp. 444–456. ▶ Partly trans. in [11], pp. 190–195.

[3] N. H. Abel, *Démonstration d'une propriété générale d'une certaine classe de fonctions transcendantes.* J. für Math. **4** (1829), 200–201. = [5] **I,** XXVII, pp. 515–517.

[4] N. H. Abel, *Sur la comparaison des fonctions transcendantes.* Posthumous fragment. = [5] **II,** X, pp. 55–66.

[5] N. H. Abel, *Œvres complètes,* **I, II.** Sylow L. and Lie S. (eds.), Grøndahl, Christiania (Oslo), 1881.

[6] A. B. Altman and S. L. Kleiman, *Compactifying the Jacobian.* Bull. Amer. Math. Soc. **82** (1976), 947–949.

[7] A. Andreotti, *Sopra le varietà di Picard di una superficie algebrica.* Rend. Accad. Nazionalle dei XL (IV) **2** (1951), 1–9.

[8] H. F. Baker, *Abelian functions. Abel's theorem and the allied theory of theta functions.* Cambridge U. Press, 1897, reissued 1995.

[9] E. T. Bell, "Men of Mathematics." Simon and Schuster, 1937, reprinted 1965.

[10] E. T. Bell, "The development of mathematics." Second ed., McGraw-Hill, 1945.

[11] G. Birkhoff and U. Merzbach, "A source book in classical analysis." Harvard U. Press, 1973.

[12] C. A. Bjerknes, "Niels Henrik Abel." Mémoire Soc. Sci. physiques et naturelles de Bordeaux, 3^e série, I, Gauthier-Villars, 1884.

[13] G. A. Bliss, *The reduction of singularities of plane curves by birational transformation.* Bull. AMS **29** (1923), 161–183.

[14] G. A. Bliss, "Algebraic functions." AMS Colloquium Publ. **XVI**, 1933, republished by Dover, 1966.

[15] A. Brill, *Max Noether.* Jahresbericht DMV **32** (1923), 211–233.

[16] A. Brill and M. Noether, *Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie.* Math Ann. **7** (1874), 269–310.

[17] A. Brill and M. Noether, *Die Entwickelung der Theorie der algebraischen Functionen in älterer und neuerer Zeit.* Jahresbericht DMV **3** (1894), 107–566.

[18] W. Burau, *Clebsch, Rudolf Friedrich Alfred.* In "Dictionary of Scientific Biography." Gillispie (ed. in chief) Vol. III. Scribner's 1971, pp. 313–315.

[19] P. Cartier, "Dérivationes et diviseurs en géométrie algébrique." Gauthier-Villars, 1959.

[20] E. Casas, "Singularities of plane curves." LMS Lecture Notes Series **276**, 2000.

[21] G. Castelnuovo, *Le corrispondenze univoche tra gruppi di p punti sopra una curva di genere p.* Rend. Ist. Lomb. (2) **25** (1893), 1189–1205. = "Memorie scelte." Zanichelli, 1937, pp. 79–94.

[22] G. Castelnuovo, *Sulle integrali semplici appartenenti ad una superficie irregolare* (3 notes). Rend. R. Accad. Lincei **V 14** (1905), 545–546, 593–598, 655–663. = "Memorie scelte." Zanichelli, 1937, pp. 473–507.

[23] G. Castelnuovo and F. Enriques, *Sur quelques résultats nouveaux dans la théorie des surfaces algébriques.* Note V. in "Théorie des fonctions algébriques de deus variables indépendantes" by E. Picard and G. Simart, t. II, Gauthier-Villars, 1906, republished by Chelsea, 1971, pp. 485–522.

[24] G. Castelnuovo and F. Enriques, *Die algebraischen Flächen vom Gesichtspunkte der birationalen Transformationen aus.* Dated Dec. 1914. In "Encyklopädie math. Wissenschaften." Vol. **III C 6b**. Teubner 1927, pp. 674–768.
[25] G. Castelnuovo, F. Enriques, and F. Severi, *Max Noether.* Math. Ann. **93** (1925), 161–181.
[26] B. Castrucci, *Prof. Giacomo Albanese.* Bull. Soc. Saõ Paulo **2** (1947), 1–5.
[27] A. Cayley, *On the transformation of plane curves*. Proc. LMS **I** (1865–1866), 1–11. = Papers VI, Cambridge U. Press, 1893, pp. 1–8.
[28] W-L. Chow, *The Jacobian variety of an algebraic curve.* Amer. J. Math. **76** (1954), 453–476.
[29] A. Clebsch, *Ueber die Anwendung der Abelschen Functionen in der Geometrie.* J. für Math. **63** (1864), 189–243.
[30] A. Clebsch, *Ueber diejenigen ebenen Curven, deren Coordinaten rationale Functionen eines Parameters sind.* J. für Math. **64** (1865), 43–65.
[31] A. Clebsch, *Ueber die Singularitäten algebraischer Curven.* J. für Math. **64** (1865), 98–100.
[32] A. Clebsch, *Ueber diejenigen Curven, deren Coordinaten sich als elliptische Functionen eines Parameters darstellen lassen.* J. für Math. **64** (1865), 210–271.
[33] A. Clebsch and P. Gordan, "Theorie der Abelschen Funktionen." Teubner, 1866.
[34] A. Clebsch and F. Lindemann, "Vorlesungen über geometrie." Vol. 1, Teubner, 1876.
[35] C. H. Clemens and P. A. Griffiths, *The intermediate Jacobian of the cubic threefold.* Ann. Math. (2) **95** (1972), 281–356.
[36] P. Colmez and J.-P. Serre, "Correspondance Grothendieck–Serre." SMF, 2001.
[37] R. Cooke, *Abel's Theorem.* In "The History of Modern Math. Vol. 1: Ideas and their reception." Rowe D. E. and McCleary J. (Eds.) Academic Press, 1989, pp. 389–421.
[38] J. L. Coolidge, "A history of geometrical methods." Oxford U. Press, 1940.
[39] D. Cox, J. Little, and D. O'Shea, "Ideals, Varieties, and Algorithms." Springer-Verlag, Second edition, 1996.
[40] L. Cremona, *Sugli integrali a differenziale algebrico.* Mem. dell'Acc. delle Scienze di Bologna, (2) **10** (1870), 3–33.
[41] A. Del Centina, *Abel's manuscripts in the Libri collection: their history and their fate.* In [42], pp. 87–103.
[42] A. Del Centina, "Abel's Parisian manuscript, preserved in the Moreniana Library of Florence." Olschki, Firenze, 2002.
[43] J. A. Dieudonné, "History of algebraic geometry." J. D. Sally (trans.) Wadsworth, 1985.
[44] F. Enriques and O. Chisini, "Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche 2." Zanichelli. Vol. 4, 1st ed., 1934. Reprinted 1989.
[45] F. Enriques and F. Severi *Intorno alle superficie iperellittiche.* Rend. Acc. Lincei **XVI** (1907), 443–453. = "Memorie Scelte." Vol. **2**, Zanichelli 1959, pp. 283–294.
[46] C. S. Fisher, *Gordan, Paul Albert.* In "Dictionary of Scientific Biography." Gillispie (ed. in chief) Vol. V. Scribner's 1972, pp. 472–473.
[47] A. R. Forsyth, "Theory of functions of a complex variable." Third edition, Cambridge 1918. Reprinted in two volumes by Dover, 1965.
[48] H. Freudenthal, *Riemann, Georg Friedrich Bernhard.* In "Dictionary of Scientific Biography." Gillispie (ed. in chief) Vol. XI. Scribner's 1975, pp. 447–456.
[49] W. Fulton, "Algebraic curves." Benjamin Inc., 1968.
[50] D. Gorenstein, *An arithmetic theory of adjoint plane curves.* Trans. AMS **72** (1952), 414–436.

[51] E. Goursat, “Functions of a complex variable. Vol. II, Part One.” E. R. Hedrick and O. Dunkel. (trans.) Ginn and Co., 1916. Reprinted by Dover, 1959.
[52] J. J. Gray, *The Riemann–Roch theorem: acceptance and rejection of geometric ideas.* Cahiers d’histoire et de philosophie des sciences **20** (1987), 139–151.
[53] J. J. Gray, *Algebraic geometry in the late nineteenth century.* In “The history of modern mathematics, Vol. I: Ideas and their reception.” Rowe D. E. and McCleary J. (Eds.) Academic Press, 1989, pp. 361–385.
[54] J. J. Gray, *The Riemann–Roch theorem and geometry, 1854–1914.* Doc. Math. J. DMV – Extra Volume ICM 1998. Vol. III, pp. 811–822.
[55] P. A. Griffiths, *Variations on a theorem of Abel.* Inv. Math. **35** (1976), 321–390.
[56] A. Grothendieck, “Fondements de la géométrie algébrique.” Séminaire Bourbaki 1957–1962. Secrétariat Math., Paris, 1962.
[57] K. Hensel and G. Landsberg, “Theorie der algebraischen Functionen einer Variabeln und ihre Anwendung auf algebraische Kurven und abelsche Integrale.” Teubner, 1902. Reprinted by Chelsea, 1965.
[58] C. Houzel, *Fonctions elliptiques et intégrales abéliennes.* In “Abrégé d’histoire des mathématiques 1700–1900, Vol. II.” J. Dieudonné (ed.) Hermann, 1978 = [61], Ch. VIII, pp. 81–190.
[59] C. Houzel, *Histoire de la théorie des équations algébriques de Lagrange à Galois.* Journées X-UPS, **4** (1987), 55–66 = [61], Ch. V, pp. 53–61.
[60] C. Houzel, *Aux origines de la géométrie algébrique: les travaux de Picard sur les surfaces (1884–1905).* Cahiers d’histoire et de philosophie des sciences. Vol. **34** Société Mathématique de France. (1991), 243–276. = [61], Ch. X, pp. 203–234.
[61] C. Houzel, “La géométrie algébrique, Recherches Historiques.” Albert Blanchard, 2002.
[62] J. Igusa, *Fibre systems of Jacobian varieties.* Amer. J. Math. **78** (1956), 171–199.
[63] C. G. J. Jacobi, Review of Legendre’s “Théorie des fonctions elliptiques, troisième supplément.” J. für Math. **8** (1832), 413–417.
[64] C. G. J. Jacobi, *Considerationes generales de transcendentibus Abelianis.* J. für Math. **9** (1832), 394–403. = “Werke II.” Reimer, 1882, pp. 5–16. ▶ Partly trans. in [11] pp. 207–212.
[65] C. G. J. Jacobi, *Notiz über A. Göpel.* J. für Math. **35** (1847), 277–282. = “Werke II.” Reimer, 1882, pp. 145–152.
[66] F. Klein, “Lectures on Mathematics.” Reported by A. Ziwet. Macmillan, 1894.
[67] F. Klein, “Vorlesungen über die Entwicklung der Mathematik im 19. Jarhundert.” Prepared by R. Courant and O. Neugebauer. Part I. Springer, 1926.
[68] M. Kline, “Mathematical thought from ancient to modern times.” Oxford U. Press, 1972.
[69] E. Kramer, *Noether, Max.* In “Dictionary of Scientific Biography.” Gillispie (ed. in chief) Vol. X. Scribner’s 1974, pp. 193–141.
[70] S. Lang, “Abelian varieties.” Interscience Tract **7**. Interscience, 1959.
[71] S. Lang, *Comments on non-references in Weil’s works.* Mitteilungen DMV. **1** (2002), 49–56.
[72] O. A. Laudal, *The mathematical significance of Abel’s Paris memoir.* In [42], pp. 65–68.
[73] S. Lefschetz, *On certain numerical invariants of algebraic varieties with applications to Abelian varieties.* (Bordin prize, Paris, 1919, and Bôcher prize, AMS, 1924.) Trans. AMS **22** (1921), 327–482. = “Selected Papers, Chelsea, 1971, pp. 41–196.
[74] S. Lefschetz, *Hyperelliptic surfaces and Abelian varieties.* In “Selected topics in algebraic geometry.” Bull. National Res. Council, No. 96 (Washington, 1934). Chelsea reprint, 1970, pp. 349–395.

[75] A. M. Legendre, *Cover letter.* J. für Math. **8** (1832), 413.

[76] A. I. Markusevich, "Introduction to the classical theory of Abelian functions." Transl. Math. Monographs **96**, AMS, 1992.

[77] D. Mumford, "Lectures on curves on an algebraic surface." Annals of Math. Studies, No. 59. Princeton U. Press, 1966.

[78] M. Noether, *Ueber Schnittpunktsysteme einer algebraischen Curve mit nicht-adjungirten Curven.* Math. Ann. **15** (1879), 507–528.

[79] M. Noether, *Rationale Ausführung der Operationen in der Theorie der algebraischen Functionen.* Math. Ann. **23** (1884), 311–358.

[80] J. J. O'Connor and E. F. Robertson, *Rudolf Friedrich Alfred Clebsch.* http://www-history.mcs.st-andrews.ac.uk/history/References/Clebsch.html

[81] F. Oort, *A construction of generalized Jacobian varieties by group extensions.* Math. Ann. **147** (1962), 277–286.

[82] O. Ore, "Niels Henrik Abel, mathematician extraordinary." U. Minn. Press, 1957, reprinted Chelsea Publishing, 1974.

[83] O. Ore, *Abel, Niels Henrik.* In "Dictionary of Scientific Biography." Gillispie (ed. in chief) Vol. I. Scribner's 1971, pp. 12–17.

[84] P. Painlevé, *Sur les fonctions qui admettent un théorème d'addition.* Acta Math. **27** (1903), 1–54.

[85] C. Parikh, "The unreal life of Oscar Zariski." Academic Press 1991.

[86] S. J. Patterson, "Abel's Theorem." R. Circ. Mat. Palermo II, Supplemento N **61** (1999), 9–48.

[87] L. Pepe, "Genius and orderliness, Abel in Berlin and Paris." In [42], pp. 87–103.

[88] E. Picard, *Mémoire sur la théorie des fonctions algébriques de deux variables.* J. Math. Pures et Appl. **5** (1889), 135–319. = "Oeuvres" Tome III, Cen. nat. rech. sc., 1980, pp. 201–385.

[89] E. Picard, *Sur la théorie des groupes et des surfaces algébriques.* R. Circ. Mat. Palermo **9** (1895), 244–255. = "Oeuvres" Tome III, Cen. nat. rech. sc., 1980, pp. 417–428.

[90] E. Picard, "Traité d'analyse." Vol. 2, 2nd ed., Gauthier-Villars, 1905.

[91] J. B. Pogrebyssky, *Brill, Alexander Wilhelm von.* In "Dictionary of Scientific Biography." Gillispie (ed. in chief) Vol. II. Scribner's 1970, p. 465.

[92] B. Riemann, *Theorie der Abelschen Functionen.* J. für Math. **54** (1857), 115–155. = "Werke." Springer, 1990, pp. 120–174. ▶ Partly trans. in [11], pp. 196–201.

[93] G. Roch, "Ueber die Anzahl willkürlichen Constanten in algebraischen Functionen." J. für Math. **64** (1864), 372–376. ▶ partly trans. in [11], pp. 201–203.

[94] M. Rosenlicht, *Equivalence relations on algebraic curves.* Annals Math. **56** (1952), 169–191.

[95] M. Rosenlicht, *Generalized Jacobian varieties* Annals Math. **59** (1954), 505–530.

[96] M. Rosenlicht, *A universal mapping property of generalized Jacobian varieties.* Annals Math. **66** (1957), 80–88.

[97] R. C. Rowe, *Memoir on Abel's theorem.* Phil. Trans. Royal Acad. **172** (1881), 713–750.

[98] G. Scorza, *Intorno alla teoria generale delle matrici di Riemann ed alcune se applicazione* R. Circ. Mat. Palermo **41** (1916), 263–380. = "Opere scelte." Cremonese, Vol. 2, 1961, pp. 126–275.

[99] C. A. Scott, *On the intersections of plane curves.* Bull. AMS **2** (1897/98), 260–273.

[100] J.-P. Serre, "Groupes algébriques et corps de classes." Hermann, 1959.

[101] F. Severi, *Sulle superficie algebriche che ammettonon un gruppo continuo permutabile a due parametri, di trasformazioni birazionali.* Atti del R. Ist. Veneto di Scienze, Lettere ed Arti, **67** (1907), 409–419.

[102] F. Severi, *Relazioni fra gl'integrali semplici e multipli di I^a specie d'una varietà algebrica.* Annali Mat. **20** (1913), 201–216.
[103] F. Severi, *Un teorema d'inversione per gli integrali semplici di 1^a specie appartenentu ad una superficie algebrica.* Atti del R. Ist. Veneto di Scienze, Lettere ed Arti, **72** (1913), 765–772.
[104] F. Severi, "Vorlesungen über algebraische Geometrie." E. Löffler (trans.) Teubner, 1921. Reprinted by Johnson 1968.
[105] F. Severi, "Funzioni Quasi Abeliane." Pontificia Accademia delle Scienze. Scipta Varia, **4**, 1947.
[106] F. Severi, "Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica. Vol. 3. sviluppo delle teorie degli integrali semplici e multipli sopra una superficie o varietà e delle teorie collegate." Cremonese, 1959.
[107] I. R. Shafarevich, "Basic algebraic geometry." K. A. Hirsch (trans.) Springer, 1974.
[108] C. L. Siegel, "Topics in complex function theory." Vol. 3. E. Gottschling and M. Tretkoff (trans.) Interscience Tracts, No. 25. Wiley, 1973.
[109] A. Stubhaug, "Niels Henrik Abel and his times: called too soon by flames afar." Translated from the second Norwegian edition of 1996 by R. H. Daly. Springer 2000.
[110] L. Sylow, *Remarques sur les nombres γ et $\mu - \alpha$.* In [5], **II**, pp. 298–300.
[111] K. Weierstrass, "Theorie der Abelschen Transcendenten." = Werke 4. Mayer & Müller, 1902.
[112] A. Weil, "Foundations of algebraic geometry." AMS Coll., Vol. **XXIX**, 1946. Second ed. 1962.
[113] A. Weil, (a) "Sur les courbes algébriques et les variétés qui s'en déduisent," Hermann, Paris, 1948; (b) Variétés abéliennes et courbes algébriques," ibid., 1948; (c) 2$^{\text{nd}}$ edition of (a) and (b), under the collective title "Courbes algébriques et variétés abéliennes," ibid., 1971.
[114] A. Weil, *Variétés abéliennes.* In "Colloque d'Algèbre et Théorie des Nombres." CNRS, Paris, pp. 125–127. = "Papers" I, Springer, 1979, pp. 437–440.
[115] A. Weil, *On Picard varieties.* Amer. J. Math. **90** (1952), 865–893. = "Papers" II, Springer, 1979, pp. 73–102.
[116] A. Weil, *On the projective embedding of Abelian varieties.* In "Algebraic geometry and topology, A symposium in honor of S. Lefschetz," Princeton U. Press, 1957. = "Papers" II, Springer, 1979, pp. 177–181.
[117] A. Weil, *Commentaire.* In "Collected Papers." Springer, 1979, Vol. I, pp. 518–574, I,I pp. 526–553, and III, pp. 443–465.
[118] A. Weil, "The Apprenticeship of a mathematician." J. Gage (trans.) Birkhäuser 1992.
[119] H. Weyl, *The concept of a Riemann surface.* 3rd ed., Teubner, 1955, G. R. Maclane (trans.) Addison-Wessley, 1964.
[120] W. Wirtinger, *Algebraische Funktionen und ihre Integrale.* Dated Oct. 1901. In "Encyklopädie math. Wissenschaften." Vol. **II B 2.** Teubner 1921, pp. 115–175.
[121] O. Zariski, "Algebraic Surfaces." Springer. First edition, Ergebnisse **III, 5,** 1935. Second supplemented edition, Ergebnisse **61**, 1971.